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Topics in Functional Analysis over Valued Division Rings

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Topics in Functional Analysis over Valued Division Rings

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To the memory of
SILVIO MACHADO

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PREFACE

In this volume we are interested mainly with topological vector spaces (over valued division rings) and with some other algebraic structures equipped with suitable topologies making the algebraic operations continuous mappings. In the applications of the general theory, the algebraic structures (vector spaces, algebras and rings) considered are sets of functions defined on a topological space X and having values either in a valued division ring $(F,|\cdot|)$, or in a topological vector space, algebra or ring (E,τ). This set of functions receives the algebraic structure given by pointwise operations and, most of the time, the topology of compact convergence.

In the first chapter we study the properties of valued division rings $(F,|\cdot|)$ that are needed in the subsequent chapters. It presupposes only a basic knowledge of division rings; in fact, very little besides the actual definition of a division ring or of a field. After establishing the elementary properties of an absolute value $\lambda \to |\lambda|$, we present Kaplansky's Lemma (see Lemma 1.23) for non-archimedean valued division rings, which is the main tool for getting Stone-Weierstrass type theorems. The proof of Kaplansky's Lemma that we have presented is due to Chernoff, Rasala and Waterhouse [14]. Kaplansky's Lemma appeared in Kaplansky [38].

The general theory of topological vector spaces, over non-trivially valued division rings, is the subject of chapter 2. Three main results are dealt with in this chapter: the closed graph theorem, the open mapping theorem and the Banach-Steinhaus theorem. We extend the basic notion of a "string" (see definition 2.29) of Adasch, Ernst and Keim [1] to topological vector spaces over non-trivially valued division rings. As in the case of [1], where the valued fields are $\mathbb{R}$ and $\mathbb{C}$ with their usual absolute values, the importance of strings stems from the fact that they permit to develop "duality free" arguments to establish, in a "non convex" setting, two out of the three so-called "fundamental principles of functional analysis". (The third one, the Hahn-Banach theorem, will be taken up in chapter 4.) These theorems are best understood when the classes of spaces for which they hold true are introduced. We extend Waelbroeck's result characterizing barrelled spaces as those for which the uniform boundedness principle, i.e. the Banach-Steinhaus theorem, holds true. The notion of strings allows a very natural definition of two other classes of topological vector spaces: the bornological and the quasi-barrelled ones. It should be remarked that in the language of the older literature they should be called ultrabarrelled, ultrabornological and quasi-ultrabarrelled. We have followed [1] by dropping the prefix "ultra".

In chapter 3 we study non-archimedean spaces: those for which a fundamental system of neighborhoods V of the origin satisfying $V + V \subset V$ can be found. They are, in a certain sense, the analogue of the locally convex spaces over $\mathbb{R}$ or $\mathbb{C}$, when the absolute value of the division ring F of scalars is non-archimedean, i.e.

$$|\lambda + \mu| \leq \max(|\lambda|, |\mu|)$$

for all λ, $\mu \in F$. When $(F, |\cdot|)$ is not trivially valued, the notions of non-archimedean and locally F-convex spaces coincide. There is a very extensive literature on the subject of locally F-convex spaces and we confined our attention only to those results needed in the application we had in mind (approximation theory), or to those results not covered in the literature. The reader will notice that the beautiful "duality theory" of Van Tiel [92] is absent from our presentation. The reason for this is twofold. Firstly, it is not needed for Stone-Weierstrass type results, and secondly, we could not give a better exposition than that of Van Tiel himself.

In chapter 4 we come to the third basic principle of functional analysis: the Hahn-Banach Theorem. The main result here is Ingleton's characterization of those non-trivially valued non-archimedean division rings for which the Hahn-Banach Theorem on the extension of linear functionals is true.

In fact, Ingleton's result deals with the problem of characterizing more generally those non-archimedean normed spaces $(E, \|\cdot\|)$ which have the "norm-preserving extension property". The notion of spherical completeness introduced by Ingleton for this purpose has the character of an intersection property: it is the analogue, in non-archimedean functional analysis, of the "binary intersection property" of L. Nachbin. In [60], Nachbin had characterized those normed spaces $(E, \|\cdot\|)$ over the reals which have the norm-preserving extension property by means of the binary intersection property: every family of closed balls such that any two of them meet has non-void intersection.

In chapter 5 we consider the space $C(X,E)$ of all continuous functions from a topological space X into a topological vector space (E,τ) over a non-archimedean valued division ring $(F,|\cdot|)$. We endow it with the compact-open topology and prove a Stone-Weierstrass Theorem, i.e. we characterize the closure in the compact-open topology of a vector subspace $M \subset C(X;E)$ which is an A-module over a subalgebra $A \subset C(X;F)$ of scalar-valued functions (being an A-module means that $fg \in M$ for all $f \in A$ and $g \in M$). As a Corollary we get Kaplansky's Stone-Weierstrass Theorem [38] for subalgebras. We emphasize that dealing with modules over algebras presents no more difficulties than dealing directly with algebras and presents many advantages in the applications. In fact, since E has no multiplication, we could not consider $C(X;E)$ as an algebra and then take subalgebras. But even in the case $E = F$, when $C(X;F)$ is naturally an algebra, the module theorem is extremely useful. Given *any* vector space $M \subset C(X;E)$ there is always a subalgebra $A \subset C(X;F)$ over which M is a module: define

$$A = \{f \in C(X;F);\ \ fg \in M \ \text{ for all } \ g \in M\}.$$

Consider the partition of X by the sets of constancy for A, say P. Then it follows from the Stone-Weierstrass Theorem for modules that, for any $f \in C(X;E)$, f belongs to the closure of

M if, and only if, its restriction $f|_Y$ belongs to the closure of $M|_Y$ for any $Y \in P$.

Now, if $E = F$ and M contains the constants, then $A \subset M$. If, moreover, M is an algebra containing the constants, then $A = M$ and we have recovered the Stone-Weierstrass Theorem for unitary algebras.

In this chapter we also present another Stone-Weierstrass Theorem due to Kaplansky: in this case (E,τ) is a topological ring with unit having a fundamental system of neighborhoods of 0 which are ideals in E and X is supposed to be a 0-dimensional T_1-space. (See Theorem 5.31).

In chapter 6 we restrict our attention to normed spaces $(E, \|\cdot\|)$. Now, if X is compact, $C(X;E)$ becomes a normed space too, and given M and A as in the preceding chapter, we ask for a formula giving the distance of any $f \in C(X;E)$ from M. This is established in Theorem 6.4:

$$\operatorname{dist}(f;M) = \sup_{Y\in P} \operatorname{dist}(f|_Y ; M_Y)$$

where P is the partition of X by the sets of constancy for A.

The analogue of Bishop's approximation Theorem is also established in this chapter, by considering a division subring $G \subset F$ and defining (see Definition 6.7) sets of antisymmetry for $A \subset C(X;F)$ with respect to G. In Bishop's case, $F = \mathbb{C}$ and $G = \mathbb{R}$. Our proof is essentially due do S. Machado [48], his version, and proof, of Bishop's generalized Stone-Weierstrass is well suited for our purposes, it uses neither measure theory (our F is not locally compact) nor Krein-Milman's Theorem (no analogue for extreme points available).

The next topic we take up is the approximation property: over a spherically complete division ring $(F,|\cdot|)$, all non-archimedean normed spaces have the metric approximation property. This result is due to Monna [58].

Chapter 6 ends with some comments on the equivalence for a non-archimedean normed space $(E, \|\cdot\|)$ between (a) spherical completeness; (b) norm-preserving extension property; (c) norm-one projection property (d) non-archimedean intersection property (see Definition 6.29).

In chapter 7 we present some results on topological rings and algebras of continuous functions. The results 7.11 through 7.25 are all due to I. Kaplansky, with some minor modifications. Theorem 7.11 characterizes maximal one-sided ideals in $C(X;E)$, when X is a 0-dimensional compact T_1-space and (E,τ) is a topological ring with identity e, which is a Q-ring (i.e. the set of invertible elements is open) and has continuous inverse. Under this hypothesis, it is proved that every proper one-sided ideal in $C(X;E)$ is fixed. (An ideal I is fixed if there exists $x \in X$ such that $I(x)$ is a proper ideal in E).

When (E,τ) is a topological algebra (over a valued division ring $(F,|\cdot|)$, and either X or E is 0-dimensional, theorem 7.30 characterizes those closed (under the compact-open topology) one sided algebra ideals which are $C(X;F)$-modules. This class includes all regular ideals. Since the kernel of any algebra homomorphism is a regular ideal, this result is used to characterize the spectrum of $C(X;E)$. If (E,τ) is any topological algebra over $(F,|\cdot|)$ the spectrum of E, denoted by $\Delta(E)$, is the set of all non-zero continuous algebra homomorphisms of E onto F. Our main result is Theorem 7.45 establishing a homeomorphism between $X \times \Delta(E)$ and $\Delta(C(X;E))$ under very general hypothesis: X is any 0-dimensional T_1-space and (E,τ) is any associative topological algebra such that $\Delta(E)$ is locally equicontinuous when topologized by the relative weak topology. As a Corollary, any two 0-dimensional T_1-spaces X and Y are homeomorphic if, and only if $C(X;F)$ and $C(Y;F')$ are isomorphic as topological algebras under their respective compact-open topologies.

In chapter 8 we give an account of some results on the Banaschewski compactification $\beta_o X$ of a 0-dimensional T_1-space X. We begin by recalling some results on ultranormal and ultraregular spaces. The space $\beta_o X$ is then shown to be in one-to-one correspondence with the set of all characters of the algebra $C^*(X;F)$, the set of all continuous functions with relatively compact range in a non-archimedean valued division ring F (A character of a linear algebra over F is a non-zero algebra homomorphism into F). The result just mentioned is due to Bachman, Beckenstein, Narici and Warner [3]. The subset $\nu_F X$ of all $x \in \beta_o X$ such that $(\beta_o f)(x)$ belongs to F for all $f \in C(X;F)$, is then used to identify the set of all characters of the bigger algebra $C(X;F)$.

Chapter 8 concludes with some of the elegant results of R. L. Ellis on the problem of extension of continuous functions. The main tool in this investigation is Ellis result on the possibility of extending open partitions of closed subsets of ultraparacompact spaces. (See Theorem 8.23). Using this result, Ellis proved that every (bounded) continuous function $f: A \to Y$ has a (bounded) continuous extension $F: X \to Y$, if $A \subset X$ is a closed subset of an ultraparacompact space X and Y is any complete metric space. (See 8.26). When Y is separable, then X may be an ultranormal space. (See 8.27). Since every 0-dimensional compact T_1-space is ultranormal, and the p-adic field is separable, Ellis theorem generalizes Théorème 1 of Dieudonné [18].

The final chapter deals with the problem of best (and best simultaneous) approximation, for non-archimedean normed spaces. If $(E, \|\cdot\|)$ is a normed space over $(F, |\cdot|)$, $G \subset E$, and $B \subset E$ is bounded, the relative *Chebyshev radius* of B (with respect to G) is the number

$$\mathrm{rad}_G(B) = \inf_{g \in G} \sup_{f \in B} \|g - f\|.$$

The elements of G where the infimum is attained are called relative *Chebyshev centers* of B (with respect to G) and their collection is denoted by $\mathrm{cent}_G(B)$. When $G = E$ we write simply $\mathrm{rad}(B)$ and $\mathrm{cent}(B)$. Notice that when $B = \{f\}$, $\mathrm{rad}_G(B) = \mathrm{dist}(f;G)$ and $\mathrm{cent}_G(B) = \mathcal{P}_G(f)$, the set of best approximants of f from elements of G. The main problems here are the following

(i) given G, determine the largest class $\mathcal{B}$ of bounded subsets of E such that $\mathrm{cent}_G(B) \neq \emptyset$ for all $B \in \mathcal{B}$.

(ii) Given $\mathcal{B}$ (e.g. one-point sets, finite sets, precompact sets, all bounded sets) determine classes of subsets $G \subset E$ such that $\mathrm{cent}_G(B) \neq \emptyset$.

Let us give two examples of solutions to such problems. (i)

If $G \subset E$ is a spherically complete linear subspace of a non-archimedean normed space $(E, \|\cdot\|)$, then the largest class $\mathcal{B}$ of bounded subsets of E such that $\mathrm{cent}_G(B) \neq \emptyset$ for all $B \in \mathcal{B}$ is the class of *all* bounded subsets of E. (See Theorem 9.24). This extends Monna's result (Monna [57]) saying that such a G is proximinal in E, i.e. that $P_G(f) \neq \emptyset$ for all $f \in E$.

(ii) Let X be a 0-dimensional compact T_1-space and let $(E, \|\cdot\|)$ be a spherically complete Banach space over $(F, |\cdot|)$. Let $\mathcal{B}$ be the class of all bounded equicontinuous subsets of $C(X;E)$ (equipped with the sup-norm). Then every Weierstrass-Stone subspace $W_\pi \subset C(X;E)$ is such that $\mathrm{cent}_{W_\pi}(B) \neq \emptyset$ for all $B \in \mathcal{B}$. A Weierstrass-Stone subspace W_π is a closed vector subspace of $C(X;E)$ of the form

$$W_\pi = \{f \circ \pi;\ f \in C(Y;E)\}$$

where $\pi : X \to Y$ is a continuous mapping from X onto another 0-dimensional compact T_1-space Y. This is a Corollary to Theorem 9.32, since by Theorem 9.24, a spherically complete Banach space $(E, \|\cdot\|)$ admits Chebyshev centers, i.e. $\mathrm{cent}_E(B) \neq \emptyset$ for all bounded $B \subset E$.

The main tool for getting both results above is a selection theorem due to E. Michael [51].

Some chapters have a section, called "Notes and Remarks" where we present some results which are natural extensions of those given in the main text. The proofs of these results are often omitted and the interested reader is referred to the appropriate bibliographical reference.

It is a pleasure to acknowledge my indebtedness to many friends, colleagues and students. Special thanks are due to Miss Elda Mortari who typed this volume.

J. B. PROLLA

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