# The Arithmetic of Polynomial Dynamical Pairs

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Aux mathématiciens et mathématiciennes qui nous ont tant inspirés, Adrien Douady, Tan Lei, et Jean-Christophe Yoccoz.

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# Preface

This project started in 2017, after we realized that the continuity of the Green function for polynomial dynamical pairs opened the door for a complete proof of the dynamical André-Oort conjecture for curves in the moduli space of polynomials. This conjecture was originally stated by Baker and DeMarco, and their beautiful proof for families parametrized by the affine line is a landmark in arithmetic dynamics. It combines equidistribution theorems in arithmetic geometry with the analysis of the Böttcher coordinate of a polynomial, and relies on a result of Medvedev and Scanlon based on Ritt theory (which is algebraic in essence). It turns out that exploring this conjecture for arbitrary families also reveals a diversity of connections to classical concepts in holomorphic dynamics (such as dynamical symmetries, rigidity theorems, and combinatorial classifications). This motivated us to present our ideas on this problem in a book form, in order to explain in depth all these aspects. We hope that our presentation will make this fascinating conjecture accessible to a broader audience of researchers working in dynamical systems.

It took us almost four years to complete this book. Along the way, we received the support from many colleagues who helped us to shape this monograph to its present form. We warmly thank Romain Dujardin, Khoa Dang Nguyen, and Junyi Xie for sharing their (long list of) thoughtful comments after reading a first version of this work. We express our gratitude to Dragos Ghioca and Mattias Jonsson for their very constructive remarks, and to Gabriel Vigny for many exchanges on the rigidity of complex polynomials that greatly improved our understanding of this matter.

Part of the book was written while the first author was a resident of the PIMS in Vancouver. He would like to thank both the CNRS for allowing him to spend a sabbatical year in this wonderful place, and the PIMS and UBC for providing him perfect working conditions. The second author was partially supported by ANR project "Fatou" ANR-17-CE40-0002-01.

Last but not least, all figures in Chapters 4 and 7 were computed by Arnaud Chéritat. We thank him heartfully for his kindness, his availability, and for showing such an interest in our project. Figure 7.3 was obtained using High Performance Computing resources that were partially provided by the mesocenter hosted by the Ecole Polytechnique.

Charles Favre and Thomas Gauthier<br>September 5, 2021

# List of abbreviations

- $\bar{K}$: the algebraic closure of a field $K$
- $K^\circ$: the ring of integers of a non-Archimedean metrized field
- $\tilde{K}$: the residue field of a non-Archimedean metrized field
- $\mathbb{C}_K$: the completion of an algebraic closure of $K$
- $\mathbb{K}$: a number field
- $\mathsf{O}(z)$: the set of Galois conjugates over an algebraic closure of $\mathbb{K}$ of a point $z \in \mathbb{K}$
- $M_{\mathbb{K}}$: the set of places of a number field, i.e., of norms extending the usual $p$-adic and real norms on $\mathbb{Q}$
- $\mathbb{K}_v$: the completion of $\mathbb{K}$ with respect to $v \in M_{\mathbb{K}}$
- $\mathbb{C}_v$: the completion of an algebraic closure of $\mathbb{K}_v$

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- $X$: an algebraic variety
- $\mathcal{O}_X$: the structure sheaf of $X$
- $X^{\mathrm{an}}$: the analytification in the sense of Berkovich of $X$
- $X_v^{\mathrm{an}}$: the analytification of $X_{\mathbb{C}_v}$ when $X$ is defined over a number field $\mathbb{K}$ and $v \in M_{\mathbb{K}}$
- $C$: an algebraic curve
- $\bar{C}$: a projective compactification of a curve $C$ such that $\bar{C} \setminus C$ is smooth
- $\hat{C}$: the normalization of $\bar{C}$, and $\mathsf{n} \colon \hat{C} \to \bar{C}$ the normalization map
- $S$: a Riemann surface

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- $\log^+ = \max\{0, \log\}$
- $\mathbb{D} = \{z \in \mathbb{C},\ |z| < 1\}$: the complex open unit disk
- $S^1$: the unit circle in $\mathbb{C}$
- $\mathbb{U}$: the group of complex numbers of modulus 1
- $\mathbb{U}_m = \{z \in \mathbb{C}, z^m = 1\}$: the group of complex $m$th root of unity
- $\mathbb{U}_\infty$: the group of all roots of unity
- $\mathbb{D}_K(z,r) = \{|w-z| < r\}$: the open disk of center $z$ and radius $r$ (as a subset of either $K$ or the Berkovich affine line $\mathbb{A}_K^{1,\mathrm{an}}$)
- $\mathbb{D}_K^*(z,r) = \{0 < |w-z| < r\}$: the punctured open disk of center $z$ and radius $r$ (as a subset of either $K$ or the Berkovich affine line $\mathbb{A}_K^{1,\mathrm{an}}$)
- $\overline{\mathbb{D}_K(z,r)} = \{|w-z| \leq r\}$: the closed disk of center $z$ and radius $r$ (as a subset of either $K$ or the Berkovich affine line $\mathbb{A}_K^{1,\mathrm{an}}$)

- $\mathbb{D}_K^N(z,r)$: the open polydisk of center $z$ and polyradius $r=(r_1,\cdots,r_N)$ (as a subset of either $K^N$ or the Berkovich affine space $\mathbb{A}_K^{N,\mathrm{an}}$)
- $\overline{\mathbb{D}_K^N(z,r)}$: the closed polydisk of center $z$ and polyradius $r=(r_1,\cdots,r_N)$ (as a subset of either $K^N$ or the Berkovich affine space $\mathbb{A}_K^{N,\mathrm{an}}$)
- $\mathbb{D}_v(z,r), \overline{\mathbb{D}_v(z,r)}, \mathbb{D}_v^N(z,r), \overline{\mathbb{D}_v^N(z,r)}$: the respective open and closed disks in $\mathbb{K}_v$ if $v$ is a place on a number field $\mathbb{K}$
- $\mathbb{M}_K$: the ring of analytic functions on the punctured unit disk $\mathbb{D}_K^*(0,1)$ that are meromorphic at 0

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- $\mathcal{C}_c^0(X)$: the space of compactly supported continuous functions on $X$
- $\mathcal{D}(U)$: the space of smooth (resp. model) functions on $U$
- $\Delta u$: the Laplacian of $u$
- $u, g$: subharmonic functions
- $h$: harmonic functions
- $o(1), O(1)$: Landau notations

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- $g_P$: the Green function associated to a polynomial $P$
- $G(P)$: the critical local height of a polynomial (the maximum of $g_P$ at all critical points)
- $\varphi_P$: the Böttcher coordinate of a polynomial $P$ at infinity
- $J(P)$: the Julia set of a polynomial $P$
- $K(P)$: the filled-in Julia set of a polynomial $P$
- $\mu_P$: the equilibrium measure of a polynomial $P$
- $\mathrm{Crit}(P)$: the critical set of a polynomial $P$
- $\Sigma(P)$: the group of dynamical symmetries of a polynomial $P$
- $\mathrm{Aut}(P)$: the group of affine transformations commuting with a polynomial $P$
- $\mathrm{Aut}(J)$: the group of affine transformations fixing a compact subset $J$ of the complex plane
- $\mathrm{Preper}(P,K)$: the set of preperiodic points *lying in* $K$ of a polynomial $P \in K[z]$

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- $\mathrm{Poly}^d$: the space of polynomials of degree $d$
- $\mathrm{Poly}^d_{\mathrm{mc}}$: the space of monic and centered polynomials of degree $d$
- $\mathrm{MPoly}^d$: the space of polynomials of degree $d$ modulo conjugacy
- $\mathrm{MPoly}^d_{\mathrm{crit}}$: the moduli space of critically marked polynomials of degree $d$ modulo conjugacy
- $\mathrm{MPair}^d$: the moduli space of dynamical pairs of degree $d$ modulo conjugacy
- $\mathrm{Stab}(P)$: the stability locus of a holomorphic family of polynomials
- $T_d$: the Chebyshev polynomial of degree $d$
- $M_d$: the monomial map of degree $d$

- $(P, a)$: a dynamical pair (either holomorphic or algebraic)
- $\mathrm{Bif}(P, a)$: the bifurcation locus of a holomorphic polynomial pair
- $g_{P,a}(t) = g_{P_t}(a(t))$: the Green function associated to a holomorphic polynomial pair
- $\mu_{P,a}$: the bifurcation measure of a dynamical pair defined over a metrized field
- $\mathrm{Preper}(P, a)$: the set of parameters $t$ such that the marked point $a(t)$ is preperiodic for $P_t$

# The Arithmetic of Polynomial Dynamical Pairs

# Introduction

This book is intended as an exploration of the moduli space $\mathrm{Poly}_d$ of complex polynomials of degree $d \geq 2$ in one variable using tools primarily coming from arithmetic geometry.

The Mandelbrot set in $\mathrm{Poly}_2$ has undoubtedly been the focus of the most comprehensive set of studies, and its local geometry is still an active research field in connection with the Fatou conjecture; see [19] and the references therein. In their seminal work, Branner and Hubbard [30, 31] gave a topological description of the space of cubic polynomials with disconnected Julia sets using combinatorial tools. In any degree, $\mathrm{Poly}_d$ is a complex orbifold of dimension $d-1$, and is therefore naturally amenable to complex analysis and in particular to pluripotential theory. This observation has been particularly fruitful to describe the locus of instability, and to investigate the boundary of the connectedness locus. DeMarco [49] constructed a positive closed $(1,1)$ current whose support is precisely the set of unstable parameters. Dujardin and the first author [68] then noticed that the Monge-Ampère measure of this current defines a probability measure $\mu_{\mathrm{bif}}$ whose support is in a way the right generalization of the boundary of the Mandelbrot set in higher degree, capturing the part of the moduli space where the dynamics is the most unstable (see also [11] for the case of rational maps). The support of $\mu_{\mathrm{bif}}$ has a very intricate structure: it was proved by Shishikura [152] in degree 2 and later generalized in higher degree by the second author [87] that the Hausdorff dimension of the support of $\mu_{\mathrm{bif}}$ is maximal equal to $2(d-1)$.

A polynomial is said to be post-critically finite (or PCF) if all its critical points have a finite orbit. The Julia set of a PCF polynomial is connected, of zero measure, and the dynamics on it is hyperbolic off the post-critical set. PCF polynomials form a countable subset of larger classes of polynomials (such as Misiurewicz, or Collet-Eckmann) for which the thermodynamical formalism is well understood [141, 142]. They also play a pivotal role in the study of the connectedness locus of $\mathrm{Poly}_d$: their distribution was described in a series of papers [76, 90, 91] and proved to represent the bifurcation measure $\mu_{\mathrm{bif}}$.

Any PCF polynomial is the solution of a system of $d-1$ equations of the form $P^n(c) = P^m(c)$ where $c$ denotes a critical point and $n, m$ are two distinct integers. In the moduli space, these equations are algebraic with integral coefficients, so that any PCF polynomial is in fact defined over a number field. Ingram [109] pushed this remark further and built a natural height $h_{\mathrm{bif}} \colon \mathrm{Poly}_d(\bar{\mathbb{Q}}) \to \mathbb{R}_+$

for which the set of PCF polynomials coincides with $\{h_{\mathrm{bif}} = 0\}$.

Height theory yields interesting new perspectives on the geometry of $\mathrm{Poly}_d$, and more specifically on the distribution of PCF polynomials. We will be mostly interested here in the so-called dynamical André-Oort conjecture, which appeared in [6]; see also [156].

This remarkable conjecture was set out by Baker and DeMarco, who were motivated by deep analogies between PCF dynamics and CM points in Shimura varieties, and more specifically by works by Masser-Zannier [27, 123, 171] on torsion points in elliptic curves. An historical account on the introduction of these ideas in arithmetic dynamics is given in [5, §1.2] and [6, §1.2]; see also [93]. We note that this analogy goes far beyond the problems considered in this book, and applies to various conjectures described in [52, 155]. We refer to the book by Zannier [171] for a beautiful discussion of unlikely intersection problems in arithmetic geometry.

Baker and DeMarco proposed characterizing irreducible subvarieties of $\mathrm{Poly}_d$ (or more generally of the moduli space of rational maps) containing a Zariski dense subset of PCF polynomials, and conjectured that such varieties were defined by critical relations. This conjecture was proven for unicritical polynomials in [97] and [98], and in degree 3 in [77] and [103].

It is our aim to give a proof of that conjecture for *curves* in $\mathrm{Poly}_d$ for any $d \geq 2$, and based on this result to attempt a classification of these curves in terms of combinatorial data encoding critical relations.

Our proof roughly follows the line of arguments devised in the original paper of Baker and DeMarco, and relies on equidistribution theorems of points of small height by Thuillier [160] and Yuan [168]; on the expansion of the Böttcher coordinates; and on Ritt's theory characterizing equalities of composition of polynomials.

We needed, though, to overcome several important technical difficulties, such as proving the continuity of metrics naturally attached to families of polynomials. We also had to inject new ingredients, most notably some dynamical rigidity results concerning families of polynomials with a marked point whose bifurcation locus is real-analytic.

For the most part in the book, we shall work in the more general context of polynomial dynamical pairs $(P, a)$ parametrized by a complex affine curve $C$, postponing the proof of the dynamical André-Oort conjecture to the last chapter. We investigate quite generally the problem of unlikely intersection that was promoted in the context of torsion points on elliptic curves by Zannier and his co-authors [123, 171], and later studied by Baker and DeMarco [5, 6] in our context. This problem amounts to understanding when two polynomial dynamical pairs $(P, a)$ and $(Q, b)$ parametrized by the same curve $C$ have an infinite set of common parameters for which the marked points are preperiodic. We obtain quite definite answers for polynomial pairs, and we prove finiteness theorems that we feel are of some interest for further exploration.

We have tried to review all the necessary material for the proof of the dynamical André-Oort conjecture, but we have omitted some technical proofs that are already available in the literature in an optimal form. On the other hand, we have made some efforts to clarify some proofs which we felt are too sketchy in the literature. The group of dynamical symmetries of a polynomial plays a very important role in unlikely intersection problems, and we have thus included a detailed discussion of this notion.

Let us now describe in more detail the content of the book.

## Polynomial dynamical pairs

In this paragraph we present the main players of our moograph. The central notion is that of a polynomial dynamical pair parametrized by a curve. Such a pair $(P, a)$ is by definition an algebraic family of polynomials $P_t$ parametrized by an irreducible affine curve $C$ defined over a field $K$, accompanied by a regular function $a \in K[C]$ which defines an algebraically varying marked point. Most of the time, these objects will be defined over the field of complex numbers $K = \mathbb{C}$, but it will also be important to consider polynomial dynamical pairs over other fields such as number fields, $p$-adic fields, or finite fields.

Any polynomial dynamical pair leaves a "trace" on the parameter space $C$, which may take different forms. Suppose first that $K$ is an arbitrary field, and let $\bar{K}$ be an algebraic closure of $K$. The first basic object to consider is the set $\mathrm{Preper}(P, a)$ of (closed) points $t \in C(\bar{K})$ such that $a(t)$ is preperiodic under $P_t$. This set is either equal to $C$ or at most countable.

A slightly more complicated but equally important object one can attach to $(P, a)$ is the following divisor. Let $\bar{C}$ be the completion of $C$, that is, the unique projective algebraic curve containing $C$ as a Zariski dense open subset, and smooth at all points $\bar{C} \setminus C$. Points in $\bar{C} \setminus C$ are called branches at infinity of $C$. Any pair $(P, a)$ induces an effective divisor $\mathsf{D}_{P,a}$ on $\bar{C}$, which is obtained by setting

$$\mathrm{ord}_{\mathfrak{c}}\left(\mathsf{D}_{P,a}\right) := \lim_{n\to\infty} -\frac{1}{d^n} \min\{0, \mathrm{ord}_{\mathfrak{c}}(P^n(a))\}, \tag{1}$$

for any branch $\mathfrak{c}$ at infinity. The limit is known to exist and is always a rational number; see §4.2.2.

When $K = \mathbb{C}$, one can associate various topological objects to a polynomial dynamical pair. One can consider the locus of stability of the pair $(P, a)$ which consists of the open set in which the family of holomorphic maps $\{P^n(a)\}_{n\geq 0}$ is normal. Its complement is the bifurcation locus, which we denote by $\mathrm{Bif}(P, a)$. This set can be characterized using potential theory as follows. Recall the definition of the Green function of a polynomial $P$ of degree $d$:

$$g_P(z) := \lim_{n\to\infty} \frac{1}{d^n} \max\{\log|P^n(z)|, 0\},$$

so that $\{g_P = 0\}$ is the filled-in Julia set of $P$ consisting of those points having bounded orbits. On the parameter space $C$, we then define the function

$$g_{P,a}(t) = g_{P_t}(a(t)).$$

It is a non-negative continuous subharmonic function on $C$, and the support of the measure $\mu_{\mathrm{bif}} = \Delta g_{P,a}$ is precisely equal to $\mathrm{Bif}(P,a)$. Of crucial technical importance is the following result from [78], which relates the function $g_{P,a}$ to the divisor defined above.

**Theorem 1.** *In a neighborhood of any branch at infinity $\mathfrak{c} \in \bar{C}$, one has the expansion*

$$g_{P,a}(t) = \mathrm{ord}_{\mathfrak{c}}\left(\mathsf{D}_{P,a}\right) \log |t|^{-1} + \tilde{g}(t)$$

*where $t$ is a local parameter centered at $\mathfrak{c}$ and $\tilde{g}$ is continuous at 0.*

This result can be interpreted in the langage of complex geometry by saying that $g_{P,a}$ induces a continuous semi-positive metrization on the $\mathbb{Q}$-line bundle $\mathcal{O}_{\bar{C}}(\mathsf{D}_{P,a})$. This fact is the key to applying techniques from arithmetic geometry.

Let us now suppose that $K = \mathbb{K}$ is a number field. For any place $v$ of $\mathbb{K}$, denote by $\mathbb{K}_v$ the completion of $\mathbb{K}$, and by $\mathbb{C}_v$ the completion of its algebraic closure. It is then possible to mimic the previous constructions at any (finite or infinite) place $v$ of $\mathbb{K}$ to obtain functions $g_{P,a,v} \colon C_v^{\mathrm{an}} \to \mathbb{R}_+$ on the analytification (in the sense of Berkovich) $C_v^{\mathrm{an}}$ of the curve $C$ over $\mathbb{C}_v$. Summing all these functions yields a height function $h_{P,a} \colon C(\bar{\mathbb{K}}) \to \mathbb{R}_+$. Alternatively, we may start from the standard Weil height $h_{\mathrm{st}} \colon \mathbb{P}^1(\bar{\mathbb{K}}) \to \mathbb{R}_+$; see e.g. [105]. Then for any polynomial with algebraic coefficients, we define its canonical height [36] to be

$$h_P(z) := \lim_{n\to\infty} \frac{1}{d^n} h_{\mathrm{st}}(P^n(z)),$$

and finally we set $h_{P,a}(t) := h_{P_t}(a(t))$. Using the Northcott theorem, one obtains that $\{h_{P,a} = 0\}$ coincides with the set $\mathrm{Preper}(P,a)$ of parameters $t \in C(\bar{\mathbb{K}})$ for which $a(t)$ is a preperiodic point of $P_t$.

It is an amazing fact that all the objects attached to a polynomial dynamical pair $(P,a)$ we have seen so far are tightly interrelated, as the next theorem due to DeMarco [51] shows.

An isotrivial pair $(P,a)$ is a pair which is conjugated to a constant polynomial and a constant marked point, possibly after a base change. A marked point is STABLY PREPERIODIC when there exist two integers $n > m$ such that $P_t^n(a(t)) = P_t^m(a(t))$.

**Theorem 2.** *Let $(P,a)$ be a polynomial dynamical pair of degree $d \geq 2$ which is parametrized by an affine irreducible curve $C$ defined over a number field $\mathbb{K}$. If the pair is not isotrivial, then the following assertions are equivalent:*

**(1)** *the set* $\mathrm{Preper}(P,a)$ *is equal to* $C(\bar{\mathbb{K}})$*;*

**(2)** *the marked point is stably preperiodic;*

**(3)** *the divisor* $\mathsf{D}_{P,a}$ *of the pair* $(P, a)$ *vanishes;*

**(4)** *for any Archimedean place* $v$*, the bifurcation measure* $\mu_{P,a,v} := \Delta g_{P,a,v}$ *vanishes;*

**(5)** *the height* $h_{P,a}$ *is identically zero.*

A pair $(P, a)$ which satisfies either one of the previous conditions is said to be passive; otherwise it is called an ACTIVE PAIR. For an active pair, $\mathrm{Preper}(P, a)$ is countable, the bifurcation measure $\mu_{P,a}$ is non-trivial, and the height $h_{P,a}$ is non-zero.

### Holomorphic rigidity for polynomial dynamical pairs

Rigidity results are pervasive in (holomorphic) dynamics. One of the most famous rigidity results was obtained by Zdunik [172] and asserts that the measure of maximal entropy of a polynomial $P$ is absolutely continuous with respect to the Hausdorff measure of its Julia set iff $P$ is conjugated by an affine transformation to either a monomial map $M_d(z) = z^d$, or to a Chebyshev polynomial $\pm T_d$ where $T_d(z+z^{-1}) = z^d + z^{-d}$. In particular, these two families of examples are the only ones having a smooth Julia set, a theorem due to Fatou [74].

The following analog of Zdunik's result for polynomial dynamical pairs is our first main result.

**Theorem A.** *Let* $(P, a)$ *be a polynomial dynamical pair of degree* $d \geq 2$ *which is parametrized by a connected Riemann surface* $S$*. Assume that* $\mathrm{Bif}(P, a)$ *is non-empty and included in a smooth real curve. Then one of the following holds:*

- *either* $P_t$ *is conjugated to* $M_d$ *or* $\pm T_d$ *for all* $t \in S$*;*
- *or there exists a univalent map* $\imath\colon \mathbb{D} \to S$ *such that* $\imath^{-1}(\mathrm{Bif}(P, a))$ *is a non-empty closed and totally disconnected perfect subset of the real line and the pair* $(P \circ \imath, a \circ \imath)$ *is conjugated to a real family over* $\mathbb{D}$*.*

We say that a polynomial dynamical pair $(P, a)$ parametrized by the unit disk is a real family whenever the power series defining the coefficients of $P$ and the marked point have all real coefficients.

The previous theorem is a crucial ingredient for handling the unlikely intersection problem that we will describe later. Its proof builds on a transfer principle from the parameter space to the dynamical plane, which can be decomposed into two parts.

The first step is to find a parameter $t_0$ at which $a(t_0)$ is preperiodic to a repelling orbit of $P_{t_0}$ and such that $t \mapsto a(t)$ is transversal at $t_0$ to the preperiodic orbit degenerating to $a(t_0)$. This step builds on an argument of Dujardin [67]. The second step relies on Tan Lei's similarity theorem [159], which shows that the bifurcation locus $\mathrm{Bif}(P, a)$ near $t_0$ is conformally equivalent at small scales to the Julia set of $P_{t_0}$.