PLASMA PHYSICS

KIP S. THORNE
and
ROGER D. BLANDFORD

VOLUME 4 OF *MODERN CLASSICAL PHYSICS*

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A NOTE TO READERS
This book is the fourth in a series of volumes that together comprise a unified work titled *Modern Classical Physics*. Each volume is designed to be read independently of the others and can be used as a textbook in an advanced undergraduate- to graduate-level course on the subject of the title or for self-study. However, as the five volumes are highly complementary to one another, we hope that reading one volume may inspire the reader to investigate others in the series—or the full, unified work—and thereby explore the rich scope of modern classical physics.

To Carolee and Liz

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BOXES

PREFACE

When we (the authors of this book) were students, some understanding of plasma physics was expected of us, and there were physics courses in which we could learn it. That is no longer true, but we believe it should be, and this book is designed to facilitate such courses. *Why?*

Plasmas play major roles, for example, (1) in attempts to achieve controlled thermonuclear fusion using magnetic and inertial confinement; (2) in explanations of radio wave propagation in the ionosphere and the observed behavior of the solar corona and wind; and (3) in astrophysics, where they are responsible for emission throughout the electromagnetic spectrum (e.g., from black holes, highly magnetized neutron stars, and ultrarelativistic outflows).

Plasmas exhibit an amazingly rich set of phenomena and behaviors, which enrich their roles in nature and vastly complicate the technology required for their control and manipulation, most importantly in controlled fusion. Among the most interesting phenomena are (1) the surfing of individual electrons and ions on plasma waves that consist of collective, collisionless oscillations of electrons, ions, and sometimes magnetic fields; (2) that surfing's amplification of some wave modes (feeding of energy from the surfing particles to the collective excitations) and damping of other wave modes (feeding of energy from the modes to the particles); and (3) nonlinear wave-wave coupling in which, for example, two incoming waves (collective oscillations of collisionless particles and magnetic field) interact to produce a new outgoing wave.

This rich behavior arises from a plasma's atomic-scale structure: A plasma is a gas that is significantly ionized (usually by heating or photons) and thus is composed of electrons and ions, and sometimes has an embedded or confining magnetic field. In plasmas, the mean-free paths of electrons and ions are often comparable to or even far longer than macroscopic length scales, so the plasma typically does not behave like a fluid. Its dynamics can be strongly influenced by the particles' velocity distributions. Rich dynamical behavior occurs both in physical space and in velocity space, and the two are strongly coupled and are also coupled to any embedded magnetic field.

QUANTUM PHYSICS IN THIS BOOK

This book deals primarily with *classical* plasma physics. Nevertheless, we make frequent reference to quantum mechanical concepts and phenomena, and we often use quantum concepts and techniques in the classical domain, for example, in our analyses of nonlinear wave-wave coupling. This is because classical physics arises from quantum physics as an approximation, and sometimes—especially in plasmas—the imprints left on classical physics by its quantum roots are so strong that classical phenomena are most powerfully discussed and analyzed in quantum language.

GUIDANCE FOR READERS

The amount and variety of material covered in this book may seem overwhelming. If so, keep in mind that

- *the primary goals of this book* are to teach the fundamental concepts of plasma physics, which are not so extensive that they should overwhelm, to illustrate those concepts in action, and, through our illustrations, to give the reader some physical understanding of how plasmas behave.

We do not intend to provide a mastery of the many illustrative applications contained in the book. To further help students and other readers who feel overwhelmed, we have labeled as "Track Two" sections that can be skipped on a first reading, or skipped entirely—but are sufficiently interesting and important that many readers may choose to browse or study them. Track-Two sections are labeled by the symbol **T2**.

We have aimed this book at advanced undergraduates and first- and second-year graduate students, of whom we expect only (1) a typical physics or engineering student's facility with applied mathematics, and (2) a typical undergraduate-level understanding of classical mechanics, electromagnetism, elementary thermodynamics, and quantum mechanics. We also target working scientists and engineers who want to learn or improve their understanding of plasma physics.

This book is appropriate for a one-quarter or one-semester course in plasma physics. We presume it will also be used as supplementary reading in other courses where plasmas are important—for example, in astrophysics, geophysics, and controlled fusion.

This book is the fourth of five volumes that together constitute a single treatise, *Modern Classical Physics* (or "MCP," as we shall call it). The full treatise was published in 2017 as an embarrassingly thick single book. (The electronic edition is a good deal lighter.) For readers' convenience, we have placed, at the end of this volume, the Table of Contents, Preface, and Acknowledgments of MCP. The five separate textbooks of this decomposition are

- Volume 1: *Statistical Physics*,
- Volume 2: *Optics*,
- Volume 3: *Elasticity and Fluid Dynamics*,

- Volume 4: *Plasma Physics*, and
- Volume 5: *Relativity and Cosmology*.

These individual volumes are much more suitable for human transport and for use in individual courses than their one-volume parent treatise, MCP.

The present volume is enriched by extensive cross-references to the other four volumes—cross-references that elucidate the rich interconnections of various areas of physics.

In this and the other four volumes, we have retained the chapter numbers from MCP and, for the body of each volume, MCP's pagination. In fact, the body of this volume is identical to the corresponding MCP chapters, aside from corrections of errata (which are tabulated at the MCP website http://press.princeton.edu/titles/MCP.html) and a small amount of updating that has not changed pagination. For readers' cross-referencing convenience, a list of the chapters in each of the five volumes appears immediately after this Preface.

EXERCISES

Exercises are a major component of this volume, as well as of the other four volumes of MCP. The exercises are classified into five types:

1. *Practice*. Exercises that provide practice at mathematical manipulations (e.g., of tensors).
2. *Derivation*. Exercises that fill in details of arguments skipped over in the text.
3. *Example*. Exercises that lead the reader step by step through the details of some important extension or application of the material in the text.
4. *Problem*. Exercises with few, if any, hints, in which the task of figuring out how to set up the calculation and get started on it often is as difficult as doing the calculation itself.
5. *Challenge*. Especially difficult exercises whose solution may require reading other books or articles as a foundation for getting started.

We urge readers to try working many of the exercises—especially the examples, which should be regarded as continuations of the text and which contain many of the most illuminating applications. Exercises that we regard as especially important are designated by **.

UNITS

Throughout this volume, we use SI units, as is customary today in plasma physics.

BRIEF OUTLINE OF THIS BOOK

When a plasma's dynamical timescales are sufficiently long, it behaves like a fluid and so can be understood and analyzed by the techniques of fluid dynamics (Volume 3 of MCP). If this fluid-like plasma has an embedded magnetic field, the plasma's (usually high) electric conductivity will strongly couple that field to the fluid. The study of such a magnetized fluid is called *magnetohydrodynamics,* or *MHD* for short, and is the subject of MCP Chap. 19. In MCP, we formally classify MHD and Chap. 19 as part of fluid dynamics, because they also accurately describe other electrically conducting, magnetized fluids (e.g., liquid metals like mercury and liquid sodium). However, by far the most important and widespread application of MHD today is to plasmas, so we have included Chap. 19 in this volume instead of in our fluid dynamics book, Volume 3 of MCP.

In Chap. 19, we derive and elucidate the basic equations and principles of MHD, and we then illustrate them for a nondynamical plasma (*magnetostatics*) in two ways: (1) We analyze the steady flow of plasma along magnetic ducts and describe applications to electric power generation and to spacecraft propulsion. (2) We describe a *tokamak*, the currently most promising configuration for magnetic confinement of a hot plasma in controlled-fusion R&D. Turning to dynamical plasmas, we use MHD to analyze the stability of various magnetostatic equilibria, most importantly the tokamak and other, simpler configurations for magnetic confinement of plasmas. We also discuss the physics of some of the many unstable modes of oscillation that plague magnetic confinement configurations. To illustrate the application of MHD to geophysics and planetary science, we discuss the generation of Earth's magnetic field by dynamo flows in its liquid core; and we analyze magnetosonic waves propagating in a magnetized plasma, such as the interstellar medium.

Important aspects of a plasma's dynamics can be understood by what is happening microscopically, with individual electrons and ions, and groups of them. In Chap. 20, we study the particle kinetics of plasmas. Among the important particle-kinetic phenomena we analyze in Chap. 20 are: (1) Debye shielding—the manner in which the electric field of an individual electron or ion is shielded by collective redistribution of other electrons and ions, so the field dies out much faster than $1/r$. (2) The remarkably long mean free paths of electrons and ions when the only impediment to their constant-velocity motion is Coulomb scattering, with the consequence that the primary impediment to unimpeded motion is scattering off plasma waves (wave-particle interactions, treated in Chap. 23). (3) The motions of individual particles (electrons or ions), and collections of them, in a magnetic field that may be inhomogeneous and time varying, and implications of these motions for behaviors of the plasma.

When a plasma's dynamical timescales are sufficiently short, and the velocities of its electrons (and those of its ions) are not greatly spread out (so the plasma is "cold"), then the negatively charged electrons are coupled together and behave like one fluid, and the positively charged ions are coupled together and behave like a second fluid. These two fluids, interacting with each other and with a magnetic field, can be analyzed using a *two-fluid formalism* that we develop in Chap. 21. We illustrate this formalism by its most important application: to the wide variety of waves that can propagate in a cold plasma like our ionosphere. We also use the two-fluid formalism to illustrate how easily waves can feed off an ordered relative motion of ions and electrons and can grow in strength by extracting kinetic energy from the ordered motions: the *two-stream instability*.

For warm plasmas (with wide spreads of electron and ion velocities), the velocity distributions can undergo interesting and complex dynamics, which couples to the particles' spatial dynamics and to the magnetic field, if present. In other words, the rich dynamics occurs in the six-dimensional phase space of particle positions and velocities, and so is best analyzed using a *kinetic theory formalism* that we develop and illustrate in Chaps. 22 and 23.

In Chap. 22, we develop this kinetic-theory formalism and then linearize it in the electron and ion distribution functions and the magnetic field. For simplicity, we focus largely but not entirely on an unmagnetized plasma, applying it most importantly to *Langmuir waves*—longitudinal oscillations of the electron distribution with the restoring force partially due to thermal pressure and partially electrostatic; and *ion-acoustic* waves—the analog, for ions, of Langmuir waves. We discover, in our linearized kinetic-theory analysis, the damping of these wave modes by electrons or ions that surf on the waves (*Landau damping*) and, under some circumstances (such as the two-stream instability), the feeding of the surfing particles' kinetic energy into the waves so the waves grow instead of damp. We explore the instabilities that result from this surfing in several ways, including a study of particles trapped in the waves, and *Nyquist's method* of analyzing the waves' dispersion relation. And we illustrate the power of Nyquist's method by exhibiting its application to a system far from plasma physics: the stability of a feedback-control system (e.g., an automobile's cruise control). We conclude this chapter with a discussion of N-particle distribution functions, which, despite their apparent formality, can lead to usable answers for practical problems, such as calculating the Coulomb correction to the equation of state of a plasma.

In Chap. 23, on the nonlinear dynamics of plasmas, we restore the nonlinear terms to the kinetic-theory formalism, one after another, and discover their effects. The first nonlinear term reveals (not surprisingly) the back action of Langmuir and ion-acoustic waves on the surfing particles, to reduce or increase the particles' kinetic energy at the same rate as the wave gains or loses energy. The formalism at this order is called *quasilinear theory*. The next nonlinear term gives rise to the nonlinear interaction of two waves to resonantly produce a third wave. This is called *three-wave mixing*, and it is a ubiquitous phenomenon that occurs in many other areas of science and technology, perhaps most importantly today, for light waves that interact in a nonlinear crystal (see Chap. 10 of MCP Volume 2)—a major foundation for nonlinear-optics technology. Although we derive our weakly nonlinear kinetic-theory formalism from the equations of classical plasma physics, we discover in Chap. 23 that the formalism can be written much more elegantly and powerfully in the language of the quantum theory of interacting bosonic plasmons (the quanta associated with the plasma waves), electrons, and ions. In the quantum formalism, we easily identify a term, missed classically, that describes a high-speed charged particle's spontaneous Cerenkov-type emission of Langmuir plasmons (and ion-acoustic plasmons). By comparing the quantum theory's stimulated emission term with that computed from the classical theory, we infer the quantitative strength of this spontaneous emission and discover that it can be as important as the purely classical processes we have been studying. This analysis exhibits the deep and powerful relationships between quantum physics and classical physics that occur widely elsewhere in science and technology. We conclude this chapter with a discussion of collisionless shock waves and their deep relationship to solitons.

In three short appendixes, we present sections from other volumes of MCP that underpin portions of this volume:

- Appendix A, Evolution of Vorticity—A good starting point for our analysis of the evolution of the magnetic field in Chap. 19 on magnetohydrodynamics.
- Appendix B, Geometric Optics—An essential foundation for some of our analysis of waves in plasmas in Chaps. 21, 22, and 23.
- Appendix C, Distribution Function and Mean Occupation Number—The link between more traditional kinetic theory and its application to plasmas in Chaps. 22 and 23.

CONTENTS OF *MODERN CLASSICAL PHYSICS*, VOLUMES 1–5

CHAPTER NINETEEN

19

Magnetohydrodynamics

. . . it is only the plasma itself which does not 'understand' how beautiful the theories are and absolutely refuses to obey them.

HANNES ALFVÉN (1970)

19.1 Overview

19.1

In preceding chapters we have described the consequences of incorporating viscosity and thermal conductivity into the description of a fluid. We now turn to our final embellishment of fluid mechanics, in which the fluid is electrically conducting and moves in a magnetic field. The study of flows of this type is known as *magnetohydrodynamics,* or MHD for short. In our discussion, we eschew full generality and with one exception just use the basic Euler equation (no viscosity, no heat diffusion, etc.) augmented by magnetic terms. This approach suffices to highlight peculiarly magnetic effects and is adequate for many applications.

The simplest example of an electrically conducting fluid is a liquid metal, for example, mercury or liquid sodium. However, the major application of MHD is in plasma physics—discussed in Part VI. (A plasma is a hot, ionized gas containing free electrons and ions.) It is by no means obvious that plasmas can be regarded as fluids, since the mean free paths for Coulomb-force collisions between a plasma's electrons and ions are macroscopically long. However, as we shall learn in Sec. 20.5, collective interactions between large numbers of plasma particles can isotropize the particles' velocity distributions in some local mean reference frame, thereby making it sensible to describe the plasma macroscopically by a mean density, velocity, and pressure. These mean quantities can then be shown to obey the same conservation laws of mass, momentum, and energy as we derived for fluids in Chap. 13. As a result, a fluid description of a plasma is often reasonably accurate. We defer to Part VI further discussion of this point, asking the reader to take it on trust for the moment. In MHD, we also implicitly assume that the average velocity of the ions is nearly the same as the average velocity of the electrons. This is usually a good approximation; if it were not so, then the plasma would carry an unreasonably large current density.

Two serious technological applications of MHD may become very important in the future. In the first, strong magnetic fields are used to confine rings or columns of hot plasma that (it is hoped) will be held in place long enough for thermonuclear fusion to occur and for net power to be generated. In the second, which is directed toward a

BOX 19.1. READERS' GUIDE

- This chapter relies heavily on Chap. 13 and somewhat on the treatment of vorticity transport in Sec. 14.2.
- Part VI, Plasma Physics (Chaps. 20–23), relies heavily on this chapter.

similar goal, liquid metals or plasmas are driven through a magnetic field to generate electricity. The study of magnetohydrodynamics is also motivated by its widespread application to the description of space (in the solar system) and astrophysical plasmas (beyond the solar system). We illustrate the principles of MHD using examples drawn from all these areas.

After deriving the basic equations of MHD (Sec. 19.2), we elucidate magnetostatic (also called "hydromagnetic") equilibria by describing a *tokamak* (Sec. 19.3). This is currently the most popular scheme for the magnetic confinement of hot plasma. In our second application (Sec. 19.4) we describe the flow of conducting liquid metals or plasma along magnetized ducts and outline its potential as a practical means of electrical power generation and spacecraft propulsion. We then return to the question of magnetostatic confinement of hot plasma and focus on the stability of equilibria (Sec. 19.5). This issue of stability has occupied a central place in our development of fluid mechanics, and it will not come as a surprise to learn that it has dominated research on thermonuclear fusion in plasmas. When a magnetic field plays a role in the equilibrium (e.g., for magnetic confinement of a plasma), the field also makes possible new modes of oscillation, and some of these MHD modes can be unstable to exponential growth. Many magnetic-confinement geometries exhibit such instabilities. We demonstrate this qualitatively by considering the physical action of the magnetic field, and also formally by using variational methods.

In Sec. 19.6, we turn to a geophysical problem, the origin of Earth's magnetic field. It is generally believed that complex fluid motions in Earth's liquid core are responsible for regenerating the field through dynamo action. We use a simple model to illustrate this process.

When magnetic forces are added to fluid mechanics, a new class of waves, called magnetosonic waves, can propagate. We conclude our discussion of MHD in Sec. 19.7 by deriving the properties of these wave modes in a homogeneous plasma and discussing how they control the propagation of cosmic rays in the interplanetary and interstellar media.

As in previous chapters, we encourage our readers to view films; on magnetohydrodynamics, for example, Shercliff (1965).

19.2

19.2 Basic Equations of MHD

The equations of MHD describe the motion of a conducting fluid in a magnetic field. This fluid is usually either a liquid metal or a plasma. In both cases, the conductivity, strictly speaking, should be regarded as a tensor (Sec. 20.6.3) if the electrons' cyclotron frequency (Sec. 20.6.1) exceeds their collision frequency (the inverse of the mean time between collisions; Sec. 20.4.1). (If there are several collisions per cyclotron orbit, then the influence of the magnetic field on the transport coefficients will be minimal.) However, to keep the mathematics simple, we treat the conductivity as a constant scalar, κ_e. In fact, it turns out that for many of our applications, it is adequate to take the conductivity as infinite, and it does not matter whether that infinity is a scalar or a tensor!

Two key physical effects occur in MHD, and understanding them well is key to developing physical intuition. The first effect arises when a good conductor moves into a magnetic field (Fig. 19.1a). Electric current is induced in the conductor, which, by Lenz's law, creates its own magnetic field. This induced magnetic field tends to cancel the original, externally supported field, thereby in effect excluding the magnetic field lines from the conductor. Conversely, when the magnetic field penetrates the conductor and the conductor is moved out of the field, the induced field reinforces the applied field. The net result is that the lines of force appear to be dragged along with the conductor—they "go with the flow." Naturally, if the conductor is a fluid with complex motions, the ensuing magnetic field distribution can become quite complex, and the current builds up until its growth is balanced by Ohmic dissipation.

The second key effect is dynamical. When currents are induced by a motion of a conducting fluid through a magnetic field, a Lorentz (or $\mathbf{j} \times \mathbf{B}$) force acts on the fluid and modifies its motion (Fig. 19.1b). In MHD, the motion modifies the field, and the field, in turn, reacts back and modifies the motion. This behavior makes the theory highly nonlinear.

Before deriving the governing equations of MHD, we should consider the choice of primary variables. In electromagnetic theory, we specify the spatial and temporal variation of either the electromagnetic field or its source, the electric charge density and current density. One choice is computable (at least in principle) from the other

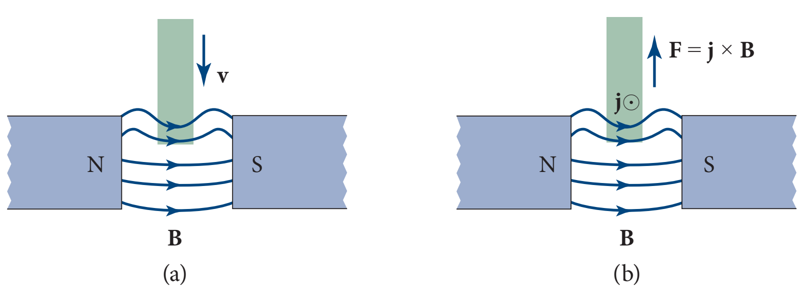

FIGURE 19.1 The two key physical effects that occur in MHD. (a) A moving conductor modifies the magnetic field by dragging the field lines with it. When the conductivity is infinite, the field lines are frozen in the moving conductor. (b) When electric current, flowing in the conductor, crosses magnetic field lines, a Lorentz force is generated that accelerates the fluid.

two key physical effects in MHD

using Maxwell's equations, augmented by suitable boundary conditions. So it is with MHD, and the choice depends on convenience. It turns out that for the majority of applications, it is most instructive to deal with the magnetic field as primary, and to use Maxwell's equations

in MHD the magnetic field is the primary variable

Maxwell's equations

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho_e}{\epsilon_0},} \tag{19.1a}$$

$$\boxed{\nabla \cdot \mathbf{B} = 0,} \tag{19.1b}$$

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},} \tag{19.1c}$$

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}} \tag{19.1d}$$

to express the electric field $\mathbf{E}$, the current density $\mathbf{j}$, and the charge density ρ_e in terms of the magnetic field (next subsection).

19.2.1

19.2.1 Maxwell's Equations in the MHD Approximation

As normally formulated, Ohm's law is valid only in the rest frame of the conductor. In particular, for a conducting fluid, Ohm's law relates the current density $\mathbf{j}'$ measured in the fluid's local rest frame to the electric field $\mathbf{E}'$ measured there:

$$\mathbf{j}' = \kappa_e \mathbf{E}', \tag{19.2}$$

where κ_e is the scalar electric conductivity. Because the fluid is generally accelerated, $d\mathbf{v}/dt \neq 0$, its local rest frame is generally not inertial. Since it would produce a terrible headache to have to transform time and again from some inertial frame to the continually changing local rest frame when applying Ohm's law, it is preferable to reformulate Ohm's law in terms of the fields $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{j}$ measured in an inertial frame. To facilitate this (and for completeness), we explore the frame dependence of all our electromagnetic quantities $\mathbf{E}$, $\mathbf{B}$, $\mathbf{j}$, and ρ_e.

Throughout our development of magnetohydrodynamics, we assume that the fluid moves with a nonrelativistic speed $v \ll c$ relative to our chosen reference frame. We can then express the rest-frame electric field in terms of the inertial-frame electric and magnetic fields as

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B}; \quad E' = |\mathbf{E}'| \ll E, \quad \text{so } \mathbf{E} \simeq -\mathbf{v} \times \mathbf{B}. \tag{19.3a}$$

In the first equation we have set the Lorentz factor $\gamma \equiv 1/\sqrt{1 - v^2/c^2}$ to unity, consistent with our nonrelativistic approximation. The second equation follows from the high conductivity of the fluid, which guarantees that current will quickly flow in whatever manner it must to annihilate any electric field $\mathbf{E}'$ that might be formed in the fluid's local rest frame. By contrast with the extreme frame dependence (19.3a) of the electric field, the magnetic field is essentially the same in the fluid's local rest frame as in the laboratory. More specifically, the analog of Eq. (19.3a) is $\mathbf{B}' = \mathbf{B} - (\mathbf{v}/c^2) \times \mathbf{E}$; and since $E \sim vB$, the second term is of magnitude $(v/c)^2 B$, which is negligible, giving

$$\mathbf{B}' \simeq \mathbf{B}. \tag{19.3b}$$

Because $\mathbf{E}$ is highly frame dependent, so is its divergence, the electric charge density ρ_e. In the laboratory frame, where $E \sim vB$, Gauss's and Ampère's laws [Eqs. (19.1a,d)] imply that $\rho_e \sim \epsilon_0 vB/L \sim (v/c^2)j$, where L is the lengthscale on which $\mathbf{E}$ and $\mathbf{B}$ vary; and the relation $E' \ll E$ with Gauss's law implies $|\rho_e'| \ll |\rho_e|$:

$$\rho_e \sim j\, v/c^2, \qquad |\rho_e'| \ll |\rho_e|. \tag{19.3c}$$

By transforming the current density between frames and approximating $\gamma \simeq 1$, we obtain $\mathbf{j}' = \mathbf{j} + \rho_e \mathbf{v} = \mathbf{j} + \mathrm{O}(v/c)^2 j$; so in the nonrelativistic limit (first order in v/c) we can ignore the charge density and write

$$\mathbf{j}' = \mathbf{j}. \tag{19.3d}$$

To recapitulate, in nonrelativistic magnetohydrodynamic flows, the magnetic field and current density are frame independent up to fractional corrections of order $(v/c)^2$, while the electric field and charge density are highly frame dependent and are generally small in the sense that $E/c \sim (v/c)B \ll B$ and $\rho_e \sim (v/c^2)j \ll j/c$ [in Gaussian cgs units we have $E \sim (v/c)B \ll B$ and $\rho_e c \sim (v/c)j \ll j$].

in MHD, magnetic field and current density are approximately frame independent; electric field and charge density are small and frame dependent

Combining Eqs. (19.2), (19.3a), and (19.3d), we obtain the nonrelativistic form of Ohm's law in terms of quantities measured in our chosen inertial, laboratory frame:

$$\mathbf{j} = \kappa_e(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{19.4}$$

Ohm's law

We are now ready to derive explicit equations for the (inertial-frame) electric field and current density in terms of the (inertial-frame) magnetic field. In our derivation, we denote by L the lengthscale on which the magnetic field changes.

We begin with Ampère's law written as $\nabla \times \mathbf{B} - \mu_0 \mathbf{j} = \mu_0\epsilon_0 \partial\mathbf{E}/\partial t = (1/c^2)\partial\mathbf{E}/\partial t$, and we notice that the time derivative of $\mathbf{E}$ is of order $Ev/L \sim Bv^2/L$ (since $E \sim vB$). Therefore, the right-hand side is $\mathrm{O}[Bv^2/(c^2 L)]$ and thus can be neglected compared to the $\mathrm{O}(B/L)$ term on the left, yielding:

$$\boxed{\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B}.} \tag{19.5a}$$

current density in terms of magnetic field

We next insert this expression for $\mathbf{j}$ into the inertial-frame Ohm's law (19.4), thereby obtaining

$$\boxed{\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \frac{1}{\kappa_e \mu_0}\nabla \times \mathbf{B}.} \tag{19.5b}$$

electric field in terms of magnetic field

If we happen to be interested in the charge density (which is rare in MHD), we can compute it by taking the divergence of this electric field:

charge density in terms of magnetic field

$$\boxed{\rho_e = -\epsilon_0 \nabla \cdot (\mathbf{v} \times \mathbf{B}).} \tag{19.5c}$$

Equations (19.5) express all the secondary electromagnetic variables in terms of our primary one, $\mathbf{B}$. This has been possible because of the high electric conductivity κ_e and our choice to confine ourselves to nonrelativistic (low-velocity) situations; it would not be possible otherwise.

We next derive an evolution law for the magnetic field by taking the curl of Eq. (19.5b), using Maxwell's equation $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and the vector identity $\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$, and using $\nabla \cdot \mathbf{B} = 0$. The result is

evolution law for magnetic field

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \left(\frac{1}{\mu_0 \kappa_e}\right) \nabla^2 \mathbf{B},} \tag{19.6}$$

which, using Eqs. (14.4) and (14.5) with $\boldsymbol{\omega}$ replaced by $\mathbf{B}$, can also be written as

$$\frac{D\mathbf{B}}{Dt} = -\mathbf{B}\nabla \cdot \mathbf{v} + \left(\frac{1}{\mu_0 \kappa_e}\right) \nabla^2 \mathbf{B}, \tag{19.7}$$

where D/Dt is the fluid derivative defined in Eq. (14.5). When the flow is incompressible (as it often will be), the $\nabla \cdot \mathbf{v}$ term vanishes.

Equation (19.6)—or equivalently, Eq. (19.7)—is called the *induction equation* and describes the temporal evolution of the magnetic field. It is the same in form as the propagation law for vorticity $\boldsymbol{\omega}$ in a flow with $\nabla P \times \nabla \rho = 0$ [Eq. (14.3), or (14.6) with $\boldsymbol{\omega}\nabla \cdot \mathbf{v}$ added in the compressible case]. The $\nabla \times (\mathbf{v} \times \mathbf{B})$ term in Eq. (19.6) dominates when the conductivity is large and can be regarded as describing the freezing of magnetic field lines in the fluid in the same way as the $\nabla \times (\mathbf{v} \times \boldsymbol{\omega})$ term describes the freezing of vortex lines in a fluid with small viscosity ν (Fig. 19.2). By analogy with Eq. (14.10), when flux-freezing dominates, the fluid derivative of $\mathbf{B}/\rho$ can be written as

for large conductivity: freezing of magnetic field into the fluid

$$\frac{D}{Dt}\left(\frac{\mathbf{B}}{\rho}\right) \equiv \frac{d}{dt}\left(\frac{\mathbf{B}}{\rho}\right) - \left(\frac{\mathbf{B}}{\rho} \cdot \nabla\right) \mathbf{v} = 0, \tag{19.8}$$

where ρ is mass density (not to be confused with charge density ρ_e). Equation (19.8) states that $\mathbf{B}/\rho$ evolves in the same manner as the separation $\Delta \mathbf{x}$ between two points in the fluid (cf. Fig. 14.4 and associated discussion).

The term $[1/(\mu_0 \kappa_e)]\nabla^2 \mathbf{B}$ in the B-field evolution equation (19.6) or (19.7) is analogous to the vorticity diffusion term $\nu \nabla^2 \boldsymbol{\omega}$ in the vorticity evolution equation (14.3) or (14.6). Therefore, when κ_e is not too large, magnetic field lines will diffuse through the fluid. The effective diffusion coefficient (analogous to ν) is

magnetic diffusion coefficient

$$\boxed{D_M = 1/(\mu_0 \kappa_e).} \tag{19.9a}$$

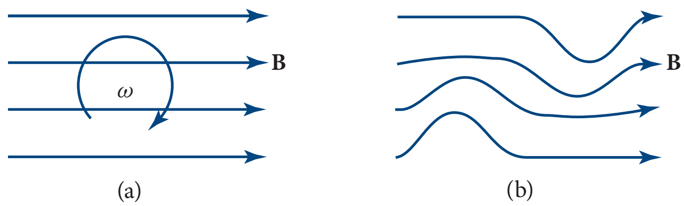

FIGURE 19.2 Pictorial representation of the evolution of the magnetic field in a fluid endowed with infinite electrical conductivity. (a) A uniform magnetic field at time $t = 0$ in a vortex. (b) At a later time, when the fluid has rotated through $\sim 30°$, the circulation has stretched and distorted the magnetic field.

Earth's magnetic field provides an example of field diffusion. That field is believed to be supported by electric currents flowing in Earth's iron core. Now, we can estimate the electric conductivity of iron under these conditions and from it deduce a value for the diffusivity, $D_M \sim 1\ \mathrm{m}^2\ \mathrm{s}^{-1}$. The size of Earth's core is $L \sim 10^4$ km, so if there were no fluid motions, then we would expect the magnetic field to diffuse out of the core and escape from Earth in a time

$$\boxed{\tau_M \sim \frac{L^2}{D_M}} \tag{19.9b}$$

magnetic decay time

~ 3 million years, which is much shorter than the age of Earth, ~ 5 billion years. The reason for this discrepancy, as we discuss in Sec. 19.6, is that there are internal circulatory motions in the liquid core that are capable of regenerating the magnetic field through dynamo action.

Although Eq. (19.6) describes a genuine diffusion of the magnetic field, to compute with confidence the resulting magnetic decay time, one must solve the complete boundary value problem. To give a simple illustration, suppose that a poor conductor (e.g., a weakly ionized column of plasma) is surrounded by an excellent conductor (e.g., the metal walls of the container in which the plasma is contained), and that magnetic field lines supported by wall currents thread the plasma. The magnetic field will only diminish after the wall currents undergo Ohmic dissipation, which can take much longer than the diffusion time for the plasma column alone.

It is customary to introduce a dimensionless number called the *magnetic Reynolds number*, R_M, directly analogous to the fluid Reynolds number Re, to describe the relative importance of flux freezing and diffusion. The fluid Reynolds number can be regarded as the ratio of the magnitude of the vorticity-freezing term, $\nabla \times (\mathbf{v} \times \boldsymbol{\omega}) \sim (V/L)\omega$, in the vorticity evolution equation, $\partial\boldsymbol{\omega}/\partial t = \nabla \times (\mathbf{v} \times \boldsymbol{\omega}) + \nu\nabla^2\boldsymbol{\omega}$, to the magnitude of the diffusion term, $\nu\nabla^2\boldsymbol{\omega} \sim (\nu/L^2)\omega$: $\mathrm{Re} = (V/L)(\nu/L^2)^{-1} = VL/\nu$. Here V is a characteristic speed, and L a characteristic lengthscale of the flow. Similarly, the magnetic Reynolds number is the ratio of the magnitude of the

TABLE 19.1: Characteristic magnetic diffusivities D_M, decay times τ_M, and magnetic Reynolds numbers R_M for some common MHD flows with characteristic length scales L and velocities V

Substance	L (m)	V (m s^{-1})	D_M (m^2 s^{-1})	τ_M (s)	R_M
Mercury	0.1	0.1	1	0.01	0.01
Liquid sodium	0.1	0.1	0.1	0.1	0.1
Laboratory plasma	1	100	10	0.1	10
Earth's core	10^7	0.1	1	10^{14}	10^6
Interstellar gas	10^{17}	10^3	10^3	10^{31}	10^{17}

magnetic-flux-freezing term, $\boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) \sim (V/L)B$, to the magnitude of the magnetic-flux-diffusion term, $D_M \nabla^2 \mathbf{B} = [1/(\mu_o \kappa_e)]\nabla^2 \mathbf{B} \sim B/(\mu_o \kappa_e L^2)$, in the induction equation (19.6):

magnetic Reynolds number and magnetic field freezing

$$\boxed{R_M = \frac{V/L}{D_M/L^2} = \frac{VL}{D_M} = \mu_0 \kappa_e V L.} \tag{19.9c}$$

When $R_M \gg 1$, the field lines are effectively frozen in the fluid; when $R_M \ll 1$, Ohmic dissipation is dominant, and the field lines easily diffuse through the fluid.

perfect MHD: infinite conductivity and magnetic field freezing

Magnetic Reynolds numbers and diffusion times for some typical MHD flows are given in Table 19.1. For most laboratory conditions, R_M is modest, which means that electric resistivity $1/\kappa_e$ is significant, and the magnetic diffusivity D_M is rarely negligible. By contrast, in space physics and astrophysics, R_M is usually very large, $R_M \gg 1$, so the resistivity can be ignored almost always and everywhere. This limiting case, when the electric conductivity is treated as infinite, is often called *perfect MHD*.

magnetic reconnection and its influence

The phrase "almost always and everywhere" needs clarification. Just as for large-Reynolds-number fluid flows, so also here, boundary layers and discontinuities can be formed, in which the gradients of physical quantities are automatically large enough to make $R_M \sim 1$ locally. An important example discussed in Sec. 19.6.3 is *magnetic reconnection*. This occurs when regions magnetized along different directions are juxtaposed, for example, when the solar wind encounters Earth's magnetosphere. In such discontinuities and boundary layers, the current density is high, and magnetic diffusion and Ohmic dissipation are important. As in ordinary fluid mechanics, these dissipative layers and discontinuities can control the character of the overall flow despite occupying a negligible fraction of the total volume.

19.2.2

19.2.2 Momentum and Energy Conservation

The fluid dynamical aspects of MHD are handled by adding an electromagnetic force term to the Euler or Navier-Stokes equation. The magnetic force density $\mathbf{j} \times \mathbf{B}$ is the sum of the Lorentz forces acting on all the fluid's charged particles in a unit volume.