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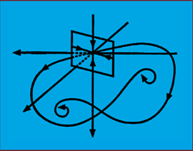
A Mathematical Treatment *of* Economic Cooperation *and* Competition Among Nations: *with* Nigeria, USA, UK, China *and* Middle East Examples



E. N. Chukwu

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A Mathematical Treatment of Economic Cooperation and Competition Among Nations: with Nigeria, USA, UK, China and Middle East Examples

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Dedication

Umu Chukwu

Professor Otomar Hájek and Mrs. Olga Hájek;

Franciscan Father Howard O'Shea

Brown University

All Heads of State and Religious Leaders

who seek peace and prosperity for their nation and all nations

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Preface

In this monograph we use economic principles to derive mathematical models of the economic state of the U.S., the U.K., China, Nigeria and the Middle East. These are validated with historical data from the International Financial Statistic Yearbook. The dynamics is a differential game of pursuit, with quarry control as government strategy, and the pursuer control as private initiative. This economic state consists of the popular ones: Gross Domestic Product (GDP), *y*; Interest Rate, *R*; Employment (or Unemployment) *L*; Values of Capital Stock, *k*; Prices (and therefore inflation $p'(t)$; and Cumulative Balance of Payment, *E*. The control strategy of government, *g* consists of autonomous government outlay, g_0, *e = exchange rate,* τ = *tariffs,* d = *transportation, distance between trading nations, trade policy, money supply and its flows* M*1*, $\dot{M}1$, and f_0 foreign credit and equalization taxes. Representative private firms' reaction to government control is *p*, which consists of productivity *n*, wages *w*, autonomous consumption C_0, autonomous investment I_0, autonomous net export x_0, autonomous money demand and intercepts associated with net export, supply and prices on the supply side. We formulate the systems of equations of the national economy of the four countries and by regression and differential equations methods test whether they are currently competing or cooperating. We explore the consequences of cooperation and deduce how this can be realized and the growth of wealth assured. More explicitly, we prove that the realistic economic models constructed for the four countries are controllable, so that from the current initial economic state a chosen better economic state can be attained using government and private strategies. Additional increase in the growth of gross national product so attained can then be used to increase the coefficient of cooperation in the four nations' dynamics. The outcome is seen to be unbounded growth of wealth for all concerned – real paradise.

In the Nigerian case to attain internal peace and harmony we consider the economy of the three regions (the North, the West, and the East) and the newer six geopolitical regions South-South, South-East, South-West, North West, North Central, and North East. We do the same type of analysis as is done for the four countries, and indicate how cooperation can ensure economic growth even under scarcity. Maple and MATLAB programs are used.

Our theory was affirmed by President Clinton in Lagos in the year 2000, He said: "So the Samaritan story is right for another reason, it is not just whatever is mine is yours if you need it, but if I give a little of it now I will get it back many times over because this old world is like a sea, and sometimes the sea is stormy and sometimes the sea is calm, sometimes the winds blow with us and sometimes the winds blow against us; sometimes one of us is the captain of the ship, and then three decades later somebody else may be the captain of the ship ......no matter what we are, we are all in the same boat. In the end the good Samaritan is better off."

President Jefferson Clinton, Vanguard Daily, Lagos. August 28, 2000.

The practical foreign policy implication of our analysis is promulgated by President Bush in July 2001 and was echoed again in March 2002 in Mexico.

"The Industrial Nations have the opportunity to include the world's poor in an expanding circle of development throughout all the Americas, all of Asia and all of Africa. This cause is a priority of the United States foreign policy."
"I also propose the World Bank and other development banks dramatically increase the share of their funding provided as grants rather than loans to the poorest countries. Specifically, I propose that up to 50 percent of the funds provided by the development banks to the poorest countries be provided as grants for education, health, nutrition, water supply, sanitation and other human needs."

President Bush, News and Observer, July 2001.

"Developed nations have a duty to not only share our wealth", Bush said. "We must tie greater aid to political and legal and economic reforms." "We must do good."

President Bush, Monterrey, Mexico, Friday, March 22, 2002 at UN Conference on Financing for Development.

In this book two major models of the economic state of a nation are derived and studied. One is an ordinary differential equation game of pursuit. We use Hájek's idea of reducing this to a control system whose controllability can easily be explored. Hájek's original formulation is contained and detailed in Hájek's Pursuit Games. Academic Press 1975. The other model is a hereditary one which utilized the principle of supply and demand and the principle of rational expectations. A full treatment is contained in the author's book, "Neutral Systems for Controlling the Wealth of Nations," World Scientific,

2001. The property of Controllability is also deduced. The book then explores the interaction of each of the group of four nations – Nigeria, US, UK, and China, and the Middle East powers of Egypt, Jordan, Israel, and the US. A theory of interreactions is postulated. The UN and IMF data are used to validate it, and test for competition and cooperation. Two different programs are used: MATLAB and MAPLE. A policy of sustained growth is deduced easily. The Nigerian internal case is presented in some detail.

The book treats in Chapters 5 and 6 the problem of Diffusion of Wealth and Control. It is seen that if the net aggregate of inflow of wealth and internally generated wealth are slow and nondecreasing the growth of wealth is unbounded. Some unsolved problems are posed. A tantalizating international wealth simulator is presented in the form of an electric circuit in Chapter 10. It mimics the behavior of wealth dynamics in Chapter 11. The book ends with a perspective and the moral basis of sustained economic growth and the triumph of cooperation. Accompanying the book is a CD containing two items: Matlab and Maple programming codes developed for validating the theory and for forecasting purposes; and Appendices covering the Matlab/Maple programs and results on the Nigeria, Middle East, and the Hereditary model.

In previous books and papers the author constructed and studied economic models of the national economies. Controllability was proved for a number of nations. The author would like to highlight what he considers extremely important and capable of leading to peace and prosperity in our world: the consequences of cooperation and competition which are studied. To the author's knowledge no such study seem to be available in the literature (in this way). The approach is new and the results are already being articulated by the world's religious leaders (Pope John Paul) and presidents of countries (Clinton, Bush).

This book requires an economic and mathematical background a little more advanced than an undergraduate mathematical analysis and macroeconomics course. However, the basic ideas presented were tried out in MATH 341, ordinary differential equation course at North Carolina State University. It is supplemented by MATLAB, MAPLE programs to identify the best coefficients that best fit the economic models and ensure the best forecasts; a computational easy-to-use approach. In Math 341, the ideas are only one application of the theory, at par with electrical, mechanical, and population dynamics which are natural in this course. By introducing the "rational expectations principle" we are led to hereditary differential equations and control. The author's books, "Optimal Control of the Growth of Wealth of Nations", and "Differential Models and Neutral Systems for Controlling the Wealth of Nations", are available as references.

The broad features of the book, the model, from economic principles, validation from data, (1MF, World Bank) and forecasts (from the predict command) attempts to provide "scientific" answers to broad national policy issues: can the economic state be controlled to a better state? Can the economic growth be maintained? How? Which is better, cooperation or competition? Does competition need be checked? Does cooperation need to be promoted in pursuit of national interest? From the economic point of view what can we do to create a sustainable society? Is it enough? The book is not meant for casual reading, for nonmathematicians or noneconomists. But the preface and the conclusions are assessable even when full basic understanding of the mathematics and the economics is not available. The fundamental argument for the role of cooperation in sustained economic growth can be described.

Suppose the growth rate of the US GDP x_1 is impacted by Egypt's GDP x_4 with a resultant $d_1x_1x_4$, i.e., a percentage growth rate $\frac{d_1x_1x_4}{x_1}\times 100 = 100d_1\, x_4$. Because Egypt's economic state is function space controllable, it is possible to double x_4. We can add a percentage $\frac{200d_1x_4}{x_1}$ to the growth rate of x_1. Thus we can increase the coefficient of cooperation from d_1 to $2d_1$ with growth values from, $d_1x_1x_4$ to $2d_1x_1x_4$. If $d_1 < 0, - d_1 > 0$ and we can increase the coefficient of cooperation from d_1 to $-2d_1$ with growth values from $d_1x_1x_4$ to $-2d_1x_1x_4$. Thus $(d_1 - 2d_1)x_1x_4 = - d_{11}x_1x_4$ $(d_{11} = - d_1 > 0)$. Then

$$\frac{dx_1}{dt} = - x_1(- a_1 + b_1x_2 + c_3x_3 + d_{11}x_4) + e$$

$$(d_{11} = - d_1 > 0)$$

The impact is a positive growth rate $d_{11}x_1x_4$ and a surplus growth rate of $100d_{11}x_4 > 0$. Thus x_1 will grow faster. Cooperation helps! The argument can also be made for x_4.

A refreshing approach in the "scientific" treatment of cooperation and competition models of the gross-domestic product of two groups of nations – Nigeria, the USA, the UK and China, and the US, Egypt, Jordan, and Israel – is the use of real data to validate the differential models. MATLAB programs and the computer run by Emeka Chukwu were used to estimate the coefficients of both the ordinary and hereditary models. An alternative Maple program is applied and run by Professor Norris. The generic programs are first displayed. The computer output in the form of graphs is embedded in the body of the book. The model seems realistic. With the "predict command" one can use the models to make prediction of the dynamics for up to 4 years with some degree of accuracy. The systems are studied for controllability stability and persistence.

Our approach opens the way to study economics as a quantitative science. As a result, good policy decisions can be deduced.

Our analysis rests on differential game theory. The duality theorem of Hajek which reduces a differential game to an equivalent control system is a linear theory. Dr. Chukwu has generalized this theory to apply to nonlinear ordinary and hereditary systems, using nonlinear variation of parameter cited in E. N. Chukwu and H. C. Simpson, "Perturbations of Nonlinear Systems of Neutral Types", J. Differential Equations 82(1989) 28-39, E. N. Chukwu, Stability and Control of Hereditary Systems with Applications to the Economic Dynamics of the U.S., 2nd Edition, World Scientific 2001, E. N. Chukwu, Universal Laws for the Control of Global Economic Growth with Nonlinear Hereditary Dynamics, Applied Mathematics and Computations 78:19-81 (1996) and G. A. Shanholt "A nonlinear Variation of Constant Formula for Functional Differential Equations", Mathematical Systems Theory 6, 1973, 343-352. The control sets P and Q are associated with two others $\boldsymbol{P}$, $\boldsymbol{Q}$ of the equivalent system.

The accompanied CD contains full details of the programs and their execution, and outputs of equations and graphs. A table of contents of the CD is given at the end of this book.

This book may help to interest economists who are not mathematicians in the mathematical and numerical modes of analysis of real-world economics. Readers with mathematical or numerical training but little economics training may find the presentation here a novel and useful applied mathematics and scientific pursuit which is useful in developing competence as technical advisers to government and economic policy makers. Such people will also assist Business Chief Executives to formulate broad profitable policies when dealing with many countries. Seniors and Graduate students in Applied Mathematics Economics and Game Theory will find rewarding the effort to master the theory and computation techniques presented here. The book will provide good thought and peaceful impulses at the United Nations the IMF, Religious Leaders and all the Agencies involved with economic growth and development, and the emergence of a new moral order.

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Introduction

PAUL SOLMAN : What would you do differently now as treasury secretary if you could go back in time? Anything?
ROBERT RUBIN: At one point Madeleine Albright came to me and she said – this is the beginning of the second term, "you and I, the secretary of state and the secretary of treasury, should go around the country and try to explain to the American people how much our self-interest is affected by what happens abroad and how important it is that we support trade and foreign assistance and crisis response, this whole range of issues." I though it was a terrific idea. I never followed up on it. We never did it.
How much difference it would have made, I don't know, but I wish we would have done it. A personal regret I have is that the six-and-a-half years I spent there – two White House, four-and-a-half treasury – were a remarkable experience in just a whole multitude of ways – to be with a group of extraordinarily talented people and the president of the United States working through these very complicated issues that affect not only our country but the rest of the world, there should be a feeling of satisfaction – there should be a feeling of fulfillment.
There should be a feeling of really engaging with all of your powers, such as they may be, on issues that matter. And because I was so caught up in what I was doing, and so consumed, in a sense, with trying to meet the challenge of what I was doing, I don't think I had that very often. I think it's too bad.

In earlier publications of Chukwu [1, 2] the ordinary differential game of pursuit

$$\dot{x}(t) = Ax(t) + Bp(t) + Cq(t) \tag{1}$$

is derived as a fairly realistic model of the development in time of the economic state of some nations

$$x = [y, R, L, k, p, E]',$$

with control strategies of the firm

$$p = [p_1, p_2, p_3, p_4, p_5, p_6]',$$

and of the government

$$g = [T, g_0, e, \tau, d, M1, \dot{M}1, p_0]'.$$

Where T denotes taxation, g_0 denotes autonomous government outlay, e denotes exchange rate, τ denotes tariffs, d denotes trade agreement and/or transportation, $M1$, $M1'$ denotes money supply and its flow – ($M1(t+1) - M1(t)$), f_0 denotes interest equalization tax.

It can be written as follows, a column vector,

$$g = [g_1\ g_2\ g_3\ g_4\ g_5\ g_6]'.$$

Here the Gross Domestic Product is denoted by y, Interest rate by R, employment by L, value of capital stock by k, prices by p, and cumulative balance of payment by E.

The control instrument of the firms (private initiative) is

$$\sigma = [C_0, I_0, x_0, m_0, n, w, y_{10}, p_0]',$$

where autonomous consumption is denoted by C_0, autonomous investment by I_0, autonomous net export by x_0, autonomous money demand by m_0, wage rate by w, autonomous income consumption intercept by y_{10} and autonomous price intercept by p_0.

Both x, σ, and g are column vectors,

$$x = [y, R, L, k, p, E]'$$

(T or ′ donates vector transpose)

$$\sigma = [C_0, I_0, x_0, m_0, n, y_{10}, p_0]'$$

Sometimes we shall designate σ by p, and g by q.

This model is confronted with data from the International Statistics Yearbook for Nigeria, the U.S.A., the U.K., and China. This modeling of four aggregate national incomes is fairly accurate. It is performed with six state differential equations, eight government control variables and nine reactions of the representative firm as detailed above. In spite of its size Professors Gandolfo and Pietro Carlo Pardoan has eloquently argued that such a macrodynamic medium term econometric model may be preferable to larger models for the purpose of policy analysis and simulation once a satisfactory estimation of their parameters has been obtained. This conclusion is also validated by Stephen Ellner and his co-workers [5].

We summarize our method and results as follows:

The state variable of the key economic indicators is the vector

$$x = [y, R, L, K, P, E]',$$

where

y = GDP, R = interest rate, L = employment, K = value of capital stock, p = prices and E = cumulative balance of payment.

The control instrument of the representative firm (private initiative) is

$$\sigma = [C_0, I_0, x_0, m_0, n, w, y_{10}, p_0]',$$

a column vector of components where

I_0 denotes autonomous investment
x_0 denotes autonomous net export
m_0 denotes autonomous money demand
n denotes labor productivity
w denotes wage rate
y_{10} denotes autonomous income consumption intercept
p_0 denotes autonomous price intercept.

The control strategy of government, the "solidarity function," g, is

$$g = [T, g_0, e, \tau, d, M1, M1', f_0]',$$

where:

T denotes Taxation
g_0 denotes autonomous government outlay,
e denotes exchange rate
τ denotes tariffs
d denotes preferential trade agreement and/or transportation or trade policy

$M1$ denotes Money supply(M1)
$M1'$ denotes rate of money supply $(M1(t+1) - M1(t))$
f_0 denotes foreign credit, interest equalization tax.

The economic indicator dynamics is in the matrix equation

$$\frac{dx(t)}{dt} = Ax(t) + B_1 q(t) + B_2 p(t).$$

Associated with this dynamics is the control system

$$\frac{dx(t)}{dt} = Ax(t) + Bu(t) \tag{AS}$$

where $u \in U$, the so called Pontryagin Difference of sets

$$U = \{u : u + B_1 Q \subset B_2 P\}.$$

$$= B_2 P \underset{-}{*} B_1 Q.$$

P private firms constraint set; and Q is government constrained control set.
Definition [1 p. 113].

The system (AS) where A, B are $n \times n$, $n \times m$ matrices is controllable on interval $[0, t_1]$. If given $x_0, x_1 \in E^n$, there is a control $u \in U$ such that the solution of (AS) with $x(t_1, t_0, x_0, u) = x_0$ satisfies $x(t_1, t_0, x_0, u) = x_1$. It is called controllable at time t_0 if it is controllable on $[t_0, t_1]$ for some $t_1 > t_0$. If it is controllable at each $t_0 \geq 0$, we say it is controllable. It is said to be fully controllable at t_0 if it is controllable on $[t_0, t_1]$ for each $t_1 > t_0$. It is said to be fully controllable if it is fully controllable at each $t_0 \geq 0$.
Theorem. (Assume that A, B are constant.) The system (SA) is fully controllable if and only if

$$\text{rank}[B, AB, \ldots, A^{n-1}, B] = 0.$$

Definition: The system

$$\dot{x}(t) = Ax(t)$$

is stable if the eigenvalues of A has no positive real part, i.e., eig(A) has no positive real part.

Definition. The system (AS) is controllable with constraints if it is controllable with admissible controls contained in U.

We use the MATLAB program described below to obtain the economic dynamics for each country.

Nigeria: Nigeria22.m

U.S.A.: US22.m

The U.S. model is controllable: rank(Mat) = 6, (rank(B_1) = 6), and Mat = [B, AB, …, A^5B]. It is unstable with controls switched off (two eigenvalues have a positive real part). Some of the components are oscillating and some exponentially growing. It is possible to obtain expression for forms of optimal controls and graphs for optimal strategies and trajectories as studied in Chapter 3 of [3].

United Kingdom: UK22.m

Since rank[B] = 6, and mat = [B, AB, … A^5B], rank(Mat) = 6.

This U.K. model is controllable. It is also unstable- four eigenvalues have a positive real part. But it is stablizable.

China: China22.m, Chinacu.m (stable, controllable) or China2.m controllable.

Since the models can be made controllable with constraints on strategies, a desired target can be attained with the strategies of the private firm and the government. Thus, the economic state of each nation can be steered to a better and higher level of GDP, full employment, etc. The increased wealth can be used to promote cooperation among the four nations. If sufficiently cooperating, sustained growth of GDP, employment, capital stock, cumulative balance of payment, low inflation, low interest rate can be guaranteed.

In this section we define some key economic variables and use them and the principle of "supply and demand" to derive an ordinary differential equation as the dynamics of the economic state, the GDP. Let C denote private consumption, I private investment, X net export, G government outlay, and Z aggregate demand.

The function Z is defined as follows:

$$Z = C + I + X + G .$$

Denote the derivative of any function f by f', i.e.,

$$f' = \frac{df}{dt};$$

and after income taxes

$$y - T, \text{ y } \textit{minus } T.$$

then $\frac{d}{dt}(y - T) = y' - T'$.

Also

$$I(t) = I_0 + I_1 y - I_2 (y - T)'$$

$$= I_0 + I_1 y - I_2 y' + I_2 T'$$

$$X(t) = X_0 + X_1 y(t) + X_{16} \tau(t) + X_{15} e(t) - X_{17} d(t)$$

$$+ y_1 (a_{11} + b_{11} y_2 + c_{11} y_3 + d_{11} y_4).$$

Net export is a function of GDP, tariff, exchange rate, distance between trading partners and trade policies. There is also interaction between one country y_1, and the other three partner countries ($y_i, i = 2, 3, 4$) which is captured by

$$y_1 (a_{11} + b_{11} y_2 + c_{11} y_3 + d_{11} y_4) \equiv h(t).$$

The function h mirrows inflow of wealth from the outside. If negative it represents outflow of wealth due to trade deficit and/or debt repayment. The government outlay, G, is given by

$$G(t) = g_0(t) + g_1 y(t) + g_2 y'(t).$$

The autonomous function g_0 is government outlay (which is independent of GDP etc) and which can be described as investment into innovation, education, and health, for example, which is independent of GDP. It has a positive influence on the rate of growth of income. The constants g_1, g_2 are all nonnegative. Thus we deduce that aggregate demand is given by

$$Z = C_0 + C_1(y - T) + C_2(y - T)' + I_0 + I_1 y - I_2(y - T)' + X_0 + X_1 y(t)$$
$$+ X_2 y'(t) + X_{16}\tau(t) + X_{15}e(t) - y_{17}d(t) + y_1(a_{11} + b_{11}y_2 + c_{11}y_3 + d_{11}y_4)$$
$$+ g_0 + g_1 y + g_2 y'(t).$$

Thus

$$Z = Z_0 + Z_1 y + Z_2 y' - C_1 T - C_2 T' - I_2 T' + y_1(a_{11} + b_{11}y_2 + C_{11}y_3 + d_{11}y_4) + x_{16}\tau(t) + X_{15}e(t) - y_{17}d(t), \tag{2}$$

where

$$\begin{aligned} Z_0 &= (C_0 + I_0 + X_0 + g_0), \\ Z_1 &= (C_1 + I_1 + X_1 + g_1), \\ Z_2 &= (C_2 - I_2 + g_2 + X_2). \end{aligned} \tag{3}$$

Using the Supply and Demand Law,

$$\frac{dy}{dt} = \lambda_1(z - y),$$

we have

$$\frac{dy_1}{dt} = \lambda_1(Z_0 + Z_1 y + Z_2 y' - C_1 T - C_2 T' - I_2 T' + b_{11}\tau(t) + X_{15}e(t) - y_{17}d(t) - y) + y_1(a_{11} + a_{12}y_2 + a_{13}y_3 + a_{14}y_4). \tag{4}$$

Let $p_1 = \lambda_1(Z_0 - g_0)$, i.e., $p_1 = \lambda_1(C_0 + I_0 + X_0)$, $\quad Z_2 = 0,$ (5)

$$g_1 = \lambda_1(g_0 + C_1 T + C_2\dot{T}(t) - I_2 T(t) - I_2\dot{T}(t) + X_{16}\tau(t) + X_{15}e(t) - X_{17}d(t)), \tag{6}$$

$$a_1 = \lambda_1(Z_1 - 1). \tag{7}$$

Then

$$\frac{dy_1(t)}{dt} = a_1 y_1(t) + p_1 + g_1 + y_1(a_{11} + a_{12}y_2(t) + a_{13}y_3(t) + a_{14}y_4(t)), \quad (8)$$

so that

$$\frac{dy_1(t)}{dt} = y_1(a_{111} + a_{12}y_2(t) + a_{13}y_3(t) + a_{14}y_4(t)) + p_1 + g_1, \quad (9)$$

where p_1 is defined in (5) and g_1 defined in (6) and $a_{111} = a_1 + a_{11}$. In a similar way

$$\frac{dy_2(t)}{dt} = y_2(a_{211} + a_{22}y_1(t) + a_{23}y_3(t) + a_{24}y_4(t)) + p_2 + g_2 \quad (10)$$

and

$$a_{211} = a_{21} + a_{21}. \quad (11)$$

The private and government control instruments are appropriately defined:

$$p_i = \lambda_i(Z_0\ i) \qquad i = 2, 3, 4 \quad (12)$$

$$g_i = \lambda_i(g_0 i + C_1 iT_i(t) + C_2 i\dot{T}_i(t) - C_2 iT_i' - I_2 iT_i'(t) + X_{16} i\tau_i(t)$$

$$+ X_{15} ie_i(t) - X_{17} id_i(t)), \qquad i = 2, 3, 4. \quad (13)$$

the constants correspond to the $i = 2, 3, 4$ countries.

$$\frac{dy_3(t)}{dt} = y_3(t)(a_{311} + a_{31}y_1(t) + a_{32}y_2(t) + a_{34}y_4(t)) + p_3 + g_3, \quad (14)$$

and $a_{311} = a_3 + a_{33}$;

$$\frac{dy_4(t)}{dt} = y_4(t)(a_{411} + a_{41}y_1(t) + a_{42}y_2(t) + a_{43}y_3(t)) + p_4 + g_4, \quad (15)$$

and $a_{411} = a_4 + a_{33}$. If $Z_2 \neq 0$, divide the left hand side and the right hand side of the equations (4-7) by $(1 - \lambda z_2)$ after $z_2 y'$ is transferred to the left to yield similar equations (9)-(15).

Let

$$y_1(t+1) - y_1(t) \equiv dy_1 \approx \frac{dy_1}{dt}.$$

Without controls, or with average controls, let

$$dy_1 = y_1(a_1 + a_{11}) + a_{12}y_2(t) + a_{13}y_3(t) + a_{14}y_4(t)) + e_1.$$

Over the interval let e_1 be the average of the function $p_1 + g_1$. We test for cooperation and competition using the Maple or MATLAB regression methods. If competitive and any a_{1i} is negative, one can use p_1 and q_1 in feedback forms,

$$p_1 = \alpha_1\, y_1\, y_2 + \alpha_2\, y_1\, y_3 + \alpha_3\, y_1\, y_4,$$

$$q_1 = \beta_1\, y_1\, y_2 + \beta_2\, y_1\, y_3 + \beta_3\, y_1\, y_4,$$

for appropriate α_i, β_i, $i = 1 \ldots 3$, to render $\frac{dy_1}{dt}$ positive and make y_1 increasing.

The coefficients are identified in the MATLAB or MAPLE programs which are given below, in the CD accompanying this book.

In what we shall describe now the GDP of the four nations will be written as y_1, y_2, y_3, y_4 and we shall denote it as $y = [y_1, y_2, y_3, y_4]'$. The equations for y_1, y_2, y_3, y_4 can be written in matrix form as follows. If

$$y = [y_1, y_2, y_3, y_4]', \tag{16}$$

(′ designates transponse)

then

$$\dot{y}(t) = A_0 y(t) + A_1(y(t))y(t) + B_1 p(t) + B_2 g(t), \tag{17}$$

where

$$A_0 = \begin{bmatrix} a_{111} & 0 & 0 & 0 \\ 0 & a_{211} & 0 & 0 \\ 0 & 0 & a_{311} & 0 \\ 0 & 0 & 0 & a_{411} \end{bmatrix}, \tag{18}$$

$$A_1(y(t)) = \begin{bmatrix} 0 & a_{12}y_1(t) & a_{13}y_1(t) & a_{14}y_1(t) \\ a_{22}y_2(t) & 0 & a_{23}y_2(t) & a_{24}y_2(t) \\ a_{31}y_3(t) & a_{32}y_3(t) & 0 & a_{34}y_3(t) \\ a_{41}y_4(t) & a_{42}y_4(t) & a_{43}y_4(t) & 0 \end{bmatrix}, \tag{19}$$

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{20}$$

$$B_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{21}$$

$$\dot{y}(t) = A_0 y(t) + A_1(y(t))y(t) + B_1 p + B_2 g \tag{22}$$

where

$$p = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}, \quad g = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix}. \tag{23}$$

Let

$$Bu = B_1 p + Bgq = [B_1, B_2]\begin{bmatrix} p \\ g \end{bmatrix}. \tag{24}$$

It is instructive to give more details of the strategies $p_i, g_i, i = 1, ..., 4$ in (12) and (13): Note that

$$g = [g_1, g_2, g_3, g_4]',$$

$$p = [p_1, p_2, p_3, p_4]',$$

so that

$$B_1 = \begin{bmatrix} \lambda_1 & \lambda_1 & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda_2 & \lambda_2 & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & \lambda_3 & \lambda_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & \lambda_4 & \lambda_4 \end{bmatrix}, \quad p = \begin{bmatrix} C_1 0 \\ I_1 0 \\ X_1 0 \\ C_2 0 \\ I_2 0 \\ X_2 0 \\ C_3 0 \\ I_3 0 \\ X_3 0 \\ C_4 0 \\ I_4 0 \\ X_4 0 \end{bmatrix}.$$

Let

$$T_{bi} = C_1 T_i + C_2 T_i'(t) - I_2 T_i'(t) - I_2 T_i'(t).$$

Then

$$B_2 = \begin{bmatrix} \lambda_1 & \lambda_1 x_{15} & \lambda_1 x_{16} & \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & \lambda_2 x_{15} & \lambda_2 x_{16} & \lambda_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_3 & \lambda_3 x_{15} & \lambda_3 x_{16} & \lambda_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda_4 & \lambda_4 x_{15} & \lambda_4 x_{16} & \lambda_4 \end{bmatrix},$$

$$g = \begin{bmatrix} g_1 0 \\ e_1 \\ \tau_1 \\ T_{B1} \\ g_2 0 \\ e_2 \\ \tau_2 \\ T_{B2} \\ g_3 0 \\ e_3 \\ \tau_3 \\ T_{B3} \\ g_4 0 \\ e_3 \\ \tau_4 \\ T_{B4} \end{bmatrix}.$$

We define

$$B = [B_1, \quad B_2] .$$

Then (22) becomes

$$\dot{x}(t) = A_0 x(t) + A_1(x(t))x(t) + Bu(t) .$$

Here B is a constant matrix and

$$\operatorname{rank} B = 4 . \tag{25}$$

It can be proved that the system (22) is locally controllable since

$$\operatorname{rank}[B, AB, A^2 B, A^3 B] = 4 . \tag{26}$$

where $A = D_1 f(0)$, and

$f(x) = A_0 x(t) + A_1(x(t))x(t)$ and $D_1 f$ is the derivative of f with respect to x. For better result (see [1, p 306-318]).

We may have assumed cooperation between private firms and government in our discussions above. The game theoretic strategies in the spirit of Hájek can make (22) to be equivalent to equation (27) as follows: In (22) let $p \in \mathcal{P}$, $g \in Q$ and for some matrix $1\mathcal{B}$

$$1\mathcal{P} = 1\mathcal{B}\mathcal{P}, \qquad 1Q = 1\mathcal{B}Q,$$

$$1\mathcal{U} = 1\mathcal{P} \underline{*} 1Q = \{u : \mathcal{B}u + 1Q \subset 1\mathcal{P}\}.$$

$$\dot{x}(t) = A_0 x(t) + A_1(x(t))x(t) + Bu, \ u \in 1\mathcal{U}$$

Here $1\mathcal{P}$ denotes the set of the possibilities of the private firms and $1Q$ that of government in the associated nonlinear control system, and $0 \in \text{Int } 1\mathcal{B}\mathcal{U}$ in

$$\dot{x}(t) = A_0 x(t) + A_1(xt))x(t) + Bu(t), u \in 1\mathcal{U} \tag{27}$$

where $1\mathcal{U}$ is the Pontryagin difference of sets: Hajek's duality Theorem whose spirit and method we have invoked is generalized and applied to nonlinear ordinary and heredity systems in E. N. Chukwu, Universal Laws for the Control of Global Economic Growth with Nonlinear Hereditary Dynamics. Applied Mathematics and Computation, 78:19-81 (1996).

We now consider the problem of controllability for ordinary differential equations. The linear case is first tackled. Let

$$\dot{x}(t) = A_0(t)x(t) + B(t)u(t) \tag{28}$$

where $A_0(t), B(t)$ are analytic $n \times n, n \times m$ matrices defined on $[0, \infty)$. The functions $u(t)$ are controls which are required to be square integrable on bounded intervals, and $x(t)$ is an n-vector.

Definition: The system (28) is controllable on an interval $[t_0, t_1]$ if, given any $x_0, x_1 \in E^n$ there is a control $u \in \Omega$ such that the solution $x(t, t_0, x_0, u)$ of (28) with $x(t_0, t_0, x_0, u) = x_0$ satisfies $x(t_1, t_0, x_0, u) = x_1$. It is called controllable at time t_0 if it is controllable on $[t_0, t_1]$ for some $t_1 > t_0$.

If A_0, B are constant matrices then (28) is fully controllable if and only if

$$\text{rank}[B, A_0, B, \cdots A_0^{n-1} B] = n . \tag{29}$$

We can also consider nonlinear differential system

$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), \qquad t \geq t_0 , \\ x(0) &= x_0 \in E^n , \end{aligned} \tag{30}$$

where $f: E \times E^n \times E^m \to E^n$ is continuous and continuously differentiable in the second and third arguments. We assume that the solution of (30) exists and is unique. Thus for each $u \in L_\infty([t_0, \infty), E^m)$ there exists an unique response $x(t_0, x, u): E = (t_0, \infty) \to E^n$ defined by $t \to x(t, t_0, x_0, u)$ which represents a point of E^n.

We shall study the mapping

$$u \to x(t, t_0, x_0, u); u: L_\infty([t_0, \infty), E^m) \to E^n . \tag{31}$$

If (28) is controllable at $t_0 \geq 0$ we say it is controllable. System (28) is said to be fully controllable at t_0 if it is controllable on $[t_0, t_1]$ for each $t_1 > t_0$. It is said to be fully controllable if it is fully controllable at each $t_0 \geq 0$.

If u is a control, the solution of (28) corresponding to this u is given by the variation of constant formula

$$x(t, t_0, x_0, u) = X(t)X^{-1}(t_0)x_0 + X(t) \int_{t_0}^{t} X^{-1}(s)B(s)u(s)ds , \tag{32}$$

where $X(t)$ is a fundamental matrix solution of

$$\dot{x}(t) = A_0(t)x(t) . \tag{33}$$

Define

$$Y(s) = X^{-1}(s)B(s) ,$$

and the operation Γ by

$$\Gamma = - A_0 + \frac{d}{dt} ,$$

Let

$$M(t_0,t)=\int_{t_0}^{t} Y(s)Y^T(s)ds=\int_{t_0}^{t} X^{-1}(s)B(s)B^T(s)X^T(s)ds\,, \tag{34}$$

In what follows we adopt the notation that the vector a transposed is denoted by a^T italics, and a' is $\frac{da}{dt}$.

where X^T is the transpose of X. The following theorem can now be stated.
Theorem 1 [1 p. 114]. The following are equivalent:

(i) $M(t_0, t_1)$ is nonsingular for all $t_1 > t_0$ and all $t_0 \geq 0$.
(ii) The system (28) is fully controllable.

If $A_0(t), B(t)$ are analytic on $(t_0\ t_1), t_1 > t_0 \geq 0$ them (28) is fully controllable at t_0 if and only if for each $t_1 > t_0$ there exists $t \in (t_0, t_1)$ such

$$\operatorname{rank}[B(t), \Gamma B(t), \cdots, \Gamma^{n-1}, B(t)] = n\,. \tag{35}$$

Theorem 2. Consider the system

$$\dot{x}(t) = A_0 x(t) + A_1(x(t))x(t) + Bu(t)\,, \tag{27}$$

where

A_0, B are constants,
A_1 is differentiable,
rank $B = n$.

Then (27) is controllable.
A Heuristic Proof. Let the matrix H be defined as follows

$$H = \int_0^{t_1} BB^T ds, \qquad t_1 > 0$$

has an inverse. B^T is the transpose of B.

The solution of (27) satisfies

$$x(t) = x_0 + \int_0^t A_0 x(s)ds + \int_0^t A_1(x(s))x(s)ds + \int_0^t Bu(s)ds .,$$

x( ) is a solution of (27) corresponding to u and x_0.

The control that does the transfer from x_0 to x_1 is defined as follows:

$$u(t)\ B^T H^{-1}\left[x_1 - x_0 - \int_0^t A_0 x(s)ds - \int_0^t Bu(s)ds - \int_0^t A_1(x(s)x(s)ds \right]$$

where x( ) is a solution of (27) corresponding to u and x_0. We now show that a u exists. It is an L_∞ function since $t \to B$ is constant, and we need to prove that it is a solution of the integral equation. Once the existence is established,

$$x(t_1) = x_0 + \int_0^{t_1} A_0 x(s)ds + \int_0^{t_1} A_1(x(s))x(s)ds + \int_0^{t_1} BB^T H^{-1}\left[x_1 - x_0 - \int_0^{t_1} A_0 x(s)ds \right.$$

$$\left. - \int_0^{t_1} A_1(x(s))x(s)ds \right] = x_1 .$$

Though this argument is plausible to conclude controllability, it can be made rigorous by a fixed point procedure. A more general proof is contained in Section 8.3 of Chukwu [1, proposition 8.3.1]. "Stability and Time Optimal Control of Hereditary System with Application to the Economic Dynamics of the U.S.," World Scientific, 2003.

Consider the nonlinear system

$$\dot{x}(t) = f(t, x(t), u(t)) + B(t, x(t))u(t) \tag{36}$$

where $B: E \times E^n \times E^m \to E^{n \times m}$ is a continuous $n \times m$ matrix function, and $f: E \times E^n \times E^m \to E^n$. We state conditions for the existence of a unique solution.

Theorem 3. In (36) assume that

(i) $B: E \times E^n \to E^{n\times m}$ is continuous, $B(t,\cdot): E^n \to E^{n\times m}$ is continuously differentiable.

(ii) There exist integrable functions $N_i : E \to [0, \infty)$ $i = 1, 2$, such that

$$\| \frac{d}{dx} B(t, x(t)) \| \le N_1(t)\,,$$

$$\| B(t, x(t)) \| \le N_2(t)$$

for $t \in E$, and $x \in C([0, t_1), E^n)$. Here and in the sequel $D_i g(\ )$ is the Fréchet derivative of g with respect to the i-th variable.

(iii) $f(t, \cdot, \cdot)$ is continuously differentiable for each t.

(iv) $f(\cdot, x, w)$ is measurable for each x and w.

(v) For each compact set $k \subset E^n$ there exists an integrable function $M_1 : E \to (0, \infty)$ and square integrable functions $M_i : E \to [0, \infty)$ $i = 2, 3$ such that

$$\| \frac{d}{dx} f(t, x, w) \| \le M_1(t) + M_2(t) \| w \|\,,$$

$$\| \frac{d}{dw} f(t, x, w) \| \le M_3(t)\,,$$

for all $t \in E$, $w(t) \in E^m$ and all $x \in C([0, \infty), E^n)$.

Under assumptions (i) to (v), for each $u \in L_\infty, x_0 \in E^n$, there exists a unique solution $x = x(t, x_0, u)$ of (36); that is an absolutely continuous function $x : [t_0, \infty) \to E^n$ such that (36) holds almost everywhere and $x_0(t_0) = x_0$. Also $(x_0, u) \to x(t_0, x_0, u) \in C$ is continuously differentiable.

The proof is outlined in Lee and Marcus [8, p. 366] and mimicked in Underwood and Young [9].

In addition to

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{37}$$

where $f: E \times E^n \times E^m \to E^n$ is continuous and, in the second and third argument, is continuously differentiable, we assume all solutions $x(t, x_0, u)$ of (37) exist and are continuous. We also consider the linearized system,

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{38}$$

where

$$\frac{d}{dx} f(t, 0, 0) = A(t), \quad \frac{d}{du} f(t, 0, 0) = B(t). \tag{39}$$

Theorem 4: Consider (37) in which

(i) $f: E \times E^n \times E^m \to E^n$ is continuous, and continuously differentiable in the second and third arguments.
(ii) $f(t, 0, 0) = 0$, for all $t \geq 0$,
(iii) System (38) with $A(t)$, $B(t)$ given by (39) is Euclidean controllable on $[0, t_1]$.

Then the domain $\mathcal{C}$ of null controllability of (37) has $0 \in \operatorname{Int} \mathcal{C}$.
Some definitions. Let

$$C^m = \{u \in E^m : |u_j| \leq 1 \quad \text{for} \quad j = 1, \ldots, m\}$$

The attainable set of (37) is the subset of E^n,

$$\mathcal{A}(t, x_0) = \{x(t, x_0, u) : x \text{ is a solution of (37)},$$

$$x(0, x_0, u) = x_0, \text{ measurable } u : [0, t] \to C^m\}.$$

We set $\mathcal{A}(t) = \mathcal{A}(t, 0)$..
The domain $\mathcal{C}$ of null controllability of (37) is the set of all initial points x_0 that can be steered to the origin at finite time:

$$C = \{x_0 \in E^n : x(t_1, x_0; u) = 0 \quad \text{for some} \quad t \geq 0,$$

$$\text{some measurable } u : [0, t] \to C^m\}.$$

System (37) is said to be controllable (with constraints) if $0 \in \text{Int}\, \mathcal{C}$.

Theorem 5. Suppose system (38) is controllable, and this holds if and only if

$$\operatorname{rank}[B, AB, \cdots, A^{n-1}B] = n,$$

when A, B are constant, and, more generally if

$$\operatorname{rank}[B(t), \Gamma B(t), \cdots, \Gamma^{n-1}B(t)] = n,$$

for each t in some $(0, \in)$, $\in > 0$. Then $0 \in \text{Int}\, \mathcal{C}$.

Proof. The solution of

$$\dot{x}(t) = f(t, x(t), u(t))$$

with initial data

$$x(0) = 0$$

satisfies the integral equation

$$x(t, u) = \int_0^t f(s, x(s), u(s))ds .$$

Consider the mapping (31)

$$T : L_\infty([0, t_1], E^m) \to E^n$$

defined by

$$Tu = x(t, u) .$$

Since f is smooth

$$\frac{d}{du} x(t, u) = \int_0^t \frac{d}{dx} f(s, x(s, u), u(s)) \frac{d}{du} x(s, u)ds$$

$$+\int_0^t \frac{d}{du} f(s, x(s, u), u(s))ds .$$

We differentiate with respect to t,

$$\frac{d}{dt}\frac{d}{du} x(t, u) = \frac{d}{dx} f(t, x(t, u), u(t)) \frac{d}{du} x(t, u) \tag{40}$$

$$+\frac{d}{du} f(t, x(t, u), u(t))$$

Since $f(t, 0, 0) = 0$, for all $t \geq 0$, $x(t, 0, 0) = 0$ is a solution of

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(0) = 0 .$$

Thus

$$T'(0)v = D\frac{d}{du} x(t, 0)v = x(t, v), \qquad v \in L_\infty([0, t_1], E^m)$$

is a solution of (38), (40). Also

$$T'(0): L_\infty([0, t_1], E^m) \to E^n$$

is a surjection if and only if the system (40) with $z(0, v) = 0$ is controllable on $[0, t_1]$. It follows from Graves' Theorem [7, p. 193] that T is locally open: there is an open ball radius ρ, $B_0 \subset L_\infty([0, t_1], C^m)$ centered at 0, and an open ball radius r, $B_r \subset E^n$ centered at 0 such that

$$B_r \subset T(B_\rho) \subset T(L_\infty([0, t_1], C^m) ,$$

Since

$$T(L_\infty([0, t_1], C^m) = \mathcal{A}(t)$$

where

$\mathcal{A}(t) = \{x(t,u) : u \in L_\infty([0,t_1], C^m)$, x is a solution of (37) with $x(0,u) = 0\}$, and we have proved that $0 \in \text{Int}\, \mathcal{A}(t)$. As a result this proves also that $0 \in \text{Int}\, \mathcal{C}$. To obtain a global result, assume the hypotheses, (i) – (ii) of Theorem 5 and suppose the solution $x(t)$ of

$$\dot{x}(t) = f(t, x(t), 0), \qquad x(0) = x_0,$$

tends to $x_1 = 0$ as $t \to \infty$, then $\mathcal{C} = E^n$ that is (37) is globally (Euclidean) null controllable with constraints. Proof [1, p. 292].

We have isolated null controllability in E^n as our objective in the analysis of the growth of x, the gross domestic products of nations. The zero target is artificial. We now incorporate nontrivial targets. Indeed if x_1 is a nontrivial target and $y(t) = y(t, x_0, u_0)$ is any solution of

$$\dot{x}(t) = f(t, x(t), u(t))$$

$$y_0 = x_0$$

and u^0 an admissible such that $y(t_1) = x_1$, one can equivalently study the null controllability of the system

$$\dot{x}(t) = f(t, z(t) + y(t), u(t)) - f(t, y(t), u(t)),$$

where $z_0 = 0$ so that $x(0) = y(0) = x_0$ and $x(t) = z(t) + y(t)$. If we can show that there is a neighborhood ϑ of the origin in z-space such that $x_0 \in \vartheta$ can be brought to $z(t_1\ u, x_0, u^0) = 0$ by some admissible control at time t_1, then

$$z(t_1, u, x_0, u^0) = x(t_1, x_0, u) - y(t_1) = 0$$

so that

$$x(t_1, x_0, u) = y(t_1) = x_1.$$

We now consider the dynamical systems of gross domestic product

$$\frac{dy_1(t)}{dt} = y_1(a_{111} + a_{12}y_2(t) + a_{13}y_3(t) + a_{14}y_4(t)) + p_1 + g_1$$