

Fluid Mechanics

2nd edition

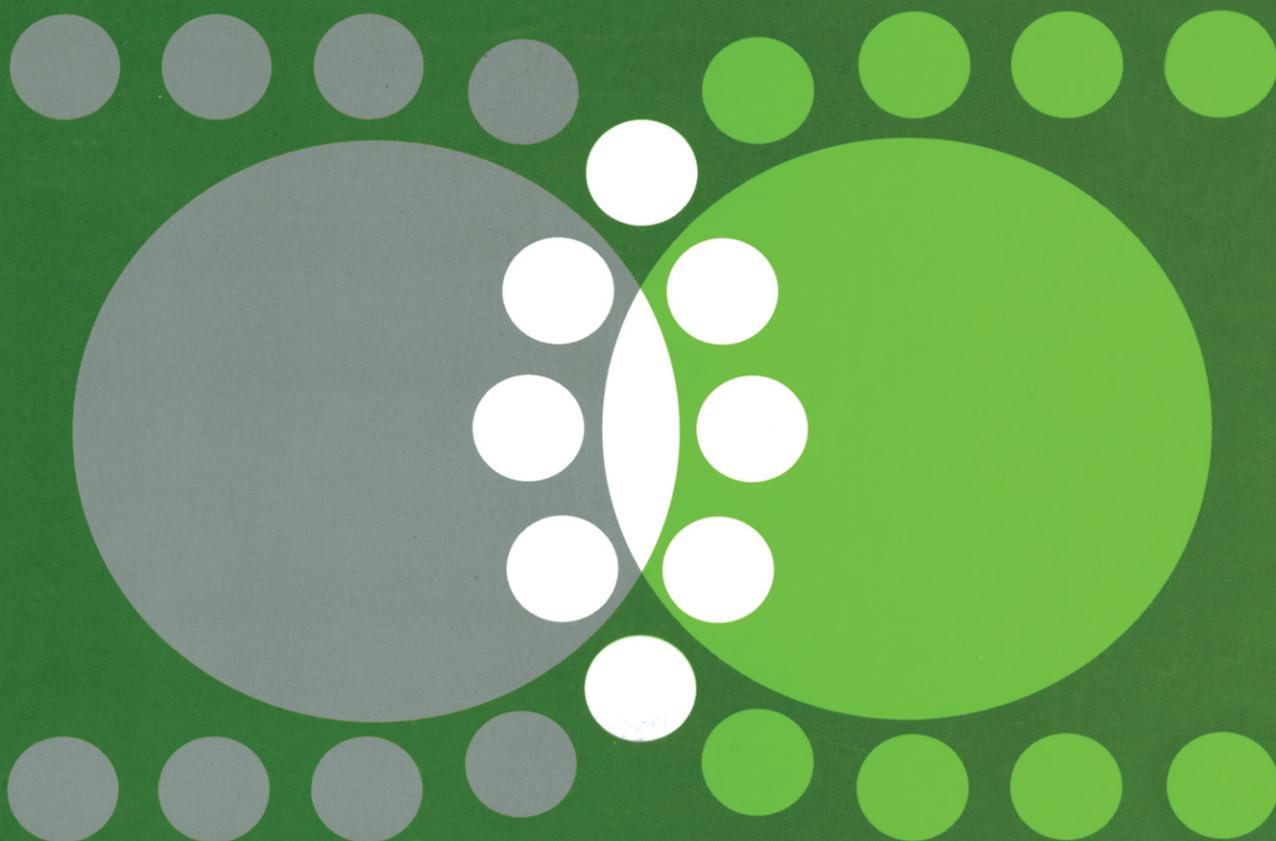
Landau and Lifshitz

Course of Theoretical Physics

Volume 6

L.D. Landau and E.M. Lifshitz

Institute of Physical Problems, USSR Academy of Sciences, Moscow



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FLUID MECHANICS

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Institute of Physical Problems, U.S.S.R. Academy of Sciences

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Second English Edition, Revised**

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J. B. SYKES and W. H. REID



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PREFACE TO THE SECOND ENGLISH EDITION

The content and treatment in this edition remain in accordance with what was said in the preface to the first edition (see below). My chief care in revising and augmenting has been to comply with this principle.

Despite the lapse of thirty years, the previous edition has, with very slight exceptions, not gone out of date. Its material has been only fairly slightly supplemented and modified. About ten new sections have been added.

In recent decades, fluid mechanics has undergone extremely rapid development, and there has accordingly been a great increase in the literature of the subject. The development has been mainly in applications, however, and in an increasing complexity of the problems accessible to theoretical calculation (with or without computers). These include, in particular, various problems of instability and its development, including non-linear regimes. All such topics are beyond the scope of our book; in particular, stability problems are discussed, as previously, mainly in terms of results.

There is also no treatment of non-linear waves in dispersive media, which is by now a significant branch of mathematical physics. The purely hydrodynamic subject of this theory consists in waves with large amplitude on the surface of a liquid. Its principal physical applications are in plasma physics, non-linear optics, various problems of electrodynamics, and so on, and in that respect they belong in other volumes of the *Course*.

There have been important changes in our understanding of the mechanism whereby turbulence occurs. Although a consistent theory of turbulence is still a thing of the future, there is reason to suppose that the right path has finally been found. The basic ideas now available and the results obtained are discussed in three sections (§§30–32) written jointly with M. I. Rabinovich, to whom I am deeply grateful for this valuable assistance. A new area in continuum mechanics over the last few decades is that of liquid crystals. This combines features of the mechanics of liquid and elastic media. Its principles are discussed in the new edition of *Theory of Elasticity*.

This book has a special place among those I had occasion to write jointly with L. D. Landau. He gave it a part of his soul. That branch of theoretical physics, new to him at the time, caught his fancy, and in a very typical way he set about thinking through it *ab initio* and deriving its basic results. This led to a number of original papers which appeared in various journals, but several of his conclusions or ideas were not published elsewhere than in the book, and in some instances even his priority was not established till later. In the new edition, I have added an appropriate reference to his authorship in all such cases that are known to me.

In the revision of this book, as in other volumes of the *Course*, I have had the help and advice of many friends and colleagues. I should like to mention in particular numerous discussions with G. I. Barenblatt, L. P. Pitaevskii, Ya. G. Sinai, and Ya. B. Zel'dovich. Several useful comments came from A. A. Andronov, S. I. Anisimov, V. A. Belokon', A. L. Fabrikant, V. P. Krainov, A. G. Kulikovskii, M. A. Liberman, R. V. Polovin, and A. V. Timofeev. To all of them I express my sincere gratitude.

Institute of Physical Problems
August 1984

E. M. LIFSHITZ

PREFACE TO THE FIRST ENGLISH EDITION

The present book deals with fluid mechanics, i.e. the theory of the motion of liquids and gases.

The nature of the book is largely determined by the fact that it describes fluid mechanics as a branch of theoretical physics, and it is therefore markedly different from other textbooks on the same subject. We have tried to develop as fully as possible all matters of physical interest, and to do so in such a way as to give the clearest possible picture of the phenomena and their interrelation. Accordingly, we discuss neither approximate methods of calculation in fluid mechanics, nor empirical theories devoid of physical significance. On the other hand, accounts are given of some topics not usually found in textbooks on the subject: the theory of heat transfer and diffusion in fluids; acoustics; the theory of combustion; the dynamics of superfluids; and relativistic fluid dynamics.

In a field which has been so extensively studied as fluid mechanics it was inevitable that important new results should have appeared during the several years since the last Russian edition was published. Unfortunately, our preoccupation with other matters has prevented us from including these results in the English edition. We have merely added one further chapter, on the general theory of fluctuations in fluid dynamics.

We should like to express our sincere thanks to Dr Sykes and Dr Reid for their excellent translation of the book, and to Pergamon Press for their ready agreement to our wishes in various matters relating to its publication.

Moscow 1958

L. D. LANDAU
E. M. LIFSHITZ

EVGENIĬ MIKHAĬLOVICH LIFSHITZ (1915–1985)†

Soviet physics suffered a heavy loss on 29 October 1985 with the death of the outstanding theoretical physicist Academician Evgeniĭ Mikhailovich Lifshitz.

Lifshitz was born on 21 February 1915 in Khar'kov. In 1933 he graduated from the Khar'kov Polytechnic Institute. He worked at the Khar'kov Physicotechnical Institute from 1933 to 1938 and at the Institute of Physical Problems of the USSR Academy of Sciences in Moscow from 1939 until his death. He was elected an associate member of the USSR Academy of Sciences in 1966 and a full member in 1979.

Lifshitz's scientific activity began very early. He was among L. D. Landau's first students and at 19 he co-authored with him a paper on the theory of pair production in collisions. This paper, which has not lost its significance to this day, outlined many methodological features of modern relativistically invariant techniques of quantum field theory. It includes, in particular, a consistent allowance for retardation.

Modern ferromagnetism theory is based on the "Landau-Lifshitz" equation, which describes the dynamics of the magnetic moment in a ferromagnet. A 1935 article on this subject is one of the best known papers on the physics of magnetic phenomena. The derivation of the equation is accompanied by development of a theory of ferromagnetic resonance and of the domain structure of ferromagnets.

In a 1937 paper on the Boltzmann kinetic equation for electrons in a magnetic field, E. M. Lifshitz developed a drift approximation extensively used much later, in the 50s, in plasma theory.

A paper published in 1939 on deuteron dissociation in collisions remains a brilliant example of the use of quasi-classical methods in quantum mechanics.

A most important step towards the development of a theory of second-order phase transitions, following the work by L. D. Landau, was a paper by Lifshitz dealing with the change of the symmetry of a crystal, of its space group, in transitions of this type (1941). Many years later the results of this paper came into extensive use, and the terms "Lifshitz criterion" and "Lifshitz point," coined on its basis have become indispensable components of modern statistical physics.

A decisive role in the detection of an important physical phenomenon, second sound in superfluid helium, was played by a 1944 paper by E. M. Lifshitz. It is shown in it that second sound is effectively excited by a heater having an alternating temperature. This was precisely the method used to observe second sound in experiment two years later.

A new approach to the theory of molecular-interaction forces between condensed bodies was developed by Lifshitz in 1954–1959. It is based on the profound physical idea that these forces are manifestations of stresses due to quantum and thermal fluctuations of an electromagnetic field in a medium. This idea was pursued to develop a very elegant and general theory in which the interaction forces are expressed in terms of electrodynamic material properties such as the complex dielectric permittivity. This theory of E. M.

† By A. F. Andreev, A. S. Borovik-Romanov, V. L. Ginzburg, L. P. Gor'kov, I. E. Dzyaloshinskiĭ, Ya. B. Zel'dovich, M. I. Kaganov, L. P. Pitaevskii, E. L. Feĭnberg, and I. M. Khalatnikov; published in Russian in *Uspekhi fizicheskikh nauk* **148**, 549–550, 1986. This translation is by J. G. Adashko (first published in *Soviet Physics Uspekhi* **29**, 294–295, 1986), and is reprinted by kind permission of the American Institute of Physics.

Lifshitz stimulated many studies and was confirmed by experiment. It gained him the M. V. Lomonosov Prize in 1958.

E. M. Lifshitz made a fundamental contribution in one of the most important branches of modern physics, the theory of gravitation. His research into this field started with a classical 1946 paper on the stability of cosmological solutions of Einstein's theory of gravitation. The perturbations were divided into distinctive classes—scalar, with variation of density, vector, describing vortical motion, and finally tensor, describing gravitational waves. This classification is still of decisive significance in the analysis of the origin of the universe. From there, E. M. Lifshitz tackled the exceedingly difficult question of the general character of the singularities of this theory. Many years of labor led in 1972 to a complete solution of this problem in papers written jointly with V. A. Belinskiĭ and I. M. Khalatnikov, which earned their authors the 1974 L. D. Landau Prize. The singularity was found to have a complicated oscillatory character and could be illustratively represented as contraction of space in two directions with simultaneous expansion in the third. The contraction and expansion alternate in time according to a definite law. These results elicited a tremendous response from specialists, altered radically our ideas concerning relativistic collapse, and raised a host of physical and mathematical problems that still await solution.

His life-long occupation was the famous Landau and Lifshitz *Course of Theoretical Physics*, to which he devoted about 50 years. (The first edition of *Statistical Physics* was written in 1937. A new edition of *Theory of Elasticity* went to press shortly before his last illness.) The greater part of the *Course* was written by Lifshitz together with his teacher and friend L. D. Landau. After the automobile accident that made Landau unable to work, Lifshitz completed the edition jointly with Landau's students. He later continued to revise the previously written volumes in the light of the latest advances in science. Even in the hospital, he discussed with visiting friends the topics that should be subsequently included in the *Course*.

The *Course of Theoretical Physics* became world famous. It was translated in its entirety into six languages. Individual volumes were published in 10 more languages. In 1972 L. D. Landau and E. M. Lifshitz were awarded the Lenin Prize for the volumes published by then.

The *Course of Theoretical Physics* remains a monument to E. M. Lifshitz as a scientist and a pedagogue. It has educated many generations of physicists, is being studied, and will continue to teach students in future generations.

A versatile physicist, E. M. Lifshitz dealt also with applications. He was awarded the USSR State Prize in 1954.

A tremendous amount of E.M. Lifshitz's labor and energy was devoted to Soviet scientific periodicals. From 1946 to 1949 and from 1955 to his death he was deputy editor-in-chief of the *Journal of Experimental and Theoretical Physics*. His extreme devotion to science, adherence to principles, and meticulousness greatly helped to make this journal one of the best scientific periodicals in the world.

E. M. Lifshitz accomplished much in his life. He will remain in our memory as a remarkable physicist and human being. His name will live forever in the history of Soviet physics.

NOTATION

ρ	density
p	pressure
T	temperature
s	entropy per unit mass
ε	internal energy per unit mass
$w = \varepsilon + p/\rho$	heat function (enthalpy)
$\gamma = c_p/c_v$	ratio of specific heats at constant pressure and constant volume
η	dynamic viscosity
$\nu = \eta/\rho$	kinematic viscosity
κ	thermal conductivity
$\chi = \kappa/\rho c_p$	thermometric conductivity
R	Reynolds number
c	velocity of sound
M	ratio of fluid velocity to velocity of sound (Mach number)

Vector and tensor (three-dimensional) suffixes are denoted by Latin letters i, k, l, \dots . Summation over repeated (“dummy”) suffixes is everywhere implied. The unit tensor is δ_{ik} :

References to other volumes in the *Course of Theoretical Physics*:

- Fields* = Vol. 2 (*The Classical Theory of Fields*, fourth English edition, 1975).
QM = Vol. 3 (*Quantum Mechanics*, third English edition, 1977).
SP 1 = Vol. 5 (*Statistical Physics, Part 1*, third English edition, 1980).
ECM = Vol. 8 (*Electrodynamics of Continuous Media*, second English edition, 1984).
SP 2 = Vol. 9 (*Statistical Physics, Part 2*, English edition, 1980).
PK = Vol. 10 (*Physical Kinetics*, English edition, 1981).

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CHAPTER I

IDEAL FLUIDS

§1. The equation of continuity

Fluid dynamics concerns itself with the study of the motion of fluids (liquids and gases). Since the phenomena considered in fluid dynamics are macroscopic, a fluid is regarded as a continuous medium. This means that any small volume element in the fluid is always supposed so large that it still contains a very great number of molecules. Accordingly, when we speak of infinitely small elements of volume, we shall always mean those which are “physically” infinitely small, i.e. very small compared with the volume of the body under consideration, but large compared with the distances between the molecules. The expressions *fluid particle* and *point in a fluid* are to be understood in a similar sense. If, for example, we speak of the displacement of some fluid particle, we mean not the displacement of an individual molecule, but that of a volume element containing many molecules, though still regarded as a point.

The mathematical description of the state of a moving fluid is effected by means of functions which give the distribution of the fluid velocity $\mathbf{v} = \mathbf{v}(x, y, z, t)$ and of any two thermodynamic quantities pertaining to the fluid, for instance the pressure $p(x, y, z, t)$ and the density $\rho(x, y, z, t)$. All the thermodynamic quantities are determined by the values of any two of them, together with the equation of state; hence, if we are given five quantities, namely the three components of the velocity \mathbf{v} , the pressure p and the density ρ , the state of the moving fluid is completely determined.

All these quantities are, in general, functions of the coordinates x, y, z and of the time t . We emphasize that $\mathbf{v}(x, y, z, t)$ is the velocity of the fluid at a given point (x, y, z) in space and at a given time t , i.e. it refers to fixed points in space and not to specific particles of the fluid; in the course of time, the latter move about in space. The same remarks apply to ρ and p .

We shall now derive the fundamental equations of fluid dynamics. Let us begin with the equation which expresses the conservation of matter. We consider some volume V_0 of space. The mass of fluid in this volume is $\int \rho dV$, where ρ is the fluid density, and the integration is taken over the volume V_0 . The mass of fluid flowing in unit time through an element $d\mathbf{f}$ of the surface bounding this volume is $\rho \mathbf{v} \cdot d\mathbf{f}$; the magnitude of the vector $d\mathbf{f}$ is equal to the area of the surface element, and its direction is along the normal. By convention, we take $d\mathbf{f}$ along the outward normal. Then $\rho \mathbf{v} \cdot d\mathbf{f}$ is positive if the fluid is flowing out of the volume, and negative if the flow is into the volume. The total mass of fluid flowing out of the volume V_0 in unit time is therefore

$$\oint \rho \mathbf{v} \cdot d\mathbf{f},$$

where the integration is taken over the whole of the closed surface surrounding the volume in question.

Next, the decrease per unit time in the mass of fluid in the volume V_0 can be written

$$-\frac{\partial}{\partial t} \int \rho \, dV.$$

Equating the two expressions, we have

$$\frac{\partial}{\partial t} \int \rho \, dV = - \oint \rho \, \mathbf{v} \cdot d\mathbf{f}. \quad (1.1)$$

The surface integral can be transformed by Green's formula to a volume integral:

$$\oint \rho \, \mathbf{v} \cdot d\mathbf{f} = \int \operatorname{div}(\rho \mathbf{v}) \, dV.$$

Thus

$$\int \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) \right] dV = 0.$$

Since this equation must hold for any volume, the integrand must vanish, i.e.

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0. \quad (1.2)$$

This is the *equation of continuity*. Expanding the expression $\operatorname{div}(\rho \mathbf{v})$, we can also write (1.2) as

$$\frac{\partial \rho}{\partial t} + \rho \operatorname{div} \mathbf{v} + \mathbf{v} \cdot \operatorname{grad} \rho = 0. \quad (1.3)$$

The vector

$$\mathbf{j} = \rho \mathbf{v} \quad (1.4)$$

is called the *mass flux density*. Its direction is that of the motion of the fluid, while its magnitude equals the mass of fluid flowing in unit time through unit area perpendicular to the velocity.

§2. Euler's equation

Let us consider some volume in the fluid. The total force acting on this volume is equal to the integral

$$-\oint p \, d\mathbf{f}$$

of the pressure, taken over the surface bounding the volume. Transforming it to a volume integral, we have

$$-\oint p \, d\mathbf{f} = - \int \operatorname{grad} p \, dV.$$

Hence we see that the fluid surrounding any volume element dV exerts on that element a force $-dV \operatorname{grad} p$. In other words, we can say that a force $-\operatorname{grad} p$ acts on unit volume of the fluid.

We can now write down the equation of motion of a volume element in the fluid by equating the force $-\operatorname{grad} p$ to the product of the mass per unit volume (ρ) and the acceleration $d\mathbf{v}/dt$:

$$\rho \, d\mathbf{v}/dt = - \operatorname{grad} p. \quad (2.1)$$

The derivative $d\mathbf{v}/dt$ which appears here denotes not the rate of change of the fluid velocity at a fixed point in space, but the rate of change of the velocity of a given fluid particle as it moves about in space. This derivative has to be expressed in terms of quantities referring to points fixed in space. To do so, we notice that the change $d\mathbf{v}$ in the velocity of the given fluid particle during the time dt is composed of two parts, namely the change during dt in the velocity at a point fixed in space, and the difference between the velocities (at the same instant) at two points $d\mathbf{r}$ apart, where $d\mathbf{r}$ is the distance moved by the given fluid particle during the time dt . The first part is $(\partial\mathbf{v}/\partial t)dt$, where the derivative $\partial\mathbf{v}/\partial t$ is taken for constant x, y, z , i.e. at the given point in space. The second part is

$$dx \frac{\partial\mathbf{v}}{\partial x} + dy \frac{\partial\mathbf{v}}{\partial y} + dz \frac{\partial\mathbf{v}}{\partial z} = (d\mathbf{r} \cdot \mathbf{grad})\mathbf{v}.$$

Thus

$$d\mathbf{v} = (\partial\mathbf{v}/\partial t)dt + (d\mathbf{r} \cdot \mathbf{grad})\mathbf{v},$$

or, dividing both sides by dt ,†

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \mathbf{grad})\mathbf{v}. \quad (2.2)$$

Substituting this in (2.1), we find

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \mathbf{grad})\mathbf{v} = -\frac{1}{\rho} \mathbf{grad} p. \quad (2.3)$$

This is the required equation of motion of the fluid; it was first obtained by L. Euler in 1755. It is called *Euler's equation* and is one of the fundamental equations of fluid dynamics.

If the fluid is in a gravitational field, an additional force $\rho\mathbf{g}$, where \mathbf{g} is the acceleration due to gravity, acts on any unit volume. This force must be added to the right-hand side of equation (2.1), so that equation (2.3) takes the form

$$\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \mathbf{grad})\mathbf{v} = -\frac{\mathbf{grad} p}{\rho} + \mathbf{g}. \quad (2.4)$$

In deriving the equations of motion we have taken no account of processes of energy dissipation, which may occur in a moving fluid in consequence of internal friction (viscosity) in the fluid and heat exchange between different parts of it. The whole of the discussion in this and subsequent sections of this chapter therefore holds good only for motions of fluids in which thermal conductivity and viscosity are unimportant; such fluids are said to be *ideal*.

The absence of heat exchange between different parts of the fluid (and also, of course, between the fluid and bodies adjoining it) means that the motion is adiabatic throughout the fluid. Thus the motion of an ideal fluid must necessarily be supposed adiabatic.

In adiabatic motion the entropy of any particle of fluid remains constant as that particle moves about in space. Denoting by s the entropy per unit mass, we can express the condition for adiabatic motion as

$$ds/dt = 0, \quad (2.5)$$

† The derivative d/dt thus defined is called the *substantial* time derivative, to emphasize its connection with the moving substance.

where the total derivative with respect to time denotes, as in (2.1), the rate of change of entropy for a given fluid particle as it moves about. This condition can also be written

$$\partial s / \partial t + \mathbf{v} \cdot \mathbf{grad} s = 0. \quad (2.6)$$

This is the general equation describing adiabatic motion of an ideal fluid. Using (1.2), we can write it as an "equation of continuity" for entropy:

$$\partial(\rho s) / \partial t + \text{div}(\rho s \mathbf{v}) = 0. \quad (2.7)$$

The product $\rho s \mathbf{v}$ is the *entropy flux density*.

The adiabatic equation usually takes a much simpler form. If, as usually happens, the entropy is constant throughout the volume of the fluid at some initial instant, it retains everywhere the same constant value at all times and for any subsequent motion of the fluid. In this case we can write the adiabatic equation simply as

$$s = \text{constant}, \quad (2.8)$$

and we shall usually do so in what follows. Such a motion is said to be *isentropic*.

We may use the fact that the motion is isentropic to put the equation of motion (2.3) in a somewhat different form. To do so, we employ the familiar thermodynamic relation

$$dw = T ds + V dp,$$

where w is the heat function per unit mass of fluid (enthalpy), $V = 1/\rho$ is the specific volume, and T is the temperature. Since $s = \text{constant}$, we have simply

$$dw = V dp = dp/\rho,$$

and so $(\mathbf{grad} p)/\rho = \mathbf{grad} w$. Equation (2.3) can therefore be written in the form

$$\partial \mathbf{v} / \partial t + (\mathbf{v} \cdot \mathbf{grad}) \mathbf{v} = -\mathbf{grad} w. \quad (2.9)$$

It is useful to notice one further form of Euler's equation, in which it involves only the velocity. Using a formula well known in vector analysis,

$$\frac{1}{2} \mathbf{grad} v^2 = \mathbf{v} \times \mathbf{curl} \mathbf{v} + (\mathbf{v} \cdot \mathbf{grad}) \mathbf{v},$$

we can write (2.9) in the form

$$\partial \mathbf{v} / \partial t - \mathbf{v} \times \mathbf{curl} \mathbf{v} = -\mathbf{grad} (w + \frac{1}{2} v^2). \quad (2.10)$$

If we take the curl of both sides of this equation, we obtain

$$\frac{\partial}{\partial t} (\mathbf{curl} \mathbf{v}) = \mathbf{curl} (\mathbf{v} \times \mathbf{curl} \mathbf{v}), \quad (2.11)$$

which involves only the velocity.

The equations of motion have to be supplemented by the boundary conditions that must be satisfied at the surfaces bounding the fluid. For an ideal fluid, the boundary condition is simply that the fluid cannot penetrate a solid surface. This means that the component of the fluid velocity normal to the bounding surface must vanish if that surface is at rest:

$$v_n = 0. \quad (2.12)$$

In the general case of a moving surface, v_n must be equal to the corresponding component of the velocity of the surface.

At a boundary between two immiscible fluids, the condition is that the pressure and the velocity component normal to the surface of separation must be the same for the two fluids, and each of these velocity components must be equal to the corresponding component of the velocity of the surface.

As has been said at the beginning of §1, the state of a moving fluid is determined by five quantities: the three components of the velocity \mathbf{v} and, for example, the pressure p and the density ρ . Accordingly, a complete system of equations of fluid dynamics should be five in number. For an ideal fluid these are Euler's equations, the equation of continuity, and the adiabatic equation.

PROBLEM

Write down the equations for one-dimensional motion of an ideal fluid in terms of the variables a, t , where a (called a *Lagrangian variable*†) is the x coordinate of a fluid particle at some instant $t = t_0$.

SOLUTION. In these variables the coordinate x of any fluid particle at any instant is regarded as a function of t and its coordinate a at the initial instant: $x = x(a, t)$. The condition of conservation of mass during the motion of a fluid element (the equation of continuity) is accordingly written $\rho \, dx = \rho_0 \, da$, or

$$\rho \left(\frac{\partial x}{\partial a} \right)_t = \rho_0,$$

where $\rho_0(a)$ is a given initial density distribution. The velocity of a fluid particle is, by definition, $v = (\partial x / \partial t)_a$, and the derivative $(\partial v / \partial t)_a$ gives the rate of change of the velocity of the particle during its motion. Euler's equation becomes

$$\left(\frac{\partial v}{\partial t} \right)_a = - \frac{1}{\rho_0} \left(\frac{\partial p}{\partial a} \right)_t,$$

and the adiabatic equation is

$$(\partial s / \partial t)_a = 0.$$

§3. Hydrostatics

For a fluid at rest in a uniform gravitational field, Euler's equation (2.4) takes the form

$$\mathbf{grad} \, p = \rho \mathbf{g}. \quad (3.1)$$

This equation describes the mechanical equilibrium of the fluid. (If there is no external force, the equation of equilibrium is simply $\mathbf{grad} \, p = 0$, i.e. $p = \text{constant}$; the pressure is the same at every point in the fluid.)

Equation (3.1) can be integrated immediately if the density of the fluid may be supposed constant throughout its volume, i.e. if there is no significant compression of the fluid under the action of the external force. Taking the z -axis vertically upward, we have

$$\partial p / \partial x = \partial p / \partial y = 0, \quad \partial p / \partial z = -\rho g.$$

Hence

$$p = -\rho g z + \text{constant}.$$

If the fluid at rest has a free surface at height h , to which an external pressure p_0 , the same at every point, is applied, this surface must be the horizontal plane $z = h$. From the condition $p = p_0$ for $z = h$, we find that the constant is $p_0 + \rho g h$, so that

$$p = p_0 + \rho g (h - z). \quad (3.2)$$

† Although such variables are usually called *Lagrangian*, the equations of motion in these coordinates were first obtained by Euler, at the same time as equations (2.3).

For large masses of liquid, and for a gas, the density ρ cannot in general be supposed constant; this applies especially to gases (for example, the atmosphere). Let us suppose that the fluid is not only in mechanical equilibrium but also in thermal equilibrium. Then the temperature is the same at every point, and equation (3.1) may be integrated as follows. We use the familiar thermodynamic relation

$$d\Phi = -sdT + Vdp,$$

where Φ is the thermodynamic potential (Gibbs free energy) per unit mass. For constant temperature

$$d\Phi = Vdp = dp/\rho.$$

Hence we see that the expression $(\mathbf{grad} p)/\rho$ can be written in this case as $\mathbf{grad} \Phi$, so that the equation of equilibrium (3.1) takes the form

$$\mathbf{grad} \Phi = \mathbf{g}.$$

For a constant vector \mathbf{g} directed along the negative z -axis we have

$$\mathbf{g} \equiv -\mathbf{grad}(gz).$$

Thus

$$\mathbf{grad}(\Phi + gz) = 0,$$

whence we find that throughout the fluid

$$\Phi + gz = \text{constant}; \quad (3.3)$$

gz is the potential energy of unit mass of fluid in the gravitational field. The condition (3.3) is known from statistical physics to be the condition for thermodynamic equilibrium of a system in an external field.

We may mention here another simple consequence of equation (3.1). If a fluid (such as the atmosphere) is in mechanical equilibrium in a gravitational field, the pressure in it can be a function only of the altitude z (since, if the pressure were different at different points with the same altitude, motion would result). It then follows from (3.1) that the density

$$\rho = -\frac{1}{g} \frac{dp}{dz} \quad (3.4)$$

is also a function of z only. The pressure and density together determine the temperature, which is therefore again a function of z only. Thus, in mechanical equilibrium in a gravitational field, the pressure, density and temperature distributions depend only on the altitude. If, for example, the temperature is different at different points with the same altitude, then mechanical equilibrium is impossible.

Finally, let us derive the equation of equilibrium for a very large mass of fluid, whose separate parts are held together by gravitational attraction—a star. Let ϕ be the Newtonian gravitational potential of the field due to the fluid. It satisfies the differential equation

$$\Delta\phi = 4\pi G\rho, \quad (3.5)$$

where G is the Newtonian constant of gravitation. The gravitational acceleration is $-\mathbf{grad} \phi$, and the force on a mass ρ is $-\rho \mathbf{grad} \phi$. The condition of equilibrium is therefore

$$\mathbf{grad} p = -\rho \mathbf{grad} \phi.$$

Dividing both sides by ρ , taking the divergence of both sides, and using equation (3.5), we obtain

$$\operatorname{div}\left(\frac{1}{\rho}\mathbf{grad} p\right) = -4\pi G\rho. \quad (3.6)$$

It must be emphasized that the present discussion concerns only mechanical equilibrium; equation (3.6) does not presuppose the existence of complete thermal equilibrium.

If the body is not rotating, it will be spherical when in equilibrium, and the density and pressure distributions will be spherically symmetrical. Equation (3.6) in spherical polar coordinates then takes the form

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dp}{dr}\right) = -4\pi G\rho. \quad (3.7)$$

§4. The condition that convection be absent

A fluid can be in mechanical equilibrium (i.e. exhibit no macroscopic motion) without being in thermal equilibrium. Equation (3.1), the condition for mechanical equilibrium, can be satisfied even if the temperature is not constant throughout the fluid. However, the question then arises of the stability of such an equilibrium. It is found that the equilibrium is stable only when a certain condition is fulfilled. Otherwise, the equilibrium is unstable, and this leads to the appearance in the fluid of currents which tend to mix the fluid in such a way as to equalize the temperature. This motion is called *convection*. Thus the condition for a mechanical equilibrium to be stable is the condition that convection be absent. It can be derived as follows.

Let us consider a fluid element at height z , having a specific volume $V(p, s)$, where p and s are the equilibrium pressure and entropy at height z . Suppose that this fluid element undergoes an adiabatic upward displacement through a small interval ξ ; its specific volume then becomes $V(p', s)$, where p' is the pressure at height $z + \xi$. For the equilibrium to be stable, it is necessary (though not in general sufficient) that the resulting force on the element should tend to return it to its original position. This means that the element must be heavier than the fluid which it "displaces" in its new position. The specific volume of the latter is $V(p', s')$, where s' is the equilibrium entropy at height $z + \xi$. Thus we have the stability condition

$$V(p', s') - V(p', s) > 0.$$

Expanding this difference in powers of $s' - s = \xi ds/dz$, we obtain

$$\left(\frac{\partial V}{\partial s}\right)_p \frac{ds}{dz} > 0. \quad (4.1)$$

The formulae of thermodynamics give

$$\left(\frac{\partial V}{\partial s}\right)_p = \frac{T}{c_p} \left(\frac{\partial V}{\partial T}\right)_p,$$

where c_p is the specific heat at constant pressure. Both c_p and T are positive, so that we can write (4.1) as

$$\left(\frac{\partial V}{\partial T}\right)_p \frac{ds}{dz} > 0. \quad (4.2)$$

The majority of substances expand on heating, i.e. $(\partial V/\partial T)_p > 0$. The condition that convection be absent then becomes

$$ds/dz > 0, \quad (4.3)$$

i.e. the entropy must increase with height.

From this we easily find the condition that must be satisfied by the temperature gradient dT/dz . Expanding the derivative ds/dz , we have

$$\frac{ds}{dz} = \left(\frac{\partial s}{\partial T}\right)_p \frac{dT}{dz} + \left(\frac{\partial s}{\partial p}\right)_T \frac{dp}{dz} = \frac{c_p}{T} \frac{dT}{dz} - \left(\frac{\partial V}{\partial T}\right)_p \frac{dp}{dz} > 0.$$

Finally, substituting from (3.4) $dp/dz = -g/V$, we obtain

$$-dT/dz < g\beta T/c_p, \quad (4.4)$$

where $\beta = (1/V)(\partial V/\partial T)_p$ is the thermal expansion coefficient. For a column of gas in equilibrium which can be taken as a thermodynamically perfect gas, $\beta T = 1$ and (4.4) becomes

$$-dT/dz < g/c_p. \quad (4.5)$$

Convection occurs if these conditions are not satisfied, i.e. if the temperature decreases upwards with a gradient whose magnitude exceeds the value given by (4.4) and (4.5).†

§5. Bernoulli's equation

The equations of fluid dynamics are much simplified in the case of steady flow. By *steady flow* we mean one in which the velocity is constant in time at any point occupied by fluid. In other words, \mathbf{v} is a function of the coordinates only, so that $\partial \mathbf{v}/\partial t = 0$. Equation (2.10) then reduces to

$$\frac{1}{2} \text{grad } v^2 - \mathbf{v} \times \text{curl } \mathbf{v} = -\text{grad } w. \quad (5.1)$$

We now introduce the concept of *streamlines*. These are lines such that the tangent to a streamline at any point gives the direction of the velocity at that point; they are determined by the following system of differential equations:

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dz}{v_z}. \quad (5.2)$$

In steady flow the streamlines do not vary with time, and coincide with the paths of the fluid particles. In non-steady flow this coincidence no longer occurs: the tangents to the streamlines give the directions of the velocities of fluid particles at various points in space at a given instant, whereas the tangents to the paths give the directions of the velocities of given fluid particles at various times.

We form the scalar product of equation (5.1) with the unit vector tangent to the streamline at each point; this unit vector is denoted by \mathbf{l} . The projection of the gradient on any direction is, as we know, the derivative in that direction. Hence the projection of $\text{grad } w$ is $\partial w/\partial l$. The vector $\mathbf{v} \times \text{curl } \mathbf{v}$ is perpendicular to \mathbf{v} , and its projection on the direction of \mathbf{l} is therefore zero.

† For water at 20°C, the right-hand side of (4.4) is about one degree per 6.7 km; for air, the right-hand side of (4.5) is about one degree per 100 m.

Thus we obtain from equation (5.1)

$$\frac{\partial}{\partial l} \left(\frac{1}{2}v^2 + w \right) = 0.$$

It follows from this that $\frac{1}{2}v^2 + w$ is constant along a streamline:

$$\frac{1}{2}v^2 + w = \text{constant.} \quad (5.3)$$

In general the constant takes different values for different streamlines. Equation (5.3) is called *Bernoulli's equation*.†

If the flow takes place in a gravitational field, the acceleration \mathbf{g} due to gravity must be added to the right-hand side of equation (5.1). Let us take the direction of gravity as the z -axis, with z increasing upwards. Then the cosine of the angle between the directions of \mathbf{g} and \mathbf{l} is equal to the derivative $-dz/dl$, so that the projection of \mathbf{g} on \mathbf{l} is

$$-g \, dz/dl.$$

Accordingly, we now have

$$\frac{\partial}{\partial l} \left(\frac{1}{2}v^2 + w + gz \right) = 0.$$

Thus Bernoulli's equation states that along a streamline

$$\frac{1}{2}v^2 + w + gz = \text{constant.} \quad (5.4)$$

§6. The energy flux

Let us choose some volume element fixed in space, and find how the energy of the fluid contained in this volume element varies with time. The energy of unit volume of fluid is

$$\frac{1}{2}\rho v^2 + \rho\varepsilon,$$

where the first term is the kinetic energy and the second the internal energy, ε being the internal energy per unit mass. The change in this energy is given by the partial derivative

$$\frac{\partial}{\partial t} \left(\frac{1}{2}\rho v^2 + \rho\varepsilon \right).$$

To calculate this quantity, we write

$$\frac{\partial}{\partial t} \left(\frac{1}{2}\rho v^2 \right) = \frac{1}{2}v^2 \frac{\partial \rho}{\partial t} + \rho \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t},$$

or, using the equation of continuity (1.2) and the equation of motion (2.3),

$$\frac{\partial}{\partial t} \left(\frac{1}{2}\rho v^2 \right) = -\frac{1}{2}v^2 \operatorname{div}(\rho \mathbf{v}) - \mathbf{v} \cdot \operatorname{grad} p - \rho \mathbf{v} \cdot (\mathbf{v} \cdot \operatorname{grad}) \mathbf{v}.$$

In the last term we replace $\mathbf{v} \cdot (\mathbf{v} \cdot \operatorname{grad}) \mathbf{v}$ by $\frac{1}{2} \mathbf{v} \cdot \operatorname{grad} v^2$, and $\operatorname{grad} p$ by $\rho \operatorname{grad} w - \rho T \operatorname{grad} s$ (using the thermodynamic relation $dw = Tds + (1/\rho)dp$), obtaining

$$\frac{\partial}{\partial t} \left(\frac{1}{2}\rho v^2 \right) = -\frac{1}{2}v^2 \operatorname{div}(\rho \mathbf{v}) - \rho \mathbf{v} \cdot \operatorname{grad} \left(\frac{1}{2}v^2 + w \right) + \rho T \mathbf{v} \cdot \operatorname{grad} s.$$

† It was derived for an incompressible fluid (§10) by D. Bernoulli in 1738.

In order to transform the derivative $\partial(\rho\varepsilon)/\partial t$, we use the thermodynamic relation

$$d\varepsilon = Tds - pdV = Tds + (p/\rho^2)d\rho.$$

Since $\varepsilon + p/\rho = \varepsilon + pV$ is simply the heat function w per unit mass, we find

$$d(\rho\varepsilon) = \varepsilon d\rho + \rho d\varepsilon = w d\rho + \rho T ds,$$

and so

$$\frac{\partial(\rho\varepsilon)}{\partial t} = w \frac{\partial\rho}{\partial t} + \rho T \frac{\partial s}{\partial t} = -w \operatorname{div}(\rho\mathbf{v}) - \rho T \mathbf{v} \cdot \operatorname{grad} s.$$

Here we have also used the general adiabatic equation (2.6).

Combining the above results, we find the change in the energy to be

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + \rho\varepsilon \right) = - \left(\frac{1}{2} v^2 + w \right) \operatorname{div}(\rho\mathbf{v}) - \rho \mathbf{v} \cdot \operatorname{grad} \left(\frac{1}{2} v^2 + w \right),$$

or, finally,

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + \rho\varepsilon \right) = - \operatorname{div} [\rho \mathbf{v} \left(\frac{1}{2} v^2 + w \right)]. \quad (6.1)$$

In order to see the meaning of this equation, let us integrate it over some volume:

$$\frac{\partial}{\partial t} \int \left(\frac{1}{2} \rho v^2 + \rho\varepsilon \right) dV = - \int \operatorname{div} [\rho \mathbf{v} \left(\frac{1}{2} v^2 + w \right)] dV,$$

or, converting the volume integral on the right into a surface integral,

$$\frac{\partial}{\partial t} \int \left(\frac{1}{2} \rho v^2 + \rho\varepsilon \right) dV = - \oint \rho \mathbf{v} \left(\frac{1}{2} v^2 + w \right) \cdot d\mathbf{f}. \quad (6.2)$$

The left-hand side is the rate of change of the energy of the fluid in some given volume. The right-hand side is therefore the amount of energy flowing out of this volume in unit time. Hence we see that the expression

$$\rho \mathbf{v} \left(\frac{1}{2} v^2 + w \right) \quad (6.3)$$

may be called the *energy flux density* vector. Its magnitude is the amount of energy passing in unit time through unit area perpendicular to the direction of the velocity.

The expression (6.3) shows that any unit mass of fluid carries with it during its motion an amount of energy $w + \frac{1}{2}v^2$. The fact that the heat function w appears here, and not the internal energy ε , has a simple physical significance. Putting $w = \varepsilon + p/\rho$, we can write the flux of energy through a closed surface in the form

$$- \oint \rho \mathbf{v} \left(\frac{1}{2} v^2 + \varepsilon \right) \cdot d\mathbf{f} - \oint p \mathbf{v} \cdot d\mathbf{f}.$$

The first term is the energy (kinetic and internal) transported through the surface in unit time by the mass of fluid. The second term is the work done by pressure forces on the fluid within the surface.

§7. The momentum flux

We shall now give a similar series of arguments for the momentum of the fluid. The momentum of unit volume is $\rho\mathbf{v}$. Let us determine its rate of change, $\partial(\rho\mathbf{v})/\partial t$. We shall use tensor notation. We have

$$\frac{\partial}{\partial t}(\rho v_i) = \rho \frac{\partial v_i}{\partial t} + \frac{\partial \rho}{\partial t} v_i.$$

Using the equation of continuity (1.2) in the form

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho v_k)}{\partial x_k},$$

and Euler's equation (2.3) in the form

$$\frac{\partial v_i}{\partial t} = -v_k \frac{\partial v_i}{\partial x_k} - \frac{1}{\rho} \frac{\partial p}{\partial x_i},$$

we obtain

$$\begin{aligned} \frac{\partial}{\partial t}(\rho v_i) &= -\rho v_k \frac{\partial v_i}{\partial x_k} - \frac{\partial p}{\partial x_i} - v_i \frac{\partial(\rho v_k)}{\partial x_k} \\ &= -\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_k}(\rho v_i v_k). \end{aligned}$$

We write the first term on the right in the form

$$\frac{\partial p}{\partial x_i} = \delta_{ik} \frac{\partial p}{\partial x_k},$$

and finally obtain

$$\frac{\partial}{\partial t}(\rho v_i) = -\frac{\partial \Pi_{ik}}{\partial x_k}, \quad (7.1)$$

where the tensor Π_{ik} is defined as

$$\Pi_{ik} = p\delta_{ik} + \rho v_i v_k. \quad (7.2)$$

This tensor is clearly symmetrical.

To see the meaning of the tensor Π_{ik} , we integrate equation (7.1) over some volume:

$$\frac{\partial}{\partial t} \int \rho v_i dV = - \int \frac{\partial \Pi_{ik}}{\partial x_k} dV.$$

The integral on the right is transformed into a surface integral by Green's formula:†

$$\frac{\partial}{\partial t} \int \rho v_i dV = - \oint \Pi_{ik} df_k. \quad (7.3)$$

The left-hand side is the rate of change of the i th component of the momentum contained in the volume considered. The surface integral on the right is therefore the

† The rule for transforming an integral over a closed surface into one over the volume bounded by that surface can be formulated as follows: the surface element df_i must be replaced by the operator $dV \cdot \partial/\partial x_i$, which is to be applied to the whole of the integrand.

amount of momentum flowing out through the bounding surface in unit time. Consequently, $\Pi_{ik} df_k$ is the i th component of the momentum flowing through the surface element df . If we write df_k in the form $n_k df$, where df is the area of the surface element, and \mathbf{n} is a unit vector along the outward normal, we find that $\Pi_{ik} n_k$ is the flux of the i th component of momentum through unit surface area. We may notice that, according to (7.2), $\Pi_{ik} n_k = pn_i + \rho v_i v_k n_k$. This expression can be written in vector form

$$p\mathbf{n} + \rho\mathbf{v}(\mathbf{v} \cdot \mathbf{n}). \quad (7.4)$$

Thus Π_{ik} is the i th component of the amount of momentum flowing in unit time through unit area perpendicular to the x_k -axis. The tensor Π_{ik} is called the *momentum flux density tensor*. The energy flux is determined by a vector, energy being a scalar; the momentum flux, however, is determined by a tensor of rank two, the momentum itself being a vector.

The vector (7.4) gives the momentum flux in the direction of \mathbf{n} , i.e. through a surface perpendicular to \mathbf{n} . In particular, taking the unit vector \mathbf{n} to be directed parallel to the fluid velocity, we find that only the longitudinal component of momentum is transported in this direction, and its flux density is $p + \rho v^2$. In a direction perpendicular to the velocity, only the transverse component (relative to \mathbf{v}) of momentum is transported, its flux density being just p .

§8. The conservation of circulation

The integral

$$\Gamma = \oint \mathbf{v} \cdot d\mathbf{l},$$

taken along some closed contour, is called the *velocity circulation* round that contour.

Let us consider a closed contour drawn in the fluid at some instant. We suppose it to be a "fluid contour", i.e. composed of the fluid particles that lie on it. In the course of time these particles move about, and the contour moves with them. Let us investigate what happens to the velocity circulation. In other words, let us calculate the time derivative

$$\frac{d}{dt} \oint \mathbf{v} \cdot d\mathbf{l}.$$

We have written here the total derivative with respect to time, since we are seeking the change in the circulation round a "fluid contour" as it moves about, and not round a contour fixed in space.

To avoid confusion, we shall temporarily denote differentiation with respect to the coordinates by the symbol δ , retaining the symbol d for differentiation with respect to time. Next, we notice that an element $d\mathbf{l}$ of the length of the contour can be written as the difference $\delta\mathbf{r}$ between the position vectors \mathbf{r} of the points at the ends of the element. Thus we write the velocity circulation as $\oint \mathbf{v} \cdot \delta\mathbf{r}$. In differentiating this integral with respect to time, it must be borne in mind that not only the velocity but also the contour itself (i.e. its shape) changes. Hence, on taking the time differentiation under the integral sign, we must differentiate not only \mathbf{v} but also $\delta\mathbf{r}$:

$$\frac{d}{dt} \oint \mathbf{v} \cdot \delta\mathbf{r} = \oint \frac{d\mathbf{v}}{dt} \cdot \delta\mathbf{r} + \oint \mathbf{v} \cdot \frac{d\delta\mathbf{r}}{dt}.$$

Since the velocity \mathbf{v} is just the time derivative of the position vector \mathbf{r} , we have

$$\mathbf{v} \cdot \frac{d\delta\mathbf{r}}{dt} = \mathbf{v} \cdot \delta \frac{d\mathbf{r}}{dt} = \mathbf{v} \cdot \delta\mathbf{v} = \delta\left(\frac{1}{2}v^2\right).$$

The integral of a total differential along a closed contour, however, is zero. The second integral therefore vanishes, leaving

$$\frac{d}{dt} \oint \mathbf{v} \cdot \delta\mathbf{r} = \oint \frac{d\mathbf{v}}{dt} \cdot \delta\mathbf{r}.$$

It now remains to substitute for the acceleration $d\mathbf{v}/dt$ its expression from (2.9):

$$d\mathbf{v}/dt = -\mathbf{grad} w.$$

Using Stokes' formula, we then have

$$\oint \frac{d\mathbf{v}}{dt} \cdot \delta\mathbf{r} = \oint \mathbf{curl} \left(\frac{d\mathbf{v}}{dt} \right) \cdot \delta\mathbf{f} = 0,$$

since $\mathbf{curl} \mathbf{grad} w \equiv 0$. Thus, going back to our previous notation, we find†

$$\frac{d}{dt} \oint \mathbf{v} \cdot d\mathbf{l} = 0,$$

or

$$\oint \mathbf{v} \cdot d\mathbf{l} = \text{constant}. \quad (8.1)$$

We have therefore reached the conclusion that, in an ideal fluid, the velocity circulation round a closed "fluid" contour is constant in time (*Kelvin's theorem* (1869) or the *law of conservation of circulation*).

It should be emphasized that this result has been obtained by using Euler's equation in the form (2.9), and therefore involves the assumption that the flow is isentropic. The theorem does not hold for flows which are not isentropic.‡

By applying Kelvin's theorem to an infinitesimal closed contour δC and transforming the integral according to Stokes' theorem, we get

$$\oint \mathbf{v} \cdot d\mathbf{l} = \int \mathbf{curl} \mathbf{v} \cdot d\mathbf{f} \cong \delta\mathbf{f} \cdot \mathbf{curl} \mathbf{v} = \text{constant}, \quad (8.2)$$

where $d\mathbf{f}$ is a fluid surface element spanning the contour δC . The vector $\mathbf{curl} \mathbf{v}$ is often called the *vorticity* of the fluid flow at a given point. The constancy of the product (8.2) can be intuitively interpreted as meaning that the vorticity moves with the fluid.

PROBLEM

Show that, in flow which is not isentropic, any moving particle carries with it a constant value of the product $(1/\rho) \mathbf{grad} s \cdot \mathbf{curl} \mathbf{v}$ (H. Ertel 1942).

† This result remains valid in a uniform gravitational field, since in that case $\mathbf{curl} \mathbf{g} \equiv 0$.

‡ Mathematically, it is necessary that there should be a one-to-one relation between p and ρ (which for isentropic flow is $s(p, \rho) = \text{constant}$); then $-(1/\rho) \mathbf{grad} p$ can be written as the gradient of some function, a result which is needed in deriving Kelvin's theorem.