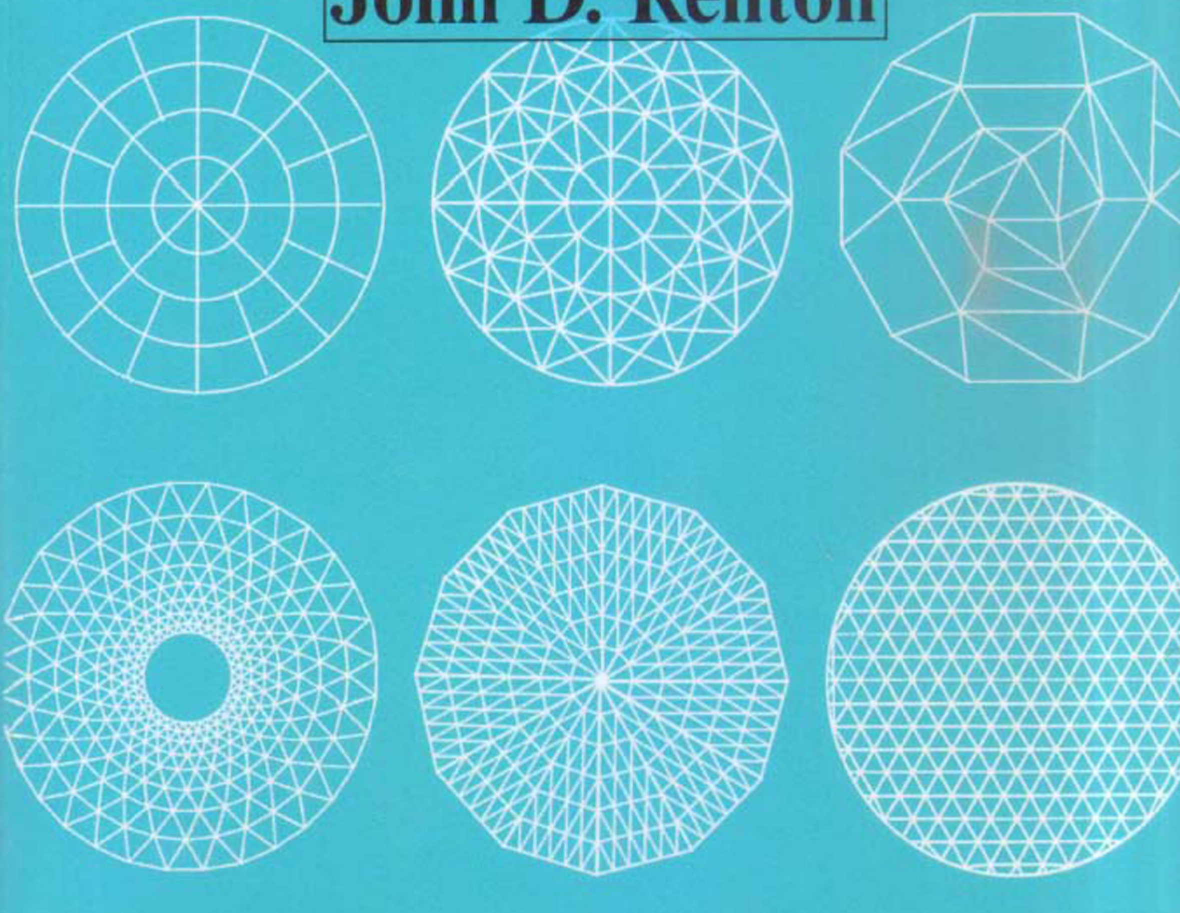
ELASTIC BEAMS AND FRAMES

Second Edition

John D. Renton



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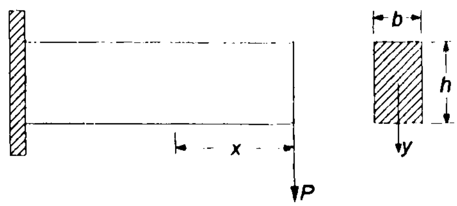
Preface

I cannot doubt but that these things, which now seem to us so mysterious, will be no mysteries at all; that the scales will fall from our eyes; that we shall learn to look on things in a different way - when that which is now a difficulty will be the only common-sense and intelligible way of looking at the subject.

(Lord Kelvin)

Early Developments in the Theory of Elasticity

1638 Galileo Galilei



1660-
1678 Robert Hooke:
(Hooke's Law)

1686 Edme Mariotte
1694 Jacob Bernoulli:
$C/R = Px$ ($C = mbh^3/3$)

1742 Daniel Bernoulli:
$\int 1/R^2 dx$ is a minimum

1744 Leonhard Euler:
$C\, d^2v/dx^2 \approx Px$ ($C \propto bh^2$)

1776 Charles Coulomb:
$\int \sigma y\, dA = M$

1788 Jacob Bernoulli (the younger):
$d^4w/dx^4 + d^4w/dy^4 = c^4\, d^2w/dt^2$

1813 Sophie Germain:
$d^2w/dt^2 + k^2\nabla^4 w = 0$

1807 Thomas Young:
(Young's modulus)——1819 Louis Navier:
$EI\, d^2v/dx^2 \approx EI/R = M$
(Modern beam theory)

1823 Siméon Poisson:
Poisson's ratio (=¼)

1850 Gustav Kirchhoff:
(Modern plate theory)

1852 Gabriel Lamé:
Material-specific λ, μ
and Poisson's ratio.

1854 Jacques Bresse:
(Principle of Superposition)

1857 Barré de Saint-Venant:
(Saint-Venant's Principle)

The history of science is one of observation, measurement, and the postulation of theories to explain the observed phenomena. As Karl Popper said, any theory worthy of being called scientific should be capable of making predictions which can be tested. With the passage of time, these processes become more precise and sophisticated. Theories which had earlier seemed to be satisfactory turn out to be inadequate, either because more precise measurements and analysis show them to be false or because they lack the scope to explain new phenomena. Also, they may be replaced by new methods and ideas which are more elegant or useful. Thus Young's modulus, E as used today, should really be attributed to Navier. Young's own definition was "The modulus of elasticity of any substance is a column of the same substance, capable of producing a pressure on its base which is to the weight causing a certain degree of compression as the length is to the diminution of its length". Poisson was originally convinced that his ratio, ν, had a fixed value of ¼ until the body of experimental evidence showed that it must take different values for different materials. Lamé (1852) was probably the first to accept that this must be so.

Galileo first examined the resistance of a cantilever beam to failure. He had no concept of elasticity, and implicitly assumed a constant stress distribution at the support with failure occurring by rotation about the bottom edge. Although this lead to poor predictions of the failure of a cantilever, it came to be known as 'Galileo's problem'. Working independently from Hooke, Mariotte found the elastic stress distribution in bending. This was the correct linear response, with the neutral axis at the centroid. However, owing to a mathematical error, he concluded that this gave the same result for the resistance of the beam as taking a tensile linear variation in stress from a fulcrum at the bottom edge. The name 'Bernoulli' is often used generically for the whole family from Jacob[1] (the elder) to Daniel. The former is usually credited with the invention of beam theory, having determined that beam curvature was proportional to the local bending moment. However, he incorporated Mariotte's mistake in his work. Even Euler, who first applied differential calculus to beam theory, wrongly estimated the stiffness of a rectangular beam to be proportional to the square of its depth. Again, it is Navier who should take the credit for beam theory as it is normally used today[2].

Perhaps surprisingly, the earliest work on the theory of plates was concerned with their vibration. Jacob Bernoulli treated plates as if they were square grids of beams (subject to bending only) and so derived an incorrect differential equation for their lateral displacement w. Sophie Germain found the correct form for the equation, but was at a loss to give an expression for the plate's bending stiffness. Poisson (1814) deduced the same equation as Sophie, but claimed that it was too complicated to be solved. He also insisted that three boundary conditions could be imposed on each edge, a debate which continued until Kirchhoff's definitive work on the subject in 1850. The Bernoulli-Euler hypothesis for beams, that plane sections remain plane, has an equivalent for plates and shells known as the Kirchhoff-Love hypothesis. This hypothesis has been challenged from the earliest times, most notably by Barré de Saint-Venant. It will be seen that his name and that of Bresse have been added to the above chart. In Chapter 8, it will be shown that their principles unify the engineering theories of flexural, torsional and axial response. Together with the concept of strain energy, the shear response of beams can be determined from exactly the same principles. They also lead to the determination of the characteristic responses of other

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[1]Also known as James or Jacques.

[2]However, it is commonly known as the Bernoulli-Euler theory. In October 1742, Daniel Bernoulli wrote to Leonhard Euler proposing his minimum principle. From this, Euler deduced his differential equation (see chart) using the calculus of variations that he had invented. It then still required Coulomb to discover the correct stress distribution and Young to devise his modulus before all the elements of bending theory as used today could be combined by Navier.

linearly-elastic structures. This is made possible by discarding the constraint of assuming that 'plane sections remain plane'. Here and elsewhere in the book it has been necessary to re-examine approaches which have been hallowed by convention.

The general reader may be more interested in practical applications rather than the more abstract aspects of the theory. The intention is to provide an introduction to more advanced ideas on the subject than is commonly available in a student text book. There are, of course, many excellent books on the basic theory of structures. The aim here will be to present the basic theory in a different light and add to it material which is useful but is not readily available. Efforts have been made to maintain an overall consistency of notation. As one form is more familiar for two-dimensional problems and another for three-dimensional problems, some compromises have been made. As far as possible, the more advanced sections which may not be required at a first reading will be enclosed in curly brackets: { } , and will normally be located after the elementary theory, even at the risk of some duplication, and specialist formulae and data consigned to the appendices.

JOHN D. RENTON

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Chapter 1 Introduction

1.1 Loads, Deflexions, Joints and Supports

The most common notation will be found in Appendix 1 and is also defined where it is first used. In addition, it is worthwhile making clear some of the distinctions in terminology which will be used in this book. A *force* is an action or influence on a body which tends to cause it to move in a particular direction. A *moment* is an action or influence on a body which has a turning effect on it, about a particular axis. The word *load* will be taken as a general term, referring to forces, moments, distributed forces (such as pressure) and distributed moments. Likewise, *deflexion* may be taken to refer to a linear displacement, a rotation or a general movement of a structure or component. Loads can have *corresponding deflexions* which are the deflexions through which they do work. Thus the corresponding deflexion to a force is its displacement along its line of action, and the corresponding deflexion to a moment acting about some axis is its rotation about that axis. The work done by a load is then given by the integral of that load with incremental changes in its corresponding deflexion.

Forces, displacements, moments and rotations are, in general, vector quantities possessing both magnitude and direction. Figure 1.1 shows the convention that will be used in two dimensions. $\boldsymbol{F}$ is a force vector and $\boldsymbol{u}$ is the deflexion vector corresponding to it. The horizontal and vertical components of $\boldsymbol{F}$ are H and V in the x and y directions respectively. If the magnitude of $\boldsymbol{F}$ is F, then the values of H and V are $F \cos \alpha$ and $F \sin \alpha$ respectively. The horizontal and vertical components of $\boldsymbol{u}$, u and v, are similarly related to the magnitude of $\boldsymbol{u}$. In a right-handed coordinate system, the z axis would come out of the paper. The moment M and the corresponding rotation θ would be clockwise about this axis, as viewed looking along it.

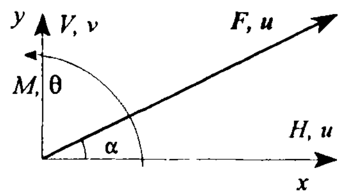

Figure 1.1 Loads and deflexions in two dimensions.

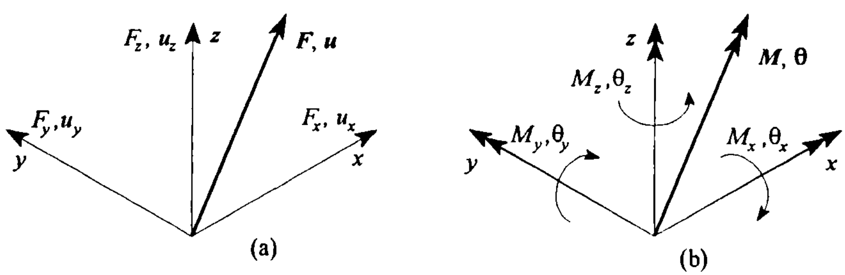

Figure 1.2 Load and deflexion vectors in three dimensions.

In three dimensions, the force $\boldsymbol{F}$ and the corresponding deflexion $\boldsymbol{u}$ have components (F_x, F_y, F_z) and (u_x, u_y, u_z) respectively in the x, y and z directions, as shown in Figure 1.2a. The moment $\boldsymbol{M}$ and the corresponding rotation $\boldsymbol{\theta}$ can be regarded as vectors too, with components (M_x, M_y, M_z) and (θ_x, θ_y, θ_z) in the x, y and z directions, as shown in Figure 1.2b. These components are taken to be positive in the clockwise sense, as viewed along the axis about which they act. The double-headed arrow convention is used to indicate that they are *rotation vectors.*

Most joints and supports are taken to be workless. That is, they neither absorb nor give out work (or energy). The most common exceptions are elastic supports or joints and frictional supports. Here, we will consider only the workless variety. Most frameworks are classified into either pin-jointed frames or rigid-jointed frames. In practice, most frames are not entirely one or the other, but from the analytic point of view, the classification is useful. Figure 1.3 shows the two types of joint. Neither permits relative displacement of the ends of the structural members joined to it, and both can sustain forces applied to them. However, the pin joint permits relative rotation of the ends of members joined to it. As it is a workless joint, it follows that it can exert no moments on these ends and also it cannot sustain an external moment applied to it. Usually, the members of a pin-jointed framework are straight and are not subject to lateral loads between their ends. In this case, they only sustain axial forces. In three dimensions, the pin joint permits relative rotation of the ends about all three axes, and is commonly referred to as a *universal joint*. A rigid joint does not permit any relative rotation of the ends attached to it, and can exert moments as well as forces on the member ends.

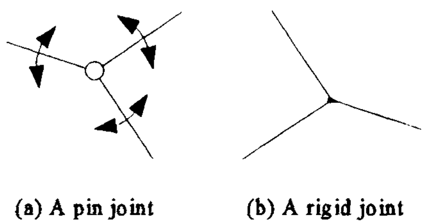

Figure 1.3 Common types of joint.

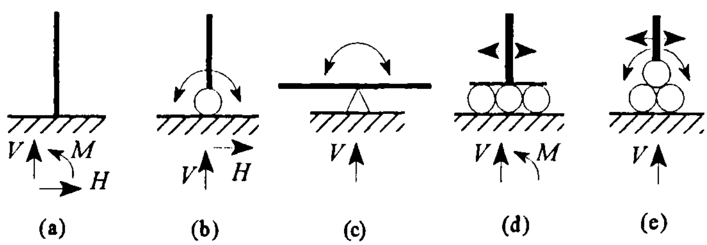

Figure 1.4 Common types of workless support (in two dimensions)

The common types of workless support are shown in Figure 1.4. These can exert either a reaction (force or moment), in which case the corresponding deflexion of the frame at that point is zero, or a particular reaction does not exist, in which case the corresponding deflexion of the frame at that point is free to take place. (The extreme example of the latter is a free end, which is subject to no reactions at all.) Other workless supports could exist, such that both reactions and deflexions took place at the support, but they would be interrelated in such a way that no net work was ever done. However, these can be simulated by rigid mechanisms attached to supports of the above kind. Figure 1.4a shows a *fixed* or *encastré* end. All rotation and displacement of the end are prevented, requiring the three reactions shown in two dimensions (or six reactions in three dimensions). Figure 1.4b shows a *pin support* which permits end rotation but no end displacement. In three dimensions, a pin support will be taken to permit rotation about all three axes, and a *rocker* support taken as one which permits rotation about a particular axis (the rock axis). Figure 1.4c shows a *knife-edge* support which permits rotation of the structure about it but no motion normal to it (in either direction). It can then exert a normal reaction but no moment reaction. This kind of support is postulated in examining the flexure of beams, when it is assumed that the axial beam loading in the beam is insignificant, so that the question of any lateral reaction provided by the knife edge does not arise. Figure 1.4d shows a *slider* support which permits transverse motion of the end, but prevents any rotation or normal motion of it with the aid of normal and moment reactions, M and V, if necessary. Finally, 1.4e shows a pinned slider support which allows rotation and lateral displacement of the end but prevents any normal motion with a normal reaction V. More possible workless support conditions exist in three dimensions, but no commonly-accepted symbols have been devised for them.

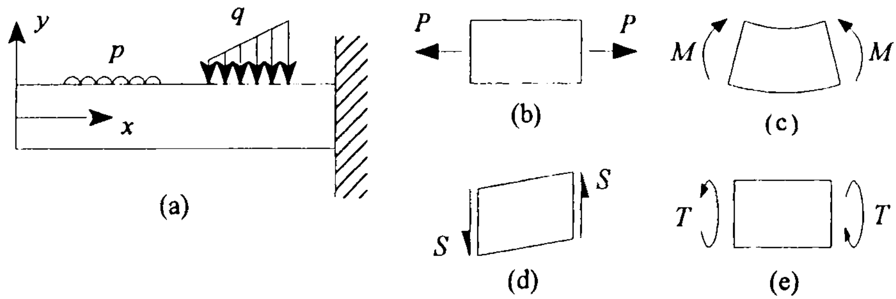

Figure 1.5 Loading on a structural member.

The above loading descriptions are related to some overall set of coordinates and not to a particular structural member. Figure 1.5a shows distributed loads p and q acting on a beam. These are usually forces per unit length, although sometimes they are used to indicate forces per unit area. The symbol used for p will be used to indicate that it is of constant intensity, whereas q, as shown, indicates a variable intensity. Figures 1.5b to 1.5e show resultant internal load pairs acting on short sections of the beam. Each pair is acting in the positive sense used in this book, which is that the corresponding deflexion tends to increase along the axis (here the x axis) of the beam. The axial force P in Figure 1.5b is tensile positive, producing an increase in the axial displacement u along the beam. In Figure 1.5c, the moment M produces 'sagging' of the beam, corresponding to a positive increase in the rotation θ along the beam. The shear force S in Figure 1.5d acts in such a way as to increase the lateral displacement v in the y direction. Lastly, the torque T is such that the beam's rotation about its longitudinal (x) axis increases along its length.

1.2 Small Deflexion Theory

In order to illustrate the mode of deflexion of a structure, this mode is usually exaggerated. The actual deflexions are usually too small to be detected with the naked eye. Typically, the maximum displacement of a beam is no more than 1/300th of its length and the induced rotations are likewise of the order of a fraction of a degree. For many purposes, the change in geometry of a structure under load can be ignored. For example, its equilibrium under load is usually examined relative to the position of the loading in the undeformed state. Also, the changes in orientation of parts of the structure, produced by rotations under load, are often (implicitly) taken as negligible. If the rotations were large, then the effect of a sequence of rotations about different axes, which themselves are affected by these rotations, would depend on the order in which the rotations took place. (The method of analysing the effects of large rotations will be discussed in setting up the stiffness matrices for beams with arbitrary orientations in three dimensions, see §11.4.) However, if the rotations are small, the effect of a sequence of rotations is given by their vector sum.

Unless specifically stated otherwise, radian measure will be used for such rotations. Then there is no need to distinguish between these rotations and their sines and tangents, as they are the same, if only the first term of each of the power expansions for these functions is taken to be significant. In particular, this means that the slope (or gradient) of a beam produced by flexure, and the flexural rotation of the beam at the same location are taken as the same.

Figure 1.6 shows the displacement of the end C of a rigid bar AC of length l resulting from a rotation θ about the end A. These are given by the horizontal and vertical components, δ and ϵ, where

$$\delta = l\sin\theta \quad ; \quad \epsilon = l(1-\cos\theta).$$

As θ is very small, δ can be taken as $l\theta$ and ϵ as zero (using the first terms in the expansions of $\sin\theta$ and $\cos\theta$ only). However, for stability analyses it will be seen later that it is necessary to use the more accurate expression $\frac{1}{2}l\theta^2$ for ϵ, given by including the second term in the expansion for $\cos\theta$.

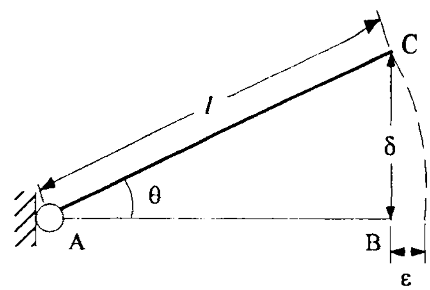

Figure 1.6 A rigid-body rotation.

1.3 Energy, Equilibrium and Stability

The laws of thermodynamics form an important part of the theoretical basis of structural analysis. They can be expressed in various ways, but put simply, they are as follows. The first law states that in any closed system, energy is conserved. That is, if we can encapsulate the structural system within a real or imaginary boundary, so that energy passes neither in nor out, the total energy within the system will remain constant. (Workless supports can form a useful part of that boundary.) However, the forms of energy within the system can change. The second law is related to what (natural) changes in these forms can take place. They tend to be such that the potential for further change to take place is reduced. The system tends to become less organised and more random (the 'entropy' increases), so that the amount of 'useful' energy is reduced. To put it another way, you cannot have your cake and eat it, or there are no free lunches.

A loaded structure is generally designed to reach a state of balance of loads (both external and internal) such that no further movement occurs. This state holds not only for the structure as a whole but for every part of it. It is then said to be in a state of *static equilibrium*. If the loaded structure tends to move away from a given deflected state, then it is not in equilibrium. The energy of movement (*kinetic energy*) may be provided from two sources. The applied loads can do work, thus reducing their *potential energy*, or the deformed structure may release *strain energy* in moving back towards its undeformed state. A clock driven by falling weights and one driven by a spring are examples of kinetic energy derived from these two sources. Then if a loaded structure is in equilibrium, it has no tendency to move resulting in the release of energy from these two sources. This can be analysed by examining the net energy released from these two sources during any imaginary small deflexion (*virtual deflexion*) from the equilibrium state. The conditions of static equilibrium of the loads on a body are that these loads have no resultant force or moment. This means that during any *rigid-body motion* of the body, in which it displaces or rotates without any change in its deformation, no net work is done and hence no kinetic energy can be generated.

A loaded structure is not always safe, even when it is in equilibrium. Although in theory it could remain in that state if undisturbed, some small imperfection or external perturbation may

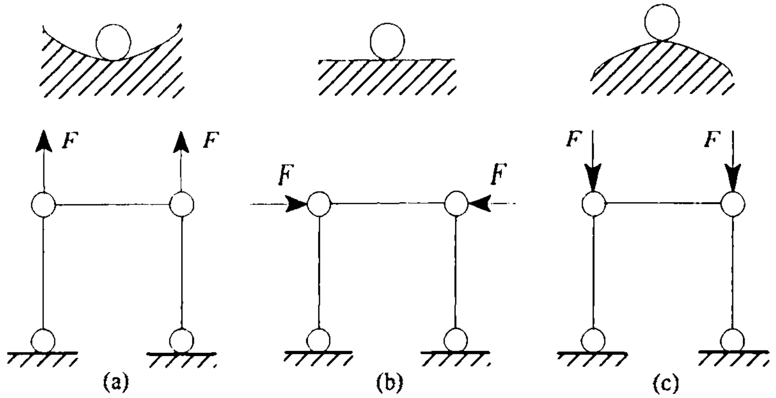

Figure 1.7 Basic types of equilibrium.

induce movements away from the assumed state. Figure 1.7 illustrates three different kinds of equilibrium. The upper diagrams show a heavy ball free to roll over a surface and the lower ones a mechanism consisting of three rigid bars linked by pin joints. The pair of diagrams shown in Figure 1.7a illustrate *stable equilibrium.* If the ball is moved slightly, it will tend to move back to its original position. Likewise, the pair of forces F will tend to pull the mechanism back to its initial state if it is perturbed slightly. In Figure 1.7b, if either the ball or the mechanism is displaced from its initial state, neither gravity acting on the ball nor the forces F have any influence on the perturbation and both systems tend to remain in the displaced position. This is known as *neutral equilibrium.* Any tendency of either the ball or the mechanism in Figure 1.7c to move from their equilibrium states is amplified by the forces acting on them. This is therefore known as *unstable equilibrium.* In the cases shown, the nature of the equilibrium state is determined from the potential of the loading in an immediately adjacent state. If this potential then permits the loading to do work in either restoring the system to its original state or enhancing the perturbation, then the state is either stable or unstable respectively. More generally, the strain energy of the system has to be considered as well. Structural systems are usually stable, because during any further growth of a small perturbation, more energy would be absorbed in straining the structure than would be released from the loss of potential energy of the applied loads in doing work. However, under certain loading conditions the structure grows unstable, as its ability to absorb strain energy becomes less than that of the loading to do work. The transition is usually marked by a state of neutral equilibrium.

{1.4 Linear Response}

{Most structural engineering analyses involve the solution of linear simultaneous equations of the form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= y_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots &\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= y_m \end{aligned} \tag{1.1}$$

or

$$\sum_{j=1}^{n} a_{ij} x_j = y_i \qquad (i = 1 \text{ to } m) \tag{1.2}$$

These equations can be written in matrix form as

$$Ax = y \tag{1.3}$$

where A is the matrix of coefficients a_{ij}, and x and y are the column vectors of the parameters x_j and y_i. Such equations arise in writing the equations of equilibrium, the relationships between (small) deflections, and the linear-elastic relationships between stresses and strains, for example. If a third column vector z of parameters z_k ($k = 1$ to p) is related to y by

$$By = z \tag{1.4}$$

where B is the matrix of coefficients b_{ki}, then

$$Cx = BAx = By = z \tag{1.5}$$

where

$$c_{kj} = \sum_{i=1}^{m} b_{ki} a_{ij} \tag{1.6}$$

so that the relationship between x and z is also linear. In any linear relationship, the following results apply.

Let x_1, x_2, x_3 be particular values of the column vector x and y_1, y_2, y_3 particular values of the column vector y such that

$$\begin{gathered} Ax_1 = y_1 \quad , \quad Ax_2 = y_2 \quad , \\ x_3 = x_1 + x_2 \quad , \quad y_3 = y_1 + y_2 \quad . \end{gathered} \tag{1.7}$$

Then

$$Ax_3 = A(x_1 + x_2) = Ax_1 + Ax_2 = y_1 + y_2 = y_3 \tag{1.8}$$

Suppose that x is the column vector of deflexions of a structure in response to a loading given by y. Then (1.8) shows that a response to a combination of loadings y_1 and y_2 is the sum of the individual responses to these loadings applied separately. This is known as the *principle of superposition* attributed to Bresse, as noted in the preface. It also follows that if y_2 is a scalar multiple, K, of y_1 then Kx_1 is a solution, for

$$A(Kx_1) = KAx_1 = Ky_1 = y_2 \tag{1.9}$$

so that this response increases in proportion to the loading.

There is not necessarily a unique solution to linear simultaneous equations such as (1.3). If there are more equations than unknowns ($m>n$), then there may be no solution. If there are fewer equations than unknowns ($m<n$), then the parameters x_j cannot be completely determined. If the structure is insufficiently constrained so that it forms a mechanism (such as that shown in Figure 1.7), the above equations will be insufficient to find the deflexions. Likewise, if too many constraints are applied to the structure, it may not be possible to find its internal loading from the equations of equilibrium alone. It is then called a *statically-indeterminate* or *redundant* structure. (This will be discussed further in Chapter 2.) Even if the number of equations is equal to the number of unknowns ($m=n$), there may still not be a unique solution. This would be because some

of the equations contain no information which could not be deduced from the other equations. In this case, some of the rows of A would be linear combinations of other rows of A, so that the determinant of A would be zero. This happens when the structure just reaches its buckling load and passes from a state of stable equilibrium to a state of neutral equilibrium where its deflexions are no longer determinate. As will be seen in §11.5, this zero determinant is a useful indicator of the loss of stability.

In engineering analyses, most problems have *unique* solutions. If a structure is constrained so that it neither forms a mechanism nor is free to move bodily in space (*rigid-body motion*), then its deflexions can be determined from the applied loading. If, in addition, there are no redundant constraints, so that the internal loading can be determined from the applied loading by means of the equations of equilibrium only, then the structure is said to be *just-stiff*. It will be assumed that the initial (unloaded) state is one of zero stress and strain, and so one of zero strain energy. As will be seen in Chapter 3, it can then be shown that the internal stresses and strains in a structure are uniquely determined for given loads and displacements applied to it. Suppose that in (1.3), x_1 and x_2 are two different responses of a structure to a particular loading y_1. Then

$$Ax_1 = y_1 \quad , \quad Ax_2 = y_1 \tag{1.10}$$

and so

$$A(x_1 - x_2) = y_1 - y_1 = 0 \tag{1.11}$$

Thus, if a unique solution exists, so that x_1 and x_2 must be equal, then the initial (unloaded) state must be one of zero response. Uniqueness implies that if *a* solution has been found which satisfies all the necessary conditions imposed on it, then it is *the* solution. The importance of uniqueness will be seen, for example, in proving Betti's reciprocal theorem in Chapter 3.}

{1.5 Symmetry and Antisymmetry}

{A structure is said to be symmetrical when, under some transformation of the coordinate system from which it is viewed, it appears unchanged. That is to say it is *invariant* with respect to the transformation. The most common type of symmetry is *mirror-symmetry*. The two-dimensional structure shown in Figure 1.8a is symmetrical about its centre line. That is, if a mirror M-M is placed on this centre line as in Figure 1.8b, the right-hand half of the structure would appear as the mirror image of the left-hand half. The system of reference shown by the (x,y) axes and the anticlockwise sense of rotation also has its mirror image, (x',y') and the clockwise rotation.

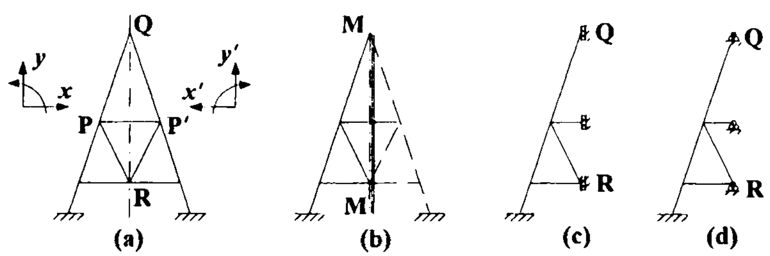

Figure 1.8 Behaviour of a symmetric framework.

The structure appears identical when viewed from either system of reference, except for changes in the joint labels. The typical joint labelled P seen with respect to the (x,y) system is the same as