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THE PYTHAGOREAN THEOREM

A 4,000-YEAR HISTORY

ELI MAOR

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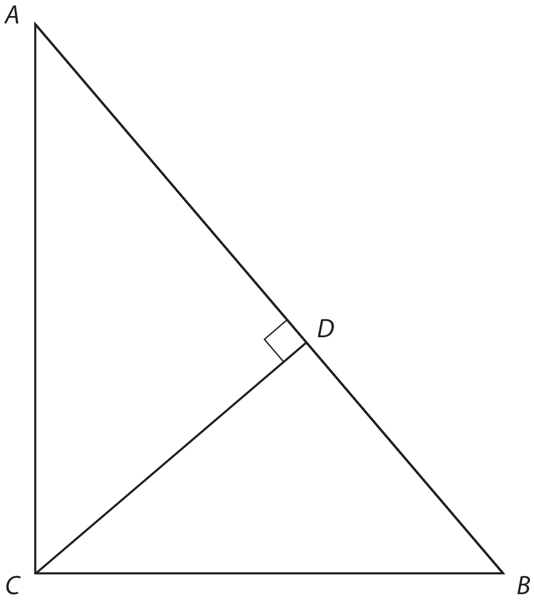

The simplest known proof of the Pythagorean theorem
(see p. 115)

The Pythagorean Theorem

❖ A 4,000-Year History

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PRINCETON UNIVERSITY PRESS Princeton and Oxford

Published by Princeton University Press, 41 William Street,
Princeton, New Jersey 08540
In the United Kingdom: Princeton University Press,
6 Oxford Street, Woodstock, Oxfordshire OX20 1TR

First Princeton Science Library paperback edition, 2019
Paperback ISBN 978-0-691-19688-6

The Library of Congress had cataloged the cloth edition as follows:
Maor, Eli.
The Pythagorean theorem : a 4,000-year history / Eli Maor.
p. cm.
Includes bibliographical references and index.
ISBN-13: 978-0-691-12526-8 (acid-free paper)
ISBN-10: 0-691-12526-0 (acid-free paper)
1. Pythagorean theorem—History. I. Title.

QA460.P8M36 2007
516.22—dc22 2006050969

British Library Cataloging-in-Publication Data is available

This book has been composed in Times

Printed on acid-free paper. ∞

press.princeton.edu

Printed in the United States of America

To our six grandchildren:

Yehuda-Leib,

Nechama-Shira,

Yechezkel,

Zahava-Gila,

Tzivia-Shalva,

and Betzalel-Shalom.

❖ *May they enjoy a healthy, prosperous, and fulfilling life.*

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Note: all photographs are courtesy of the author, except where noted.

Preface

To this day, the theorem of Pythagoras remains the most important single theorem in the whole of mathematics.

—Jacob Bronowski, *The Ascent of Man*, p. 160

Though its roots are in geometry, the theorem universally attributed to Pythagoras has found its way into nearly every branch of science, pure or applied. Well over four hundred proofs of it are known, and their number is still growing; the list includes an original proof by a future American president, another by twelve-year-old Albert Einstein, and still another by a young blind girl. Some of these proofs are breathtaking in their simplicity, while others are incredibly complex. The theorem itself is known by various names: *the theorem of Pythagoras*, *the hypotenuse theorem*, or simply *Euclid I 47*, so called because it is listed as Proposition 47 in Book I of Euclid's *Elements*. Its characteristic figure (fig. P1), known in some traditions as "the windmill" and in others as "the bride's chair," has been proposed as a cosmic identity card with which we might introduce ourselves to extraterrestrial beings, if and when we find them. The theorem plays a central role in numerous applications; occasionally it has been overused, even misused. And perhaps uniquely for a discipline not known for its popular appeal, it has found its way into our daily culture, appearing on postage stamps and on T-shirts, in works of art and literature, even in the lyrics of a famous musical. By any measure, it is the most famous theorem in all of mathematics, the one statement that every student, no matter how math-phobic, can recall from his or her high school days.

Today we think of the Pythagorean theorem as an algebraic relation, $a^2 + b^2 = c^2$, from which the length of one side of a right triangle can be found, given the lengths of the other two sides. But that is not how Pythagoras viewed it; to him it was a geometric statement about areas. It was only with the rise of modern algebra, about 1600 CE, that the theorem assumed its familiar algebraic form. It is important to bear this in mind if we are to trace the evolution of the theorem over the 2,500 years since Pythagoras supposedly first proved it and made it immortal. And he was not even the first to discover it: the theorem had been known to the Babylonians, and possibly to the Chinese, at least a thousand years before him.

Many writers have commented on the beauty of the Pythagorean theorem. Charles Lutwidge Dodgson, better known by his literary name Lewis Carroll,

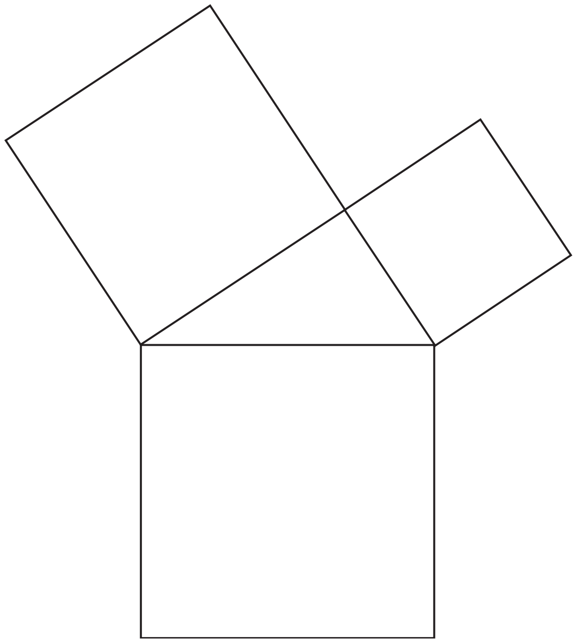

Figure P1. The Pythagorean theorem: Euclid's view

wrote in 1895: "It is as dazzlingly beautiful now as it was in the day when Pythagoras first discovered it."[1] He was certainly qualified to say this, being a talented mathematician besides gaining fame as the author of *Alice's Adventures in Wonderland* and *Through the Looking-Glass*. But who is to say what is beautiful? In 2004, the journal *Physics World* asked readers to nominate the twenty most beautiful equations in science. The top winner was Euler's formula $e^{i\pi} + 1 = 0$, followed in order by Maxwell's four electromagnetic field equations, Newton's second law of motion $F = ma$, and $a^2 + b^2 = c^2$, the Pythagorean theorem; it won only fourth place.[2]

Note that the contest was for the most beautiful *equations*, not the laws or theorems they represent. Beauty, of course, is a subjective attribute, but there is a fairly broad consensus among mathematicians as to what qualifies a theorem, or the proof thereof, to be called beautiful. A paramount criterion is symmetry. Consider, for example, the three altitudes of a triangle: they always meet at one point (as do the medians and the angle bisectors). This statement has a certain elegance to it, with its sweeping symmetry: no side or vertex takes precedence over any other; there is a complete democracy among the constituents. Or consider the theorem: If through a point P inside a circle a chord AB is drawn, the product $PA \times PB$ is constant—it has the same value for all chords through P (fig. P2). Again we have perfect democracy: every chord has the same status in relation to P as any other.

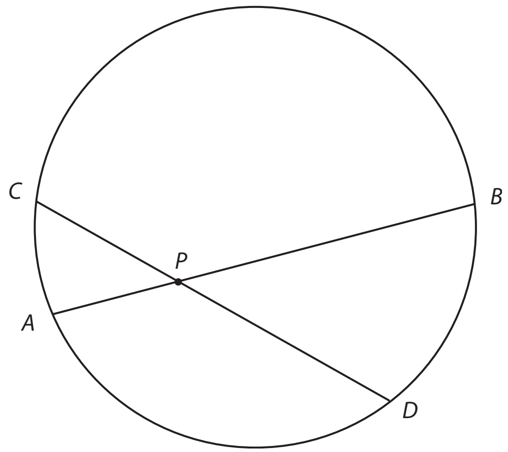

Figure P2. $PA \times PB = PC \times PD$

In this sense, the Pythagorean theorem is decidedly undemocratic. In the first place, it applies only to a very special case, that of a right triangle; and even then it singles out one side, the hypotenuse, as playing a distinctly different role from the other two sides. The word *hypotenuse* comes from the Greek words *hypo*, meaning "under," "beneath," or "down," and *teinen*, "to stretch"; this makes sense if we view the triangle with the hypotenuse at the bottom, the way it appears in Euclid's *Elements* (see again fig. P1). The Chinese call it *hsien*, a string stretched between two points (as in a lute). The Hebrew word for hypotenuse is *'yeter*, which may derive either from *mei'tar*, a string, or from *yo'ter*, "more than" (the length of each leg). But even if we look at the triangle through modern eyes, with one leg placed horizontally and the other vertically (fig. P3), the square on the hypotenuse leaps out of the figure at an odd angle. A beautiful theorem? Perhaps, but not exactly a candidate for Miss America.

If not elegance, what then is it that gives the Pythagorean theorem its universal appeal? Part of it, no doubt, has to do with the great number of proofs that have been proposed over the centuries. Elisha Scott Loomis (1852–1940), an eccentric mathematics teacher from Ohio, spent a lifetime collecting all known proofs—371 of them—and writing them up in *The Pythagorean Proposition* (1927).[3] Loomis claimed that in the Middle Ages, it was required that a student taking his Master's degree in mathematics offer a new and original proof of the Pythagorean theorem; this, he claimed, had spurred students and teachers to come up with ever new and innovative proofs. Some of these proofs are based on the similarity of triangles, others on dissection, still others on algebraic formulas, and a few make use of vectors. There are even "proofs" ("demonstrations" would be a better word) based on physical devices; in a science museum in Tel Aviv, Israel, I saw a demonstration in which colored liquid flowed freely between the squares built on the hypotenuse and on the two

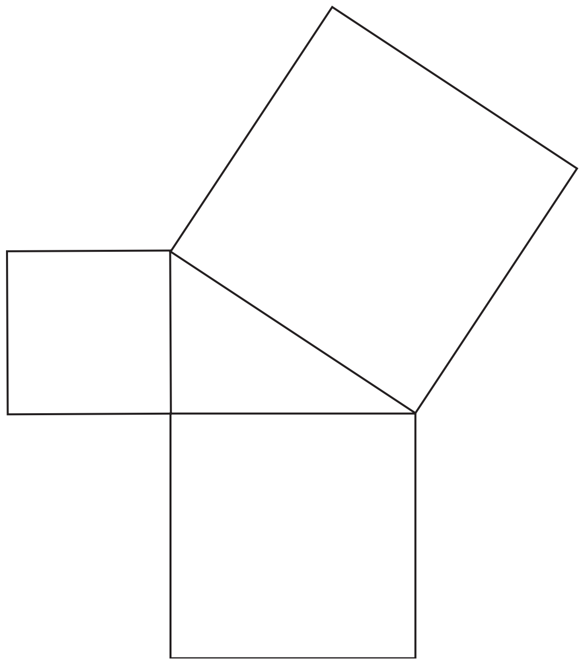

Figure P3. The Pythagorean theorem: a modern view

sides of a rotating, plexiglass-made right triangle, showing that the volume of liquid in the first square equals the combined volume in the other two.

But there is another reason for the universal appeal of the Pythagorean theorem, for it is arguably the most frequently used theorem in all of mathematics. Open any handbook of mathematical formulas; you will find the expression $x^2 + y^2$ in nearly every chapter, often tucked inside a larger expression; and it is almost always $x^2 + y^2$, not $x^3 + y^3$ or any other power of the variables. Directly or indirectly, this expression can be traced to the Pythagorean theorem. Take, for example, trigonometry, a subject notorious for its seemingly endless supply of formulas. Whether it is $\sin^2 x + \cos^2 x = 1$, or $1 + \tan^2 x = \sec^2 x$, or $1 + \cot^2 x = \csc^2 x$, these identities are the ghosts of the Pythagorean theorem—indeed, they are called the *Pythagorean identities*. The same is true in almost every branch of mathematics, from number theory and algebra to calculus and probability: in all of them, the Pythagorean theorem reigns supreme.

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In this book I have traced the evolution of the Pythagorean theorem and its impact on mathematics and on our culture in general, starting with the Babylonians nearly four thousand years ago and continuing up to our own time. I have not attempted to give a comprehensive account of the hundreds of existing proofs—a nearly impossible task, and a fruitless one too, as many of these proofs are but slight variations of one another. Even Loomis's monumental compilation remains incomplete; many new proofs have been proposed since the second edition of his book appeared in 1940 (the year of his death), and new ones continue to be offered even at the time of this writing.[4]

As with my previous books, this one is aimed at the reader with an interest in the history of mathematics. Mostly, a good knowledge of high school algebra and geometry, and an occasional smattering of calculus, will be sufficient. Several subjects that require more detailed mathematical treatment have been relegated to the appendixes. Because I am making occasional reference to my earlier books, I will refer to them simply by their titles: *To Infinity and Beyond: A Cultural History of the Infinite* (1991), *e: the Story of a Number* (1994), and *Trigonometric Delights* (1998; all published by Princeton University Press, Princeton, N.J.). Two other frequently mentioned sources are Howard Eves, *An Introduction to the History of Mathematics* (Fort Worth, Texas: Saunders, 1992), and David Eugene Smith, *History of Mathematics*, vol. 1: *General Survey of the History of Elementary Mathematics*; vol. 2: *Special Topics of Elementary Mathematics* (New York, 1923–1925; rpt. New York: Dover, 1958). These will be referred to as Eves and Smith, respectively.

Many thanks go to my dear wife Dalia for encouraging me to see this work through and for her meticulous proofreading of the manuscript; to Robert Langer, for his critical review of the text and his very useful suggestions; to Vickie Kearn, my editor at Princeton University Press, for her unwavering support and encouragement to guide this book from its inception to its completion; to Debbie Tegarden, Carmina Alvarez, and Dimitri Karetnikov, and to all at the Press for their good care of the manuscript during its production phase; to Alice Calaprice, my trusted copy editor for the past fifteen years; to Joseph L. Teeters for providing me with some hard-to-find sources of useful information; to Howard Zvi Weiss for his help in translating several verses of poetry from the German; to Barbara and Jeff Niemic and to Deborah Ward for their special effort to locate and photograph the plaque in Dublin, Ireland, commemorating Sir William Rowan Hamilton's discovery of the law of quaternion multiplication; and to the staff of the Skokie Public Library in Illinois for their efforts to locate a number of obscure sources. Their help is greatly appreciated.

July 2006

Notes and Sources

1. *A New Theory of Parallels* (London, 1895).
2. *New York Times*, Ideas and Trends, October 24, 2004, p. 12.

3. (Washington, D.C.: National Council of Teachers of Mathematics, 1968.) More on this work will be found in chapter 8.

4. Several Web sites are devoted to the Pythagorean theorem and give an account of recent proofs. The Bibliography gives a partial listing of these sites.

The Pythagorean Theorem

Cambridge, England, 1993

Remember Pythagoras?
—*New York Times*, June 24, 1993

Mathematical news rarely makes the headlines, let alone front-page coverage, but June 24, 1993, was an exception. On that day, the *New York Times* ran a front-page story headed, “At Last, Shout of ‘Eureka!’ in Age-Old Math Mystery.” Across the Atlantic the day before, a forty-year-old English mathematician announced that he had solved math’s most celebrated problem, a simple-looking proposition that had kept mathematicians busy for the past 350 years.

The mathematician at the center of the excitement was Dr. Andrew Wiles, a native of Cambridge, England, and a professor at Princeton University in New Jersey. He made the sensational announcement at the end of a three-lecture series entitled “Modular Forms, Elliptic Curves, and Galois Representations.” The subject was not a household term even among mathematicians, let alone laypeople. But there were rumors that the speaker would pull a surprise, and the lecture hall was packed with listeners. As the talk drew to its conclusion, the tension in the audience was palpable. Then, almost casually, Dr. Wiles ended his lecture with these words: “And by the way, this means that Fermat’s Last Theorem was true. Q.E.D.”[1] There was a rush to the nearest computer terminals, and those with access to e-mail services—still a novelty in 1993—flashed the news around the globe.

The circumstances behind Wiles’s announcement had all the hallmarks of a human drama. Pierre de Fermat (1601–1665), a French lawyer by profession who practiced mathematics as a pastime, made a conjecture in 1637 about the possible solutions of the simple-looking equation $x^n + y^n = z^n$, where all the variables, including the exponent n, stand for positive integers. When $n = 1$, the equation is trivial: the sum of any two integers is obviously a third integer, so we have $x^1 + y^1 = z^1$. The case $n = 2$ is of greater interest. There are many triples of integers (x, y, z) for which $x^2 + y^2 = z^2$, in fact infinitely many; two examples are (3, 4, 5) and (5, 12, 13). Such triples, of course, immediately remind us of the Pythagorean theorem: they represent right triangles in which all three sides have integer lengths. So it was only natural that mathematicians tried to go to the next step—find integer solutions of the equations $x^3 + y^3 = z^3$, $x^4 + y^4 = z^4$, and so on. None were ever found.

Fermat thought he had a proof that no integer solutions of the equation $x^n + y^n = z^n$ exist for any value of n greater than 2. In the margins of his copy of the works of the third-century CE mathematician Diophantus of Alexandria, Fermat scribbled a few words that would become immortal:

> To divide a cube into two cubes, a fourth power, or in general any power whatever into two powers of the same denomination above the second is impossible. I have found an admirable proof of this, but the margin is too narrow to contain it.[2]

For the next 350 years, numerous mathematicians, laypeople, and cranks tried to reconstruct Fermat's "admirable proof." All of them failed. Two huge monetary awards, one by the French Academy of Sciences and another by its German counterpart, were offered to the first person to come up with a valid proof; both remained unclaimed.[3]

Not that the quest for a proof was entirely futile. The great Swiss mathematician Leonhard Euler (1707–1783) in 1753 proved Fermat's claim for the special case $n = 3$. Other special cases followed, and with the advent of electronic computers, all cases for n under 100,000 have been proven to be correct. But that is not the same as proving it for *all* values of n. Fermat's Last Theorem (FLT), as it became known, remained unresolved.[4]

When Wiles jumped into the fray, he already had something to start from: in 1954, a Japanese mathematician, Yutaka Taniyama (1927–1958), made a conjecture about a class of objects called *elliptic curves*. Subsequent work, particularly by Dr. Gerhard Frey of the University of the Saarland in Germany and Dr. Kenneth Ribet of the University of California at Berkeley, showed a clear connection between Taniyama's conjecture and Fermat's Last Theorem: if the former is true, then so is the latter. Wiles, after working in his attic in near seclusion for seven years, showed that the Taniyama conjecture was indeed true; and almost as an afterthought, so was FLT.

But not all was well. After submitting a 200-page-long proof to the scrutiny of mathematicians able and willing to sift through it, a tiny hole in the logic was found. Undeterred, Wiles went back into seclusion, and after another year of hard work, with the help of Cambridge lecturer Richard Taylor, he managed to fix the hole. FLT is now considered proven, finally worthy of being called a theorem.[5]

But why was this one problem singled out as the most famous unsolved problem in mathematics? For one, there was its deceiving simplicity: any high school student would be able to understand it. And the mystery of Fermat's enigmatic note only added spice to the story (most mathematicians are convinced he did not have a valid proof; the tools needed to crack the problem simply were not available in his time). But beyond these reasons, FLT leaves us with a sense that history was closing a circle. For the very same type of equation that Fermat was investigating had already been studied by the Babylonians nearly four thousand years earlier. It is here that our story really begins.

Notes and Sources

1. This is a free quotation based on the *New York Times* article of June 24, 1993, p. D22. Wiles's exact words were not reported.

2. Fermat's famous scribble, originally written in Latin, has appeared in numerous English versions. The one used here is from Eves, p. 355.

3. The French award, a gold medal and 300 francs, was offered twice, in 1815 and again in 1860. Its German counterpart was announced in 1908 and amounted to 100,000 marks—a huge sum at the time. This sum has been reduced in value by the 1929 German inflation to a paltry 7,500 marks (about $4,400 in today's value). The two prizes brought in thousands of claims, many by amateurs and cranks with little or no knowledge of mathematics.

4. The name is a misnomer in two respects: until Wiles's proof, the "theorem" was really a conjecture; and it was not Fermat's last, but rather the last of his many conjectures that mathematicians were unable to resolve.

5. Needless to say, the description of FLT given here is only the briefest of sketches. For a more detailed account, see Simon Singh's excellent book, *Fermat's Enigma: The Epic Quest to Solve the World's Greatest Mathematical Problem* (New York: Walker, 1997).

1

Mesopotamia, 1800 BCE

> **We would more properly have to call "Babylonian" many things which the Greek tradition had brought down to us as "Pythagorean."**
>
> **—Otto Neugebauer, quoted in Bartel van der Waerden, *Science Awakening*, p. 77**

The vast region stretching from the Euphrates and Tigris Rivers in the east to the mountains of Lebanon in the west is known as the Fertile Crescent. It was here, in modern Iraq, that one of the great civilizations of antiquity rose to prominence four thousand years ago: Mesopotamia. Hundreds of thousands of clay tablets, found over the past two centuries, attest to a people who flourished in commerce and architecture, kept accurate records of astronomical events, excelled in the arts and literature, and, under the rule of Hammurabi, created the first legal code in history. Only a small fraction of this vast archeological treasure trove has been studied by scholars; the great majority of tablets lie in the basements of museums around the world, awaiting their turn to be deciphered and give us a glimpse into the daily life of ancient Babylon.

Among the tablets that have received special scrutiny is one with the unassuming designation "YBC 7289," meaning that it is tablet number 7289 in the Babylonian Collection of Yale University (fig. 1.1). The tablet dates from the Old Babylonian period of the Hammurabi dynasty, roughly 1800–1600 BCE. It shows a tilted square and its two diagonals, with some marks engraved along one side and under the horizontal diagonal. The marks are in cuneiform (wedge-shaped) characters, carved with a stylus into a piece of soft clay which was then dried in the sun or baked in an oven. They turn out to be numbers, written in the peculiar Babylonian numeration system that used the base 60. In this *sexagesimal system*, numbers up to 59 were written in essentially our modern base-ten numeration system, but without a zero. Units were written as vertical Y-shaped notches, while tens were marked with similar notches written horizontally. Let us denote these symbols by | and —, respectively. The number 23, for example, would be written as — — | | |. When a number exceeded 59,

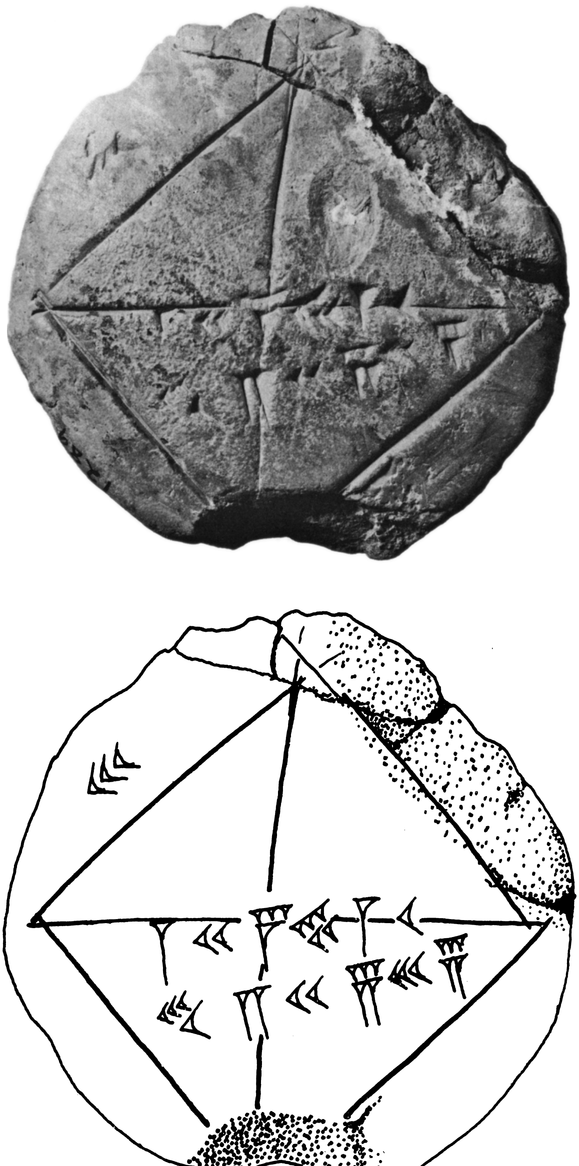

Figure 1.1. YBC 7289

it was arranged in groups of 60 in much the same way as we bunch numbers into groups of ten in our base-ten system. Thus, 2,413 in the sexagesimal system is $40 \times 60 + 13$, which was written as — — — —   — | | | (often a group of several identical symbols was stacked, evidently to save space).

Because the Babylonians did not have a symbol for the "empty slot"—our modern zero—there is often an ambiguity as to how the numbers should be grouped. In the example just given, the numerals — — — —   — | | | could also stand for $40 \times 60^2 + 13 \times 60 = 144{,}780$; or they could mean $40/60 + 13 = 13.666$, or any other combination of powers of 60 with the coefficients 40 and 13. Moreover, had the scribe made the space between — — — — and — | | | too small, the number might have erroneously been read as — — — — — | | |, that is, $50 \times 60 + 3 = 3{,}003$. In such cases the correct interpretation must be deduced from the context, presenting an additional challenge to scholars trying to decipher these ancient documents.

Luckily, in the case of YBC 7289 the task was relatively easy. The number along the upper-left side is easily recognized as 30. The one immediately under the horizontal diagonal is 1;24,51,10 (we are using here the modern notation for writing Babylonian numbers, in which commas separate the sexagesimal "digits," and a semicolon separates the integral part of a number from its fractional part). Writing this number in our base-10 system, we get $1 + 24/60 + 51/60^2 + 10/60^3 = 1.414213$, which is none other than the decimal value of $\sqrt{2}$, accurate to the nearest one hundred thousandth! And when this number is multiplied by 30, we get 42.426389, which is the sexagesimal number 42;25,35—the number on the second line below the diagonal. The conclusion is inescapable: the Babylonians knew the relation between the length of the diagonal of a square and its side, $d = a\sqrt{2}$. But this in turn means that they were familiar with the Pythagorean theorem—or at the very least, with its special case for the diagonal of a square ($d^2 = a^2 + a^2 = 2a^2$)—more than a thousand years before the great sage for whom it was named.

Two things about this tablet are especially noteworthy. First, it proves that the Babylonians knew how to compute the square root of a number to a remarkable accuracy—in fact, an accuracy equal to that of a modern eight-digit calculator.[1] But even more remarkable is the probable purpose of this particular document: by all likelihood, it was intended as an example of how to find the diagonal of *any* square: simply multiply the length of the side by 1;24,51,10. Most people, when given this task, would follow the "obvious" but more tedious route: start with 30, square it, double the result, and take the square root: $d = \sqrt{30^2 + 30^2} = \sqrt{1800} = 42.4264$, rounded to four places. But suppose you had to do this over and over for squares of different sizes; you would have to repeat the process each time with a new number, a rather tedious task. The anonymous scribe who carved these numbers into a clay tablet nearly four thousand years ago showed us a simpler way: just multiply the side of the square by $\sqrt{2}$ (fig. 1.2). Some simplification!

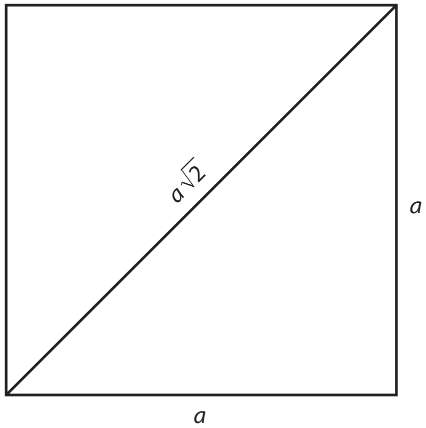

Figure 1.2. A square and its diagonal

But there remains one unanswered question: why did the scribe choose a side of 30 for his example? There are two possible explanations: either this tablet referred to some particular situation, perhaps a square field of side 30 for which it was required to find the length of the diagonal; or—and this is more plausible—he chose 30 because it is one-half of 60 and therefore lends itself to easy multiplication. In our base-ten system, multiplying a number by 5 can be quickly done by halving the number and moving the decimal point one place to the right. For example, $2.86 \times 5 = (2.86/2) \times 10 = 1.43 \times 10 = 14.3$ (more generally, $a \times 5 = \frac{a}{2} \times 10$). Similarly, in the sexagesimal system multiplying a number by 30 can be done by halving the number and moving the "sexagesimal point" one place to the right $(a \times 30 = \frac{a}{2} \times 60)$.

Let us see how this works in the case of YBC 7289. We recall that 1;24,51,10 is short for $1 + 24/60 + 51/60^2 + 10/60^3$. Dividing this by 2, we get $\frac{1}{2} + \frac{12}{60} + \frac{25\frac{1}{2}}{60^2} + \frac{5}{60^3}$, which we must rewrite so that each coefficient of a power of 60 is an integer. To do so, we replace the 1/2 in the first and third terms by by 30/60, getting $\frac{30}{60} + \frac{12}{60} + \frac{25+\frac{30}{60}}{60^2} + \frac{5}{60^3} = \frac{42}{60} + \frac{25}{60^2} + \frac{35}{60^3} = 0;42,25,35$. Finally, moving the sexagesimal point one place to the right gives us 42;25,35, the length of the diagonal. It thus seems that our scribe chose 30 simply for pragmatic reasons: it made his calculations that much easier.

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If YBC 7289 is a remarkable example of the Babylonians' mastery of elementary geometry, another clay tablet from the same period goes even further: it shows that they were familiar with algebraic procedures as well.[2] Known as

Figure 1.3. Plimpton 322

Plimpton 322 (so named because it is number 322 in the G. A. Plimpton Collection at Columbia University; see fig. 1.3), it is a table of four columns, which might at first glance appear to be a record of some commercial transaction. A close scrutiny, however, has disclosed something entirely different: the tablet is a list of *Pythagorean triples*, positive integers (a, b, c) such that $a^2 + b^2 = c^2$. Examples of such triples are (3, 4, 5), (5, 12, 13), and (8, 15, 17). Because of the Pythagorean theorem,[3] every such triple represents a right triangle with sides of integer length.

Unfortunately, the left edge of the tablet is partially missing, but traces of modern glue found on the edges prove that the missing part broke off after the tablet was discovered, raising the hope that one day it may show up on the antiquities market. Thanks to meticulous scholarly research, the missing part has been partially reconstructed, and we can now read the tablet with relative ease. Table 1.1 reproduces the text in modern notation. There are four columns, of which the rightmost, headed by the words “its name” in the original text, merely gives the sequential number of the lines from 1 to 15. The second and third columns (counting from right to left) are headed “solving number of the diagonal” and “solving number of the width,” respectively; that is, they give the length of the diagonal and of the short side of a rectangle, or equivalently, the length of the hypotenuse and the short leg of a right triangle. We will label these columns with the letters c and b, respectively. As

TABLE 1.1
Plimpton 322

$(c/a)^2$	b	c	
[1,59,0,]15	1,59	2,49	1
[1,56,56,]58,14,50,6,15	56,7	3,12,1	2
[1,55,7,]41,15,33,45	1,16,41	1,50,49	3
[1,]5[3,1]0,29,32,52,16	3,31,49	5,9,1	4
[1,]48,54,1,40	1,5	1,37	5
[1,]47,6,41,40	5,19	8,1	6
[1,]43,11,56,28,26,40	38,11	59,1	7
[1,]41,33,59,3,45	13,19	20,49	8
[1,]38,33,36,36	9,1	12,49	9
1,35,10,2,28,27,24,26,40	1,22,41	2,16,1	10
1,33,45	45	1,15	11
1,29,21,54,2,15	27,59	48,49	12
[1,]27,0,3,45	7,12,1	4,49	13
1,25,48,51,35,6,40	29,31	53,49	14
[1,]23,13,46,40	56	53	15

Note: The numbers in brackets are reconstructed.

an example, the first line shows the entries $b = 1{,}59$ and $c = 2{,}49$, which represent the numbers $1 \times 60 + 59 = 119$ and $2 \times 60 + 49 = 169$. A quick calculation gives us the other side as $a = \sqrt{169^2 - 119^2} = \sqrt{14400} = 120$; hence (119, 120, 169) is a Pythagorean triple. Again, in the third line we read $b = 1{,}16{,}41 = 1 \times 60^2 + 16 \times 60 + 41 = 4601$, and $c = 1{,}50{,}49 = 1 \times 60^2 + 50 \times 60 + 49 = 6649$; therefore, $a = \sqrt{6649^2 - 4601^2} = \sqrt{23\,040\,000} = 4800$, giving us the triple (4601, 4800, 6649).

The table contains some obvious errors. In line 9 we find $b = 9{,}1 = 9 \times 60 + 1 = 541$ and $c = 12, 49 = 12 \times 60 + 49 = 769$, and these do not form a Pythagorean triple (the third number a not being an integer). But if we replace the 9,1 by $8{,}1 = 481$, we do indeed get an integer value for a: $a = \sqrt{769^2 - 481^2} = \sqrt{360\,000} = 600$, resulting in the triple (481, 600, 769). It seems that this error was simply a "typo"; the scribe may have been momentarily distracted and carved nine marks into the soft clay instead of eight; and once the tablet dried in the sun, his oversight became part of recorded history.

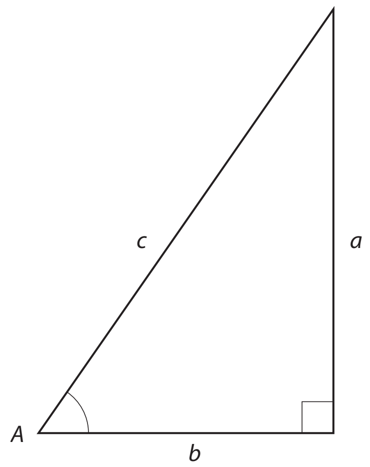

Figure 1.4. The cosecant of an angle: csc $A = c/a$

Again, in line 13 we have $b = 7,12,1 = 7 \times 60^2 + 12 \times 60 + 1 = 25\,921$ and $c = 4,49 = 4 \times 60 + 49 = 289$, and these do not form a Pythagorean triple; but we may notice that 25 921 is the square of 161, and the numbers 161 and 289 do form the triple (161, 240, 289). It seems the scribe simply forgot to take the square root of 25 921. And in row 15 we find $c = 53$, whereas the correct entry should be twice that number, that is, 106 = 1,46, producing the triple (56, 90, 106).[4] These errors leave one with a sense that human nature has not changed over the past four thousand years; our anonymous scribe was no more guilty of negligence than a student begging his or her professor to ignore "just a little stupid mistake" on the exam.[5]

The leftmost column is the most intriguing of all. Its heading again mentions the word "diagonal," but the exact meaning of the remaining text is not entirely clear. However, when one examines its entries a startling fact comes to light: this column gives the square of the ratio c/a, that is, the value of $\csc^2 A$, where A is the angle opposite side a and csc is the cosecant function studied in trigonometry (fig. 1.4). Let us verify this for line 1. We have $b = 1,59 = 119$ and $c = 2,49 = 169$, from which we find $a = 120$. Hence $(c/a)^2 = (169/120)^2 = 1.983$, rounded to three places. And this indeed is the corresponding entry in column 4: $1;59,0,15 = 1 + 59/60 + 0/60^2 + 15/60^3 = 1.983$. (We should note again that the Babylonians did not use a symbol for the "empty slot" and therefore a number could be interpreted in many different ways; the correct interpretation must be deduced from the context. In the example just cited, we assume that the leading 1 stands for units rather than sixties.) The reader may check other entries in this column and confirm that they are equal to $(c/a)^2$.

Several questions immediately arise: Is the order of entries in the table random, or does it follow some hidden pattern? How did the Babylonians find those particular numbers that form Pythagorean triples? And why were they interested in these numbers—and in particular, in the ratio $(c/a)^2$—in the first place? The first question is relatively easy to answer: if we compare the values of $(c/a)^2$ line by line, we discover that they decrease steadily from 1.983 to 1.387, so it seems likely that the order of entries was determined by this sequence. Moreover, if we compute the square root of each entry in column 4—that is, the ratio $c/a = \csc A$—and then find the corresponding angle A, we discover that A *increases* steadily from just above 45° to 58°. It therefore seems that the author of this text was not only interested in finding Pythagorean triples, but also in determining the ratio c/a of the corresponding right triangles. This hypothesis may one day be confirmed if the missing part of the tablet shows up, as it may well contain the missing columns for a and c/a. If so, Plimpton 322 will go down as history's first trigonometric table.

As to how the Babylonian mathematicians found these triples—including such enormously large ones as (4601, 4800, 6649)—there is only one plausible explanation: they must have known an algorithm which, 1,500 years later, would be formalized in Euclid's *Elements*: Let u and v be any two positive integers, with $u > v$; then the three numbers

$$a = 2uv, \quad b = u^2 - v^2, \quad c = u^2 + v^2 \tag{1}$$

form a Pythagorean triple. (If in addition we require that u and v are of opposite parity—one even and the other odd—and that they do not have any common factor other than 1, then (a, b, c) is a *primitive* Pythagorean triple, that is, a, b, and c have no common factor other than 1.) It is easy to confirm that the numbers a, b, and c as given by equations (1) satisfy the equation $a^2 + b^2 = c^2$:

$$\begin{aligned} a^2 + b^2 &= (2uv)^2 + (u^2 - v^2)^2 \\ &= 4u^2v^2 + u^4 - 2u^2v^2 + v^4 \\ &= u^4 + 2u^2v^2 + v^4 \\ &= (u^2 + v^2)^2 = c^2. \end{aligned}$$

The converse of this statement—that *every* Pythagorean triple can be found in this way—is a bit harder to prove (see Appendix B).

Plimpton 322 thus shows that the Babylonians were not only familiar with the Pythagorean theorem, but that they knew the rudiments of number theory and had the computational skills to put the theory into practice—quite remarkable for a civilization that lived a thousand years before the Greeks produced their first great mathematician.

Notes and Sources

1. For a discussion of how the Babylonians approximated the value of $\sqrt{2}$, see Appendix A.

2. The text that follows is adapted from *Trigonometric Delights* and is based on