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An Introduction to Independence Proofs

KENNETH KUNEN

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Kenneth KUNEN
University of Texas, Austin, U.S.A.

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ELSEVIER B.V.
Radarweg 29
P.O. Box 211, 1000 AE
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ELSEVIER Inc.
525 B Street
Suite 1900, San Diego
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To Anne, Isaac, and Adam

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PREFACE

This book provides an introduction to relative consistency proofs in axiomatic set theory, and is intended to be used as a text in beginning graduate courses in that subject. It is hoped that this treatment will make the subject accessible to those mathematicians whose research is sensitive to axiomatics. We assume that the reader has had the equivalent of an undergraduate course on cardinals and ordinals, but no specific training in logic is necessary.

The author is grateful to the large number of people who have suggested improvements in the original manuscript for this book. In particular we would like to thank John Baldwin, Eric van Douwen, Peter Nyikos, and Dan Velleman. Special thanks are due to Jon Barwise, who tried out the manuscript in a course at the University of Wisconsin.

CONTENTS

INTRODUCTION

Set theory is the foundation of mathematics. All mathematical concepts are defined in terms of the primitive notions of set and membership. In axiomatic set theory we formulate a few simple axioms about these primitive notions in an attempt to capture the basic "obviously true" set-theoretic principles. From such axioms, all known mathematics may be derived. However, there are some questions which the axioms fail to settle, and that failure is the subject of this book.

§1. Consistency results

The specific axiom system we discuss is ZFC, or Zermelo–Frankel set theory with the Axiom of Choice. We say that a statement ϕ is independent of ZFC if neither ϕ nor $\neg\phi$ (the negation of ϕ) is provable from ZFC; this is equivalent to saying that both ZFC + $\neg\phi$ and ZFC + ϕ are consistent. The most famous example of such a ϕ is the Continuum Hypothesis (CH), but within the past few years, a large number of statements, coming from various branches of mathematics, have been shown to be independent of ZFC.

In this book, we study the techniques for showing that a statement ϕ is consistent with ZFC. ϕ will be shown to be independent if we can successfully apply these techniques to ϕ and to $\neg\phi$; this will always involve two separate arguments. There are also many statements which have been shown to be consistent but whose independence has remained unsettled.

Some of the statements known to be consistent with ZFC are "quotable" principles of abstract set theory, such as CH, or $\neg$CH, or Martin's Axiom, or Suslin's Hypothesis, or $\diamondsuit$. Workers in the more abstract areas of analysis and topology are well aware of these principles and often apply them. Since any consequence of a consistent statement is also consistent, this provides a source for many consistency proofs in mathematics. In addition, those mathematicians with a background in set theory often return to the basic methods to prove consistency results for specific mathematical statements which do not follow from one of the known "quotable" principles.

The purpose of this book is to explain the basic techniques for proving statements consistent with ZFC. We include consistency proofs for many of the "quotable" principles. More importantly, we hope to enable mathematicians to produce new consistency proofs of their own, as needed.

§2. Prerequisites

We assume that the reader has seen a development of axiomatic set theory through the basic properties of von Neumann ordinals and cardinals. This material is contained in set theory texts such as [Enderton 1977] or [Halmos 1960], as well as in appendices to books in other areas of mathematics which use set theory, such as [Chang–Keisler 1973] or [Kelley 1955]. This material is also reviewed in Chapter I.

It is not necessary for the reader to have seen the particular axiom system ZFC. There are other systems which differ from ZFC in the formal way proper classes are handled (see I §12). A reader familiar with one of these should have no trouble with ZFC, but should bear in mind that in ZFC proper classes have no formal existence, and all variables range over sets.

The reader need not be knowledgeable about very picky axiomatic questions—such as which axioms of ZFC are used to prove which theorems. In those few cases where such questions are of any importance, they are reviewed quite extensively in Chapter I. However, we do presume some sophistication in the way set theory is handled in its mathematical applications, as one would see in a course in general topology or measure theory.

Our prerequisite in formal logic is elastic. A book whose main results involve consistency of axiomatic systems cannot avoid logic entirely. We have included a sketch of background material on formal logic to enable readers with no training in the subject to understand independence proofs, but such readers might be suspicious about the complete mathematical rigor of our methods. A good undergraduate course in logic would dispel that suspicion. On a higher level, there are many foundational questions raised by our subject which are of interest to the student of logic per se, and we have collected such material in appendices to the various chapters. In these appendices, we have felt free to assume as much logical sophistication as is needed for the particular argument at hand.

§3. Outline

Chapter I contains some logical background and a sketch of the development of the axioms of ZFC, excluding Foundation (Regularity). Since this material is partly a review, we have omitted many proofs. We have been fairly pedantic about the fact that for many of the theorems, certain axioms, especially Choice and Power Set, are not needed, and we have indicated explicitly where these axioms are used; such considerations are not important for the development of mathematics within ZFC, but will be useful when we get to independence proofs.

Chapter II covers some special topics in combinatorial set theory. In part, this chapter provides some combinatorial lemmas needed in Chapters VI–VIII, but its main purpose is to introduce the reader to the vast array of set-theoretic questions that one might try to prove independent of ZFC.

We have departed from tradition in basing our treatment of forcing in Chapter VII upon the discussion of Martin's Axiom in Chapter II. This has the advantage of separating the mathematical difficulties in handling forcing from the metamat~~hematical ones. It~~ has the disadvantage of requiring those readers (if there are any) who wish to learn forcing without learning Martin's Axiom to do some extra work.

The Axiom of Foundation is discussed in Chapter III. This axiom is never used in mathematics, but it leads to a much clearer picture of the set theoretic universe.

Chapter IV develops the basic methods used in producing consistency proofs, including inner models, relativization, and absoluteness. We also discuss the Reflection Theorem and related results.

Chapter V discusses the formalization of the logical notion of definability within ZFC. These ideas are used in defining the class **L** of constructible sets in Chapter VI. In Chapter VI we establish the consistency of the Generalized Continuum Hypothesis by showing that it holds in **L**. We also show that the combinatorial principles $\diamondsuit$ and $\diamondsuit^+$ are true in **L**.

Chapter VII introduces forcing and uses it to prove the consistency of $\neg$CH and various related statements of cardinal arithmetic. Chapter VIII covers iterated forcing and the consistency of Martin's Axiom with $\neg$CH.

§4. How to use this book

In internal cross referencing, chapters are denoted by Roman numerals and § denotes section number. Thus, VII §5 is the fifth section of Chapter Seven, and VII 5.16 is the sixteenth numbered enunciation in that section.

~~The exerci~~ses range from routine verifications to additional development of the material in the chapter. The more difficult ones are starred. The exercises are not necessary for understanding later material in the text, although they are sometimes required for later exercises. There are probably more exercises in some of the chapters than most readers will want to do.

It is not necessary to read the book straight through from cover to cover. In particular, the material in Chapter II is not used at all until the end of

Chapter VI, so the reader may simply skip Chapter II and refer back to it as needed. Also, a knowledge of constructibility is not necessary to understand forcing, so it is possible to read Chapters VII and VIII without reading Chapters V and VI, although the reader doing this would have to take on faith the existence of models of GCH. Furthermore, the appendices of all chapters may be omitted without loss of continuity.

§5. What has been omitted

We have two goals in writing this book. First, we hope to bridge the gap between the current literature and the elementary texts on cardinals and ordinals. Second, we hope to emphasize the interplay between classical combinatorial set theory and modern independence proofs. Much important material in set theory which is secondary to these goals has been omitted.

Specifically, topics which are already well covered in the literature by texts or survey articles have often been omitted. We have little here on large cardinals: the interested reader may consult [Drake 1974] or [Solovay–Reinhardt–Kanamori 1978]. Likewise, we do not treat the fine-structure methods in **L**; see [Devlin 1973] for this.

We have also avoided topics which require some sophistication in logic. In particular, we do not discuss model-theoretic applications of large cardinals (see [Drake 1974]), or results in descriptive set theory, or the relationship between these fields (see [Martin 1977] or [Moschovakis 1980]).

This book gives short shrift to the Axiom of Choice (AC). We consider AC to be one of the basic axioms of set theory, although we do indicate proofs that it is neither provable (see VII Exercise E4) nor refutable (see V 2.14 and VI 4.9) from the other axioms. For more on set theory without AC, see [Jech 1973].

§6. On references

Since this is a text and not a research monograph, we have not attempted to give references to the literature for every theorem we prove. Our bibliography is intended primarily as suggestions for further reading, and not as a source for establishing priority. We apologize to those mathematicians who are chagrined at not seeing their name mentioned more often. Aside from a few trivial exercises, none of the results in this book are due to the author.

§7. The axioms

For reference, we list here the axioms of ZFC and of some related theories; these are explained in much greater detail in Chapters I and III. After each axiom we list the section in Chapters I or III where it first occurs.

Axiom 0. *Set Existence* (I §5).

$$\exists x\,(x = x).$$

Axiom 1. *Extensionality* (I §5).

$$\forall x\,\forall y\,\big(\forall z\,(z \in x \leftrightarrow z \in y) \rightarrow x = y\big).$$

Axiom 2. *Foundation* (III §4).

$$\forall x\,\big[\exists y\,(y \in x) \rightarrow \exists y\,(y \in x \;\wedge\; \neg\exists z\,(z \in x \;\wedge\; z \in y))\big].$$

Axiom 3. *Comprehension Scheme* (I §5). For each formula ϕ with free variables among $x, z, w_1, \ldots, w_n$,

$$\forall z\,\forall w_1, \ldots, w_n\,\exists y\,\forall x\,(x \in y \leftrightarrow x \in z \;\wedge\; \phi).$$

Axiom 4. *Pairing* (I §6).

$$\forall x\,\forall y\,\exists z\,(x \in z \;\wedge\; y \in z).$$

Axiom 5. *Union* (I §6).

$$\forall \mathscr{F}\,\exists A\,\forall Y\,\forall x\,(x \in Y \;\wedge\; Y \in \mathscr{F} \;\rightarrow x \in A).$$

Axiom 6. *Replacement Scheme* (I §6). For each formula ϕ with free variables among $x, y, A, w_1, \ldots, w_n$,

$$\forall A\,\forall w_1, \ldots, w_n\,\big[\forall x \in A\,\exists!y\,\phi \rightarrow \exists Y\,\forall x \in A\,\exists y \in Y\,\phi\big].$$

On the basis of Axioms 0, 1, 3, 4, 5 and 6, one may define $\subset$ (subset), 0 (empty set), S (ordinal successor; $S(x) = x \cup \{x\}$), and the notion of well-ordering. The following axioms are then defined.

Axiom 7. *Infinity* (I §7).

$$\exists x\,\big(0 \in x \;\wedge\; \forall y \in x\,(S(y) \in x)\big).$$

Axiom 8. *Power Set* (I §10).

$$\forall x\,\exists y\,\forall z\,(z \subset x \rightarrow z \in y).$$

AXIOM 9. *Choice* (I §6).

$$\forall A\, \exists R\, (R \textit{ well-orders } A).$$

ZFC is the system of Axioms 0–9.

For technical reasons, it will sometimes be important to know that some of the results which we prove from ZFC do not in fact require all the axioms of ZFC; the reason for this is discussed at the end of I §4. We list here some abbreviations for commonly used subtheories of ZFC. ZF consists of Axioms 0–8, ZF $-$ P consists of Axioms 0–7, and ZFC $-$ P consists of Axioms 0–7 plus Axiom 9. By ZFC$^-$, ZF$^-$, ZF$^-$ $-$ P, and ZFC$^-$ $-$ P, we mean the respective theory (ZFC, ZF, ZF $-$ P, and ZFC $-$ P) with Axiom 2 (Foundation) deleted. Other abbreviations for weakenings of ZFC are usually self-explanatory. For example, ZF$^-$ $-$ P $-$ Inf is ZF$^-$ $-$ P with the Axiom of Infinity deleted.