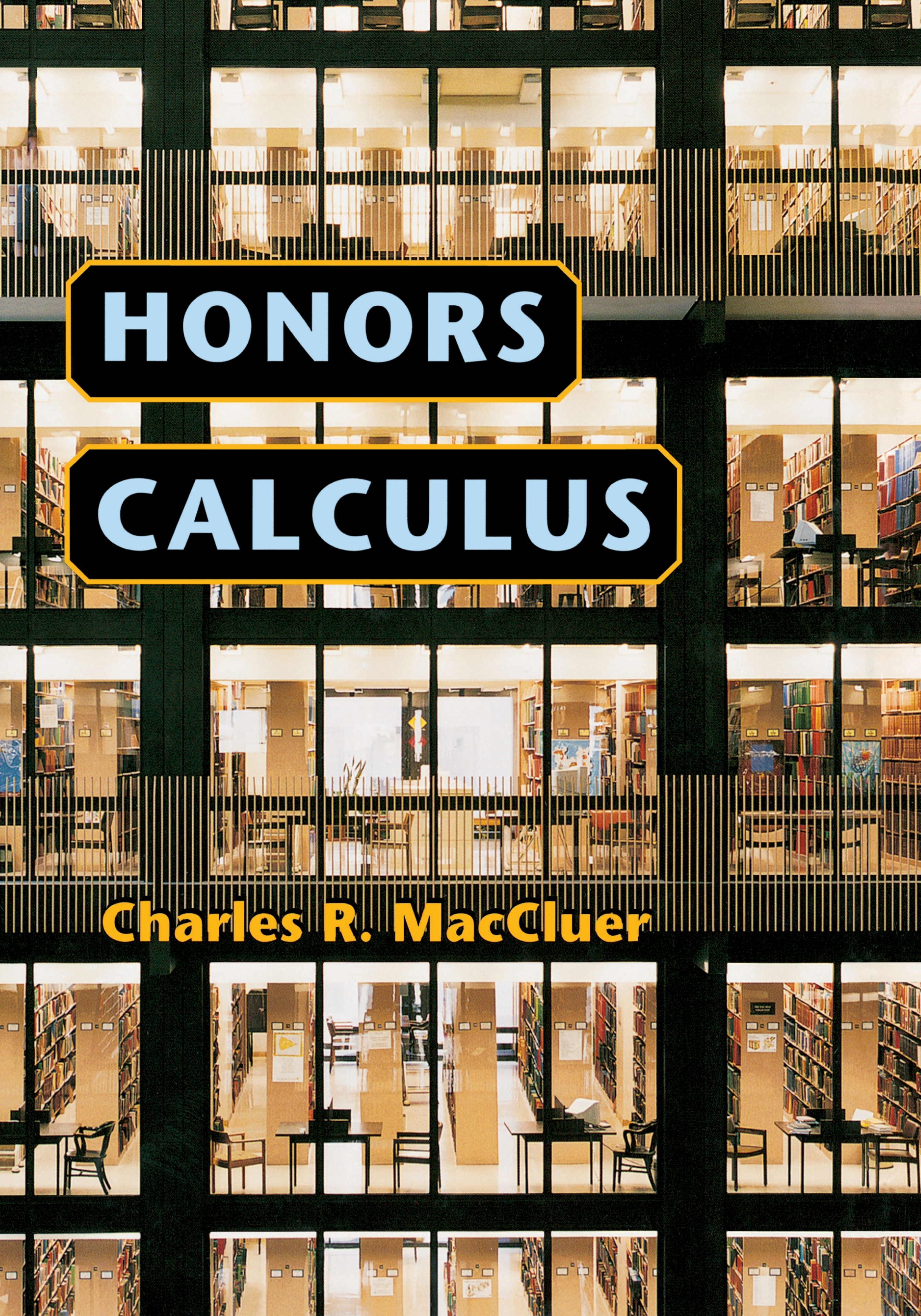

HONORS CALCULUS

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This book is dedicated to my son Joshua.

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Preface

This book is for the honors calculus course that many universities offer to well-prepared entering students. As a rule, all of these students will have had an earlier calculus course where they have been exposed to many of the standard ideas and have been drilled in the standard calculations. These students are therefore primed for an exceptional experience, yet most universities deliver the same material again in much the same style. Instead, this book can be used to catapult these exceptional students to a much higher level of understanding.

To the Student

Mathematics is more than calculations. It is also about constructing abstract structures that can make laborious calculations unnecessary. And it is about *understanding* with a precision and clarity unequaled by other disciplines. In this book you will be exposed to how mathematics is practiced by mathematicians. It is my hope that you may leapfrog the usual twice relearning of calculus as a junior and as a graduate student by learning it correctly the first time (although I suspect your presence in honors calculus signals that you have studied some calculus before).

I will assume your algebra skills are excellent. I will also assume that you are sensitive to the precision of language used in science; that is your key to success in our journey—ponder until you understand *exactly* what is being said.

Look carefully at definitions and results. What function does each word play within the statement? Can you think of an example that satisfies all but one of the criteria but where the definition or result fails? Can you think of a simple example where it holds? Much of the understanding and recalling of mathematics lies in amassing a large personal zoo of examples that you can trot out to test definitions and results.

Like any discipline, much drill is necessary for mastery, but here, both computational and conceptual drill is required. Here are suggestions for study: Take scratch notes during lectures, then reconstruct and expand these notes later into your own personal text. Next, read the corresponding section of the text, and as you read, follow along with a pen and pad checking all details. Finally, attack the exercises. Spend a portion of every study time reviewing. There seems to be no shortcut around this time-consuming study technique.

Like you would in French class, repeatedly practice the italicized epigrams of the results out loud, like "continuous functions vanish on closed sets" or "the continuous image of a compact set is compact." Repeat these mantras often.

To the Instructor

Given the correct clientele it is possible to teach calculus right the first time. Tiresome repetition and incremental advance is not suitable for the honors student. It is most important not to bore such students. They must be challenged.

All of your students will have been exposed to the routine calculus material. If you begin trudging through the same material, albeit at a higher level, they will tune out and spend their intellectual energies elsewhere. The approach and material of this book will be new to your students and very demanding because of its abstraction. But they will master it.

Details are often thrown into the exercises. You may present this detail or assign it. The overarching objective is to reveal the flow of the ideas, and so not all proofs need be presented, but it is crucial that the students believe that all proofs are accessible given the requisite time. Although the analysis is built up axiomatically, examples are drawn from the assumed past experience of the students, and only later, as the analysis builds, is this background material verified.

The usual repetitious drill in routine calculations is missing from this book. You may need to augment it with problems from a standard calculus book. Honors students pick up techniques from working only one or two standard problems; the usual dreary repetition is not necessary. The goal has been to construct a framework upon which you may tailor your course to your tastes and to the talent present.

By design, the writing style is brief, breezy, and *au courant*. The intent is to charm and engage the young clientele.

Solutions are available to instructors from the publisher upon request. Additions and corrections to the text will be updated at http://www.math.msu.edu/~maccluer/HonorsCalculus.

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1

Functions on Sets

We take as a working definition that *mathematics is the study of functions on sets*. In this chapter we take up the primitive notions of sets, functions from one set to another, and injective, surjective, and bijective functions. Sets are classified as finite and infinite, countable and uncountable. All areas of mathematics use these fundamental concepts. This is the core of mathematics.

1.1 Sets

Suppose X is a collection (*set*) of objects (*points*) generically denoted by x. A *subset* S of X is a collection consisting of some of the objects of X, in symbols, $S \subset X$. For any two subsets S and T of X, their *intersection* $S \cap T$ is the set of all objects x in both S and T. Of course it is quite possible that S and T share no common points, in which case we say their intersection is the *empty set* $\emptyset$, and we write $S \cap T = \emptyset$.

The *union* $S \cup T$ is the set of all objects x in either S or T.[1] Two subsets S and T are *equal*, in symbols, $S = T$, if[2] they consist of exactly the same objects.

Subsets A, B, C, ... of X under "cap" and "cup" enjoy an arithmetic of sorts:

Theorem A. (Boolean algebra)

$$\begin{array}{ll}
A \cap B = B \cap A & A \cup B = B \cup A \\
A \cap (B \cap C) = (A \cap B) \cap C & A \cup (B \cup C) = (A \cup B) \cup C \\
A \cap A = A & A \cup A = A \\
A \cap X = A & A \cup \emptyset = A \\
A \cap \emptyset = \emptyset & A \cup X = X
\end{array}$$

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C). \end{aligned} \tag{1.1}$$

[1]The word "or" is used in mathematics in the inclusive sense—either one or the other or both.

[2]We continue the age-old practice when defining terms of using "if" when we really should use "exactly when" or "if and only if." See [Euclid].