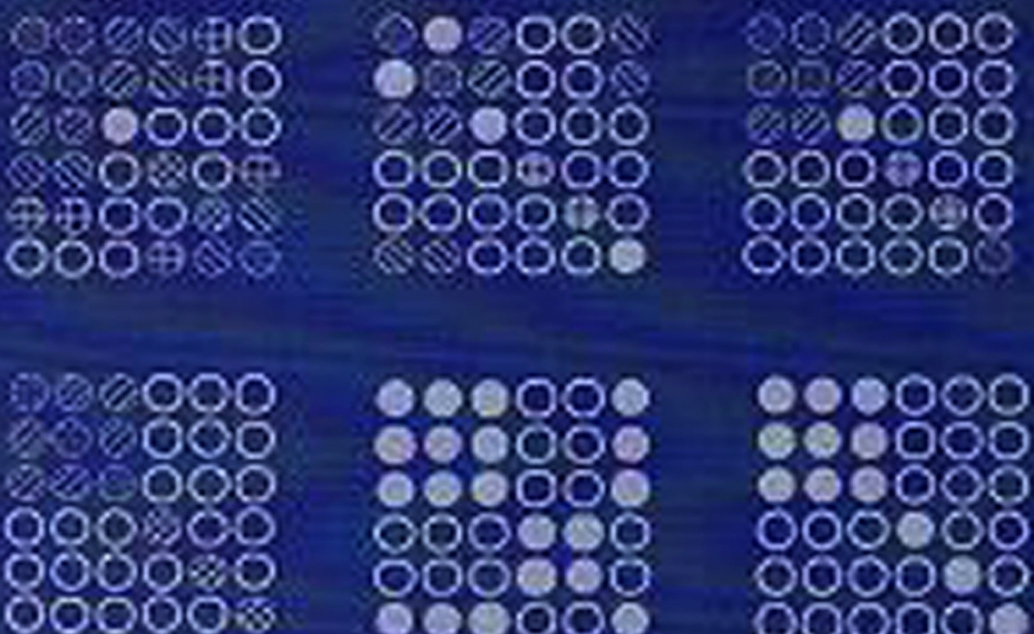
APPLIED ELASTICITY

Matrix and Tensor Analysis of Elastic Continua

2nd Edition

John D. Renton



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Talking of education, "People have now a-days" (said he) "got a strange opinion that every thing should be taught by lectures. Now, I cannot see that lectures can do so much good as reading the books from which the lectures are taken. I know nothing that can be best taught by lectures, expect where experiments are to be shewn. You may teach chymestry by lectures. — You might teach making of shoes by lectures!"

James Boswell: *Life of Samuel Johnson, 1766* [1709-1784]

ABOUT THE AUTHOR

In 1947 John D Renton was admitted to a Reserved Place (entitling him to free tuition) at King Edward's School in Edgbaston, Birmingham which was then a Grammar School. After six years there, followed by two doing National Service in the RAF, he became an undergraduate in Civil Engineering at Birmingham University, and obtained First Class Honours in 1958. He then became a research student of Dr A H Chilver (now Lord Chilver) working on the stability of space frames at Fitzwilliam House, Cambridge. Part of the research involved writing the first computer program for analysing three-dimensional structures, which was used by the consultants Ove Arup in their design project for the roof of the Sydney Opera House. He won a Research Fellowship at St John's College Cambridge in 1961, from where he moved to Oxford University to take up a teaching post at the Department of Engineering Science in 1963. This was followed by a Tutorial Fellowship to St Catherine's College in 1966.

Two main strands of research have been the behaviour of regular structures (such as trusses and plates) and the stability of continua. The former led to a general beam theory, equally applicable to continuous beams and trusses (see *Elastic Beams and Frames*, 2nd Ed. 2002 Horwood Publishing). The stability of continua, being the only way to establish the correct governing equations in terms of tensor calculus, gave rise to the present book (see Chapter 4). Both books contain much of the work in the author's published papers.

The author's other interests include judo (he was in both the Birmingham and Cambridge University teams) and photography (he does his own chemical colour printing). He thought it possible that special photographic techniques might rescue the Pre-Raphaelite murals illustrating King Arthur and the Knights of the Round Table which were in the Oxford Union from total obscurity, and with the aid of the head photographer at the Physics laboratory, Cyril Band, this was done. A booklet *The Oxford Union Murals* the author wrote on them in 1976 is now in its fourth edition, the murals having been fully restored and illuminated for the many visitors who now come to see them.

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John D. Renton
Department of Engineering Science
Oxford University

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Chichester

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Preface

This book was resulted from a need to solve certain three-dimensional problems in an organised manner. for those familiar with matrix algebra, much can be done without the explicit use of tensors. Chapter 1 was written with such readers in mind, so that only the briefest summary of matrix methods is given. Those unfamiliar with these techniques might well start at Chapter 3.

Elastic analysis using Cartesian tensor notation follows quite naturally from matrix notation and so forms the subject of Chapter 2. The concepts of thermoelasticity are examined more readily at this level. For most purposes, the slight differences between adiabatic and isothermal behaviour can be ignored, but temperature changes under adiabatic conditions can now be used to give thermal maps of stress fields. Elsewhere in the book, the material is understood to be hyperelastic. That is, the state of stress is given by the rate of potential with strain, as in Green's formula (2.52).

Matrix algebra is inadequate for analysing problems related to curvilinear coordinates. They are best solved using the curvilinear tensors discussed in Chapter 3. Care has been taken to give a physical and geometrical grounding to the quantities used. For example, all too often the Christoffel symbol is defined by a formula.

Large deformation theory is left until the last chapter. Here, the reference state is expressed in terms of the undeformed geometry. This is because the undeformed configuration of the body is known and the deformed state is sought. A consequence of this approach is that lower case letters refer to the undeformed geometry and upper case letters to the deformed geometry, which is the opposite of the notation often used elsewhere. Also, attention has tended to focus on the exact analysis of mathematically defined materials. However, these only approximate to the behaviour of real materials. Examples of such exact analyses are given in section 4.5. An alternative approach is to start from a description based on the known properties of real materials, accepting that this is likely to be incomplete. For many purposes such a description may well suffice and in section 4.6 it is used to derive a small deflection theory of the stability of elastic continua.

Other topics which do not fit readily into the main flow of the book will be found in

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Matrix methods

This chapter starts with a brief résumé of the properties of matrices, a topic which will already be familiar to most readers. Those requiring a fuller exposition may consult Barnett (1979), Bell (1975) or Graham (1980) for example. Some readers will also have encountered the matrix representation of a cross product, and possibly that of differential vector operators. An introduction to subscript notation is also outlined; this notation becomes essential in later chapters.

By using matrix methods, a number of equations can be encapsulated in a single matrix equation and readily transformed from one Cartesian coordinate system to another. These useful properties are also intrinsic in the tensor notation used in later chapters. For some purposes, anisotropic behaviour can be examined more easily in terms of matrix notation than in tensor notation†. This is because the symmetry of the components of stress and strain can be used implicitly in writing the equations, as in section 1.8.

1.1 SUMMARY OF MATRIX PROPERTIES

An $m \times n$ matrix **A** is an array of m rows and n columns of elements a_{ij}, where the subscripts i and j indicate that the element is in the ith row and jth column of the matrix. For example, if **A** is a 2×3 matrix, it is given by

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{23} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \qquad \textbf{(1.1)}$$

A matrix with only one row is called a **row vector**, and a matrix with only one column is

†Even some mathematicians who should know better seem unaware of the distinction. Tensors are like vectors and their elements have transformation properties which do not necessarily apply to matrices.

called a **column vector.** Two matrices **A** and **B** can be added and subtracted if m and n is the same both of them. Then if the elements of **B** are b_{ij}, the **sum** of the matrices **A** and **B** is the array of elements $a_{ij}+b_{ij}$ and the **difference** is the array of elements $a_{ij}-b_{ij}$. For 2×3 matrices, these take the forms

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \end{bmatrix} = \mathbf{B} + \mathbf{A} \tag{1.2}$$

$$\mathbf{A} - \mathbf{B} = \begin{bmatrix} a_{11}-b_{11} & a_{12}-b_{12} & a_{13}-b_{13} \\ a_{21}-b_{21} & a_{22}-b_{22} & a_{23}-b_{23} \end{bmatrix} = -(\mathbf{B} - \mathbf{A}) \tag{1.3}$$

A matrix can be multiplied by a **scalar**, λ say, (a simple quantity possessing magnitude only). This has the effect of multiplying each element by λ. Thus

$$\lambda\mathbf{A} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} & \lambda a_{13} \\ \lambda a_{21} & \lambda a_{22} & \lambda a_{23} \end{bmatrix} \tag{1.4}$$

A **matrix product CD** of two matrices **C** and **D** can be formed if the number of columns of **C** is the same as the number of rows of **D**. If **C** is an $m \times n$ matrix and **D** is an $n \times p$ matrix, then the product is an $m \times p$ matrix. A typical element of the ith row and jth column of the product is given by the sum of the products $c_{ir}d_{rj}$ for all values of r from 1 to n. For example

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{bmatrix} = \begin{bmatrix} c_{11}d_{11}+c_{12}d_{21} & c_{11}d_{12}+c_{12}d_{22} & c_{11}d_{13}+c_{12}d_{23} \\ c_{21}d_{11}+c_{22}d_{21} & c_{21}d_{12}+c_{22}d_{22} & c_{21}d_{13}+c_{22}d_{23} \end{bmatrix} \tag{1.5}$$

The rows of the **transpose** of a matrix are formed from the columns of the original matrix, and conversely the rows of the original matrix become the columns of the transpose. The transpose of a matrix will be denoted by the superscript $^{\mathrm{T}}$. Thus the transpose of the matrix **A** in (1.1) is

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{bmatrix} \tag{1.6}$$

The transpose of a product of matrices is equal to the product of their transposes in the reverse order. For example

$$(\mathbf{ABC})^{\mathrm{T}} = \mathbf{C}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} \tag{1.7}$$