# THE CLASSICAL AND QUANTUM $6j$-SYMBOLS

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**Dedicated to:**
Our Wives

# Contents

# Foreword

This book discusses the representation theory of classical and quantum $U(sl(2))$ with an eye towards topological applications of the latter. We use the Temperley-Lieb algebra and the quantum spin-networks to organize the computations. We define the $6j$-symbols in the classical, quantum, and quantum-root-of-unity cases, and use these computations to define the Turaev-Viro invariants of closed 3-dimensional manifolds. Our approach is elementary and fairly self-contained. We develop the spin-networks from an algebraic point of view.

# The Classical and Quantum 6*j*-symbols

# 1 Introduction

These notes grew out of a series of seminars held at the University of South Alabama during 1993 that were enhanced by regular e-mail among the three of us. We became interested in quantum diagrammatic representation theory following visits from Ruth Lawrence and Lou Kauffman to Mobile.

We develop the Clebsch-Gordan theory and the recoupling theory for representations of classical and quantum $U(sl(2))$ via the spin networks of Penrose [27] and Kauffman [16]. In these theories, the finite dimensional irreducible representations are realized in spaces of homogeneous polynomials in two variables. In the quantum case the variables commute up to a factor of $q$; *i.e.* $yx = qxy$. The tensor product of two representations is decomposed as a direct sum of irreducibles, and the coefficients of the various weight vectors are computed explicitly. In the quantum case, when the parameter is a root of unity, we only decompose the representations modulo those that have trace 0.

We use the spin networks to develop the theory in the classical case for two reasons. First, they simplify and unify many of the tricky combinatorial facts. The simplification of the proofs is nowhere more apparent than in Theorem 2.7.14 where a plethora of identities is proven via diagram manipulations. Second, the spin networks are currently useful and quite popular in the quantum case (see for example [23], [18], [28]). One of our goals here is to explain the representation theory of quantum $sl(2)$ in the spin network framework. We know of no better explanation than to run through the classical case (which should be more familiar), and then to imitate the classical theory in the quantum case.

Here we give an overview. The set of (2 by 2) matrices of determinant 1 over the complex numbers forms a group called $SL(2)$. The finite dimensional irreducible representations of $SL(2)$ are well understood. In particular, it is known how to decompose the tensor product of two such representations into a direct sum of irreducibles. In this decomposition one can compute explicitly the image of weight vectors and such computations form the heart of the so-called Clebsch-Gordan theory. The finite dimensional representations of $SL(2)$ are the same as those of $U(sl(2))$ which is an algebra generated by symbols $E, F$ and $H$ subject to certain relations.

Furthermore, the tensor product of three representations can be decomposed in two natural ways. The comparison of these two decompositions is sometimes called *recoupling theory*, and the recoupling coefficients are known as the $6j$-symbols. These symbols satisfy two fundamental identities (orthogonality and the Elliott-Biedenharn identity) that can be interpreted in terms of the decomposition of the union of two tetrahedra. In the Elliott-Biedenharn identity the tetrahedra are glued along a single face and recomposed as the union of three tetrahedra glued along an edge. For orthogonality the tetrahedra are glued along two faces, and the recomposition is not simplicial.

The symmetry of the $6j$-symbols and their relationship to tetrahedra was for the most part a mystery, until Turaev and Viro [32] constructed 3-manifold invariants based on the analogous theory for quantum $sl(2)$. The identities satisfied by the $6j$-symbols are also satisfied by their quantum analogues. The Elliott-Biedenharn identity corresponds to an Alexander [1] move on triangulations of a 3-manifold while the orthogonality condition can be interpreted as a Matveev [25] move on the dual 2-skeleton of a triangulation.

The Turaev-Viro invariants were based on work of Kirillov and Reshetikhin on the representation of quantum groups [19]. This work together with Reshetikhin-Turaev [29] formed a mathematically rigorous framework for the invariants of Witten [34]. Meanwhile Kauffman and Lins [18] gave a simple combinatoric approach to the invariants based on the Kauffman bracket and the spin networks of Penrose [27]. Piunikhin [28] showed that the Kauffman-Lins approach and the Turaev-Viro approach coincide.

Some of Kauffman's contributions to the subject can also be found in the papers [14], [15], and [17]. A more traditional algebraic approach to quantum groups can be found in [30]; in particular, they discuss from the outset the Hopf-algebra structures.

Lickorish's [23] definition of the Reshetikhin-Turaev invariants is of a combinatorial nature. The Kauffman-Lins [18] definition of the Turaev-Viro invariants is defined similarly. Neither of these combinatorial approaches relied on representation theory. However, the remarkable feature of quantum topology is that there are close connections between algebra and topology that were heretofore unimagined. The purpose of this paper is to explore these relations by examining the algebraic meaning of the diagrams and by using diagrams to prove algebraic results.

Here is our outline. Section 2 reviews the classical theory of representations of $U(sl(2))$. There is nothing new here, but we do show how the Clebsch-Gordan coefficients and the $6j$-symbols are computed in terms of the bracket expansion (at $A = 1$). In Section 3 we mimic these constructions to obtain the quantum Clebsch-Gordan and $6j$-symbols. In Section 4 we will define the quantum trace and discuss the recoupling theory in the root of unity case. Section 5 reviews the definitions of the Turaev-Viro invariants and proves that the definition is independent of the triangulation by means of the Pachner Theorem [26].

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