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Complex Topological *K*-Theory

EFTON PARK

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COMPLEX TOPOLOGICAL K-THEORY

Topological K-theory is a key tool in topology, differential geometry, and index theory, yet this is the first contemporary introduction for graduate students new to the subject. No background in algebraic topology is assumed; the reader need only have taken the standard first courses in real analysis, abstract algebra, and point-set topology.

The book begins with a detailed discussion of vector bundles and related algebraic notions, followed by the definition of K-theory and proofs of the most important theorems in the subject, such as the Bott periodicity theorem and the Thom isomorphism theorem. The multiplicative structure of K-theory and the Adams operations are also discussed and the final chapter details the construction and computation of characteristic classes.

With every important aspect of the topic covered, and exercises at the end of each chapter, this is the definitive book for a first course in topological K-theory.

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Complex Topological *K*-Theory

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To Alex, Connor, Nolan, and Rhonda

Contents

Preface

Topological K-theory first appeared in a 1961 paper by Atiyah and Hirzebruch; their paper adapted the work of Grothendieck on algebraic varieties to a topological setting. Since that time, topological K-theory (which we will henceforth simply call K-theory) has become a powerful and indespensible tool in topology, differential geometry, and index theory. The goal of this book is to provide a self-contained introduction to the subject.

This book is primarily aimed at beginning graduate students, but also for working mathematicians who know little or nothing about the subject and would like to learn something about it. The material in this book is suitable for a one semester course on K-theory; for this reason, I have included exercises at the end of each chapter. I have tried to keep the prerequisites for reading this book to a minimum; I will assume that the reader knows the following:

- Linear Algebra: Vector spaces, bases, linear transformations, similarity, trace, determinant.

- Abstract Algebra: Groups, rings, homomorphisms and isomorphisms, quotients, products.

- Topology: Metric spaces, completeness, compactness and connectedness, local compactness, continuous functions, quotient topology, subspace topology, partitions of unity.

To appreciate many of the motivating ideas and examples in K-theory, it is helpful, but not essential, for the reader to know the rudiments of differential topology, such as smooth manifolds, tangent bundles, differential forms, and de Rham cohomology. In Chapter 4, the theory of characteristic classes is developed in terms of differential forms and de

Rham cohomology; for readers not familiar with these topics, I give a quick introduction at the beginning of that chapter. I do not assume that the reader has any familiarity with homological algebra; the necessary ideas from this subject are developed at the end of Chapter 1.

To keep this book short and as easy to read as possible (especially for readers early in their mathematical careers), I have kept the scope of this book very limited. Only complex K-theory is discussed, and I do not say anything about equivariant K-theory. I hope the reader of this book will be inspired to learn about other versions of K-theory; see the bibliography for suggestions for further reading.

It is perhaps helpful to say a little bit about the philosophy of this book, and how this book differs from other books on K-theory. The fundamental objects of study in K-theory are vector bundles over topological spaces (in the case of K^0) and automorphisms of vector bundles (in the case of K^1). These concepts are discussed at great length in this book, but most of the proofs are formulated in terms of the equivalent notions of idempotents and invertible matrices over Banach algebras of continuous complex-valued functions. This more algebraic approach to K-theory makes the presentation "cleaner"(in my opinion), and also allows readers to see how K-theory can be extended to matrices over general Banach algebras. Because commutativity of the Banach algebras is not necessary to develop K-theory, this generalization falls into an area of mathematics that is often referred to as *noncommutative topology*. On the other hand, there are important aspects of K-theory, such as the existence of operations and multiplicative structures, that do not carry over to the noncommutative setting, and so we will restrict our attention to the K-theory of topological spaces.

I thank my colleagues, friends, and family for their encouragement while I was writing this book, and I especially thank Scott Nollet and Greg Friedman for reading portions of the manuscript and giving me many helpful and constructive suggestions.

1
Preliminaries

The goal of K-theory is to study and understand a topological space X by associating to it a sequence of abelian groups. The algebraic properties of these groups reflect topological properties of X, and the overarching philosophy of K-theory (and, indeed, of all algebraic topology) is that we can usually distinguish groups more easily than we can distinguish topological spaces. There are many variations on this theme, such as homology and cohomology groups of various sorts. What sets K-theory apart from its algebraic topological brethren is that not only can it be defined directly from X, but also in terms of matrices of continuous complex-valued functions on X. For this reason, we devote a significant part of this chapter to the study of matrices of continuous functions.

Our first step is to look at complex vector spaces equipped with an inner product. The reader is presumably familiar with inner products on real vector spaces, but possibly not the complex case. For this reason, we begin with a brief discussion of complex inner product spaces.

1.1 Complex inner product spaces

Definition 1.1.1 *Let $\mathcal{V}$ be a finite-dimensional complex vector space and let $\mathbb{C}$ denote the complex numbers. A* (complex) inner product *on $\mathcal{V}$ is a function $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{C}$ such that for all elements v, v', and v'' in $\mathcal{V}$ and all complex numbers α and β:*

(i) $\langle \alpha v + \beta v', v'' \rangle = \alpha \langle v, v'' \rangle + \beta \langle v', v'' \rangle$;
(ii) $\langle v, \alpha v' + \beta v'' \rangle = \overline{\alpha} \langle v, v' \rangle + \overline{\beta} \langle v, v'' \rangle$;
(iii) $\langle v', v \rangle = \overline{\langle v, v' \rangle}$;
(iv) $\langle v, v \rangle \geq 0$, *with* $\langle v, v \rangle = 0$ *if and only if* $v = 0$.

For each v in $\mathcal{V}$, the nonnegative number $\|v\|_{in} = \sqrt{\langle v, v\rangle}$ is called the magnitude *of v. A vector space equipped with an inner product is called a* (complex) inner product space*. A vector space basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathcal{V}$ is* orthogonal *if $\langle v_j, v_k\rangle = 0$ for $j \neq k$, and* orthonormal *if it is orthogonal and $\|v_k\|_{in} = 1$ for all $1 \leq k \leq n$.*

Proposition 1.1.2 *Every complex inner product space $\mathcal{V}$ admits an orthonormal basis.*

Proof The proof of this proposition follows the same lines as the corresponding fact for real inner product spaces. Start with any vector space basis $\{v_1, v_2, \ldots, v_n\}$ of $\mathcal{V}$ and apply the Gram–Schmidt process inductively to define an orthogonal basis

$$\begin{aligned} v_1' &= v_1 \\ v_2' &= v_2 - \frac{\langle v_2, v_1'\rangle}{\langle v_1', v_1'\rangle} v_1' \\ &\vdots \\ v_n' &= v_n - \frac{\langle v_n, v_1'\rangle}{\langle v_1', v_1'\rangle} v_1' - \frac{\langle v_n, v_2'\rangle}{\langle v_2', v_2'\rangle} v_2' - \cdots - \frac{\left\langle v_n, v_{n-1}'\right\rangle}{\left\langle v_{n-1}', v_{n-1}'\right\rangle} v_{n-1}'. \end{aligned}$$

Then

$$\left\{ \frac{v_1'}{\|v_1'\|_{in}}, \frac{v_2'}{\|v_2'\|_{in}}, \ldots, \frac{v_n'}{\|v_n'\|_{in}} \right\}$$

is an orthonormal basis of $\mathcal{V}$. □

For elements $(z_1, z_2, \ldots, z_n)$ and $(z_1', z_2', \ldots, z_n')$ in the vector space $\mathbb{C}^n$, the formula

$$\langle (z_1, z_2, \ldots, z_n), (z_1', z_2', \ldots, z_n')\rangle = z_1 \overline{z'}_1 + z_2 \overline{z'}_2 + \cdots + z_n \overline{z'}_n$$

defines the *standard inner product* on $\mathbb{C}^n$. For each $1 \leq k \leq n$, define e_k to be the vector that is 1 in the kth component and 0 elsewhere. Then $\{e_1, e_2, \cdots, e_n\}$ is the *standard orthonormal basis* for $\mathbb{C}^n$.

Proposition 1.1.3 (Cauchy–Schwarz inequality) *Let $\mathcal{V}$ be an inner product space. Then*

$$|\langle v, v'\rangle| \leq \|v\|_{in} \, \|v'\|_{in}$$

for all v and v' in $\mathcal{V}$.

Proof If $\langle v, v' \rangle = 0$, the proposition is trivially true, so suppose that $\langle v, v' \rangle \neq 0$. For any α in $\mathbb{C}$, we have

$$\begin{aligned} 0 \leq \|\alpha v + v'\|_{in}^2 &= \langle \alpha v + v', \alpha v + v' \rangle \\ &= |\alpha|^2 \|v\|_{in}^2 + \|v'\|_{in}^2 + \alpha \langle v, v' \rangle + \overline{\alpha \langle v, v' \rangle} \\ &= |\alpha|^2 \|v\|_{in}^2 + \|v'\|_{in}^2 + 2 \operatorname{Re}(\alpha \langle v, v' \rangle), \end{aligned}$$

where $\operatorname{Re}(\alpha \langle v, v' \rangle)$ denotes the real part of $\alpha \langle v, v' \rangle$. Take α to have the form $t\overline{\langle v, v' \rangle} | \langle v, v' \rangle |^{-1}$ for t real. Then the string of equalities above yields

$$\|v\|_{in}^2 t^2 + 2| \langle v, v' \rangle |t + \|v'\|_{in}^2 \geq 0$$

for all real numbers t. This quadratic equation in t has at most one real root, implying that

$$4| \langle v, v' \rangle |^2 - 4 \|v\|_{in}^2 \|v'\|_{in}^2 \leq 0,$$

whence the proposition follows. □

Proposition 1.1.4 (Triangle inequality) *Let $\mathcal{V}$ be an inner product space. Then*

$$\|v + v'\|_{in} \leq \|v\|_{in} + \|v'\|_{in}$$

for all v and v' in $\mathcal{V}$.

Proof Proposition 1.1.3 gives us

$$\begin{aligned} \|v + v'\|_{in}^2 &= \langle v + v', v + v' \rangle \\ &= \langle v, v \rangle + \langle v, v' \rangle + \langle v', v \rangle + \langle v', v' \rangle \\ &= \|v\|_{in}^2 + \|v'\|_{in}^2 + 2 \operatorname{Re} \langle v, v' \rangle \\ &\leq \|v\|_{in}^2 + \|v'\|_{in}^2 + 2| \langle v, v' \rangle | \\ &\leq \|v\|_{in}^2 + \|v'\|_{in}^2 + 2 \|v\|_{in} \|v'\|_{in} \\ &= (\|v\|_{in} + \|v'\|_{in})^2 . \end{aligned}$$

We get the desired result by taking square roots. □

Definition 1.1.5 *Let $\mathcal{V}$ be an inner product space and let $\mathcal{W}$ be a vector subspace of $\mathcal{V}$. The vector subspace*

$$\mathcal{W}^\perp = \{v \in \mathcal{V} : \langle v, w \rangle = 0 \text{ for all } w \in \mathcal{W}\}$$

is called the orthogonal complement *of $\mathcal{W}$ in $\mathcal{V}$.*

Proposition 1.1.6 *Let $\mathcal{V}$ be an inner product space and suppose that $\mathcal{W}$ is a vector subspace of $\mathcal{V}$. Then $\mathcal{V} \cong \mathcal{W} \oplus \mathcal{W}^\perp$.*

Proof If u is in the intersection of $\mathcal{W}$ and $\mathcal{W}^\perp$, then $\|u\|_{in} = \langle u, u \rangle = 0$, whence $u = 0$. Take v in $\mathcal{V}$, and suppose that $v = w_1 + w_1^\perp = w_2 + w_2^\perp$ for w_1, w_2 in $\mathcal{W}$ and $w_1^\perp, w_2^\perp$ in $\mathcal{W}^\perp$. Then $w_1 - w_2 = w_2^\perp - w_1^\perp$ is in $\mathcal{W} \cap \mathcal{W}^\perp$ and therefore we must have $w_1 = w_2$ and $w_1^\perp = w_2^\perp$. To show that such a decomposition of v actually exists, choose an orthonormal basis $\{w_1, w_2, \dots, w_m\}$ of $\mathcal{W}$ and set $w = \sum_{k=1}^m \langle v, w_k \rangle w_k$. Clearly w is in $\mathcal{W}$. Moreover, for every $1 \le j \le m$, we have

$$\begin{aligned}\langle v - w, w_j \rangle &= \langle v, w_j \rangle - \langle w, w_j \rangle \\ &= \langle v, w_j \rangle - \sum_{k=1}^m \langle v, w_k \rangle \langle w_k, w_j \rangle \\ &= \langle v, w_j \rangle - \langle v, w_j \rangle = 0,\end{aligned}$$

which implies that $v - w$ is in $\mathcal{W}^\perp$. □

Definition 1.1.7 *Let $\mathcal{V}$ be an inner product space, let $\mathcal{W}$ be a vector subspace of $\mathcal{V}$, and identify $\mathcal{V}$ with $\mathcal{W} \oplus \mathcal{W}^\perp$. The linear map $\mathsf{P} : \mathcal{V} \longrightarrow \mathcal{W}$ given by $\mathsf{P}(w, w^\perp) = w$ is called the* orthogonal projection *of $\mathcal{V}$ onto $\mathcal{W}$.*

We close this section with a notion that we will need in Chapter 3.

Proposition 1.1.8 *Let $\mathcal{V}$ and $\mathcal{W}$ be inner product spaces, and suppose that $\mathsf{A} : \mathcal{V} \longrightarrow \mathcal{W}$ is a vector space homomorphism; i.e., a linear map. Then there exists a unique vector space homomorphism $\mathsf{A}^* : \mathcal{W} \longrightarrow \mathcal{V}$, called the* adjoint *of A, for which $\langle \mathsf{A}v, w \rangle = \langle v, \mathsf{A}^* w \rangle$ for all v in $\mathcal{V}$ and w in $\mathcal{W}$.*

Proof Fix orthonormal bases $\{e_1, e_2, \dots, e_m\}$ and $\{f_1, f_2, \dots, f_n\}$ for $\mathcal{V}$ and $\mathcal{W}$ respectively. For each $1 \le i \le m$, write $\mathsf{A}e_i$ in the form $\mathsf{A}e_i = \sum_{j=1}^n a_{ji} f_j$ and set $\mathsf{A}^* f_j = \sum_{i=1}^m \overline{a}_{ji} e_i$ Then

$$\langle \mathsf{A}e_i, f_j \rangle = a_{ji} = \langle e_i, \mathsf{A}^* f_j \rangle$$

for all i and j, and parts (i) and (ii) of Definition 1.1.1 imply that $\langle \mathsf{A}v, w \rangle = \langle v, \mathsf{A}^* w \rangle$ for all v in $\mathcal{V}$ and w in $\mathcal{W}$.

To show uniqueness, suppose that $B : \mathcal{W} \longrightarrow \mathcal{V}$ is a linear map with the property that $\langle \mathsf{A}v, w \rangle = \langle v, \mathsf{A}^* w \rangle = \langle v, \mathsf{B}w \rangle$ for all v and w. Then $\langle v, (\mathsf{A}^* - \mathsf{B})w \rangle = 0$, and by taking $v = (\mathsf{A}^* - \mathsf{B})w$ we see that $(\mathsf{A}^* - \mathsf{B})w = 0$ for all w. Thus $\mathsf{A}^* = \mathsf{B}$.

To prove that A^* is a vector space homomorphism, note that

$$\begin{aligned}\langle v, \mathsf{A}^*(\alpha w + \beta w')\rangle &= \langle \mathsf{A}v, \alpha w + \beta w'\rangle \\ &= \overline{\alpha}\,\langle \mathsf{A}v, w\rangle + \overline{\beta}\,\langle \mathsf{A}v, w'\rangle \\ &= \overline{\alpha}\,\langle v, \mathsf{A}^* w\rangle + \overline{\beta}\,\langle v, \mathsf{A}^* w'\rangle \\ &= \langle v, \alpha\mathsf{A}^* w + \beta\mathsf{A}^* w'\rangle\end{aligned}$$

for all v in $\mathcal{V}$, all w and w' in $\mathcal{W}$, and all complex numbers α and β. Therefore $\mathsf{A}^*(\alpha w + \beta w') = \alpha\mathsf{A}^* w + \beta\mathsf{A}^* w'$. □

Proposition 1.1.9 *Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be inner product spaces, and suppose that $\mathsf{A} : \mathcal{U} \longrightarrow \mathcal{V}$ and $\mathsf{B} : \mathcal{V} \longrightarrow \mathcal{W}$ are vector space homomorphisms. Then:*

(i) $(\mathsf{A}^*)^* = \mathsf{A}$*;*
(ii) $\mathsf{A}^*\mathsf{B}^* = \mathsf{B}\mathsf{A}^*$*;*
(iii) A^* *is an isomorphism if and only if* A *is an isomorphism.*

Proof The uniqueness of the adjoint and the equalities

$$\langle \mathsf{A}^* v, u\rangle = \overline{\langle u, \mathsf{A}^* v\rangle} = \overline{\langle \mathsf{A}v, u\rangle} = \langle v, \mathsf{A}u\rangle$$

for all u in $\mathcal{U}$ and v in $\mathcal{V}$ give us (i), and the fact that

$$\langle \mathsf{B}\mathsf{A}u, w\rangle = \langle \mathsf{A}u, \mathsf{B}^* w\rangle = \langle u, \mathsf{A}^*\mathsf{B}^* w\rangle$$

for all u in $\mathcal{U}$ and w in $\mathcal{W}$ establishes (ii).

If A is an isomorphism, then $\mathcal{U}$ and $\mathcal{V}$ have the same dimension and thus we can show A^* is an isomorphism by showing that A^* is injective. Suppose that $\mathsf{A}^* v = 0$. Then $0 = \langle u, \mathsf{A}^* v\rangle = \langle \mathsf{A}u, v\rangle$ for all u in $\mathcal{U}$. But A is surjective, so $\langle v, v\rangle = 0$, whence $v = 0$ and A^* is injective. The reverse implication in (iii) follows from replacing A by A^* and invoking (i). □

1.2 Matrices of continuous functions

Definition 1.2.1 *Let X be a compact Hausdorff space. The set of all complex-valued continuous functions on X is denoted $C(X)$. If m and n are natural numbers, the set of m by n matrices with entries in $C(X)$ is written $\mathrm{M}(m, n, C(X))$. If $m = n$, we shorten $\mathrm{M}(m, n, C(X))$ to $\mathrm{M}(n, C(X))$.*

Each of these sets of matrices has the structure of a *Banach space*:

Definition 1.2.2 *A* Banach space *is a vector space $\mathcal{V}$ equipped with a function $\|\cdot\| : \mathcal{V} \longrightarrow [0, \infty)$, called a* norm*, satisfying the following properties:*

(i) *For all v and v' in $\mathcal{V}$ and α in $\mathbb{C}$:*

  (a) $\|\alpha v\| = |\alpha|\ \|v\|$;
  (b) $\|v + v'\| \leq \|v\| + \|v'\|$.

(ii) *The formula $d(v, v') = \|v - v'\|$ is a distance function on $\mathcal{V}$ and $\mathcal{V}$ is complete with respect to d.*

The topology generated by $d(v, w) = \|v - w\|$ is called the *norm topology* on $\mathcal{V}$; an easy consequence of the axioms is that scalar multiplication and addition are continuous operations in the norm topology.

Note that when X is a point we can identify $C(X)$ with $\mathbb{C}$.

Lemma 1.2.3 *For all natural numbers m and n, the set of matrices $\mathrm{M}(m, n, \mathbb{C})$ is a Banach space in the operator norm*

$$\begin{aligned} \|\mathsf{A}\|_{op} &= \sup\left\{ \frac{\|\mathsf{A}\vec{z}\|_{in}}{\|\vec{z}\|_{in}} : \vec{z} \in \mathbb{C}^n, \vec{z} \neq 0 \right\} \\ &= \sup\left\{ \|\mathsf{A}\vec{z}\|_{in} : \|\vec{z}\|_{in} = 1 \right\}. \end{aligned}$$

Proof For each A in $\mathrm{M}(m, n, \mathbb{C})$, we have

$$\begin{aligned} \|\mathsf{A}\|_{op} &= \sup\left\{ \frac{\|\mathsf{A}\vec{w}\|_{in}}{\|\vec{w}\|_{in}} : \vec{w} \neq 0 \right\} \\ &= \sup\left\{ \left\| \mathsf{A}\left(\frac{\vec{w}}{\|\vec{w}\|_{in}} \right) \right\|_{in} : \vec{w} \neq 0 \right\} \\ &= \sup\{ \|\mathsf{A}\vec{z}\|_{in} : \|\vec{z}\|_{in} = 1 \}, \end{aligned}$$

and thus the two formulas for the operator norm agree. The equation $\|\mathsf{A}(\lambda\vec{z})\|_{in} = |\lambda|\ \|\mathsf{A}\vec{z}\|_{in}$ yields $\|\lambda\mathsf{A}\|_{op} = |\lambda|\ \|\mathsf{A}\|_{op}$, and the inequality $\|\mathsf{A}_1 + \mathsf{A}_2\|_{op} \leq \|\mathsf{A}_1\|_{op} + \|\mathsf{A}_2\|_{op}$ is a consequence of Proposition 1.1.4.

To show completeness, let $\{\mathsf{A}_k\}$ be a Cauchy sequence in $\mathrm{M}(m, n, \mathbb{C})$. Then for each $\vec{z}$ in $\mathbb{C}^n$, the sequence $\{\mathsf{A}_k\vec{z}\}$ in $\mathbb{C}^m$ is Cauchy and therefore has a limit. Continuity of addition and scalar multiplication imply that the function $\vec{z} \mapsto \lim_{k\to\infty} \mathsf{A}_k\vec{z}$ defines a linear map from $\mathbb{C}^n$ to $\mathbb{C}^m$. Take the standard vector space bases of $\mathbb{C}^m$ and $\mathbb{C}^n$ and let A denote the corresponding matrix in $\mathrm{M}(m, n, \mathbb{C})$; we must show that $\{\mathsf{A}_k\}$ converges in norm to A.

Fix $\epsilon > 0$ and choose a natural number N with the property that $\|\mathsf{A}_k - \mathsf{A}_l\|_{op} < \epsilon/2$ for $k, l > N$. Then

$$\begin{aligned}\|\mathsf{A}_k \vec{z} - \mathsf{A}\vec{z}\|_{in} &= \lim_{l\to\infty} \|\mathsf{A}_k\vec{z} - \mathsf{A}_l\vec{z}\|_{in} \\ &\le \limsup_{l\to\infty} \|\mathsf{A}_k - \mathsf{A}_l\|_{op} \|\vec{z}\|_{in} \\ &< \epsilon \|\vec{z}\|_{in}\end{aligned}$$

for all $\vec{z} \neq 0$ in $\mathbb{C}^n$. Hence $\|\mathsf{A}_k - \mathsf{A}\|_{op} < \epsilon$ for $k > N$, and the desired conclusion follows. □

For the case where $m = n = 1$, the norm on each z in $\mathrm{M}(1, \mathbb{C}) = \mathbb{C}$ defined in Lemma 1.2.3 is simply the modulus $|z|$.

Proposition 1.2.4 *Let X be a compact Hausdorff space and let m and n be natural numbers. Then $\mathrm{M}(m, n, C(X))$ is a Banach space in the supremum norm*

$$\|\mathsf{A}\|_\infty = \sup\{\|\mathsf{A}(x)\|_{op} : x \in X\}.$$

Proof The operations of pointwise matrix addition and scalar multiplication make $\mathrm{M}(m, n, C(X))$ into a vector space. Note that

$$\begin{aligned}\|\alpha\mathsf{A}\|_\infty &= \sup\{\|\alpha\mathsf{A}(x)\|_{op} : x \in X\} \\ &= \sup\{|\alpha|\ \|\mathsf{A}(x)\|_{op} : x \in X\} = |\alpha|\ \|\mathsf{A}\|_\infty\end{aligned}$$

and

$$\begin{aligned}\|\mathsf{A} + \mathsf{B}\|_\infty &= \sup\{\|\mathsf{A}(x) + \mathsf{B}(x)\|_{op} : x \in X\} \\ &\le \sup\{\|\mathsf{A}(x)\|_{op} : x \in X\} + \sup\{\|\mathsf{B}(x)\|_{op} : x \in X\} \\ &= \|\mathsf{A}\|_\infty + \|\mathsf{B}\|_\infty\end{aligned}$$

for all A and B in $\mathrm{M}(m, n, C(X))$ and α in $\mathbb{C}$, and thus $\|\cdot\|_\infty$ is indeed a norm.

To check that $\mathrm{M}(m, n, C(X))$ is complete in the supremum norm, let $\{\mathsf{A}_k\}$ be a Cauchy sequence in $\mathrm{M}(m, n, C(X))$. For each x in X, the sequence $\{\mathsf{A}_k(x)\}$ is a Cauchy sequence in $\mathrm{M}(m, n, \mathbb{C})$ and therefore by Lemma 1.2.3 has a limit $\mathsf{A}(x)$. To show that this construction yields an element A in $\mathrm{M}(m, n, C(X))$, we need to show that the (i, j) entry A_{ij} of A is in $C(X)$ for all $1 \le i \le m$ and $1 \le j \le n$.

Fix i and j. To simplify notation, let $f = \mathsf{A}_{ij}$, and for each natural number k, let $f_k = (\mathsf{A}_k)_{ij}$; note that each f_k is an element of $C(X) =$

$\mathrm{M}(1, C(X))$. Endow $\mathbb{C}^m$ and $\mathbb{C}^n$ with their standard orthonormal bases. For each x in X, we have $f_k(x) = \langle \mathsf{A}_k(x)e_j, e_i\rangle$ and $f(x) = \langle \mathsf{A}(x)e_j, e_i\rangle$. Then for all natural numbers k and l, Proposition 1.1.3 gives us

$$\begin{aligned}
|f_k(x) - f_l(x)| &= |\langle \mathsf{A}_k e_j, e_i\rangle - \langle \mathsf{A}_l e_j, e_i\rangle| \\
&= |\langle (\mathsf{A}_k - \mathsf{A}_l)e_j, e_i\rangle| \\
&\leq \|(\mathsf{A}_k - \mathsf{A}_l)e_j\|_{in} \, \|e_i\|_{in} \\
&\leq \|\mathsf{A}_k - \mathsf{A}_l\|_{op} \, \|e_j\|_{in} \, \|e_i\|_{in} \\
&= \|\mathsf{A}_k - \mathsf{A}_l\|_{op} \,.
\end{aligned}$$

Therefore $\{f_k(x)\}$ is Cauchy and thus converges to $f(x)$.

To show that f is continuous, fix $\epsilon > 0$ and choose a natural number M with the property that $\|f_k - f_M\|_\infty < \epsilon/3$ for all $k > M$. Next, choose x' in X and let U be an open neighborhood of x' with the property that $|f_M(x') - f_M(x)| < \epsilon/3$ for all x in U. Then

$$\begin{aligned}
|f(x') - f(x)| &\leq |f(x') - f_M(x')| + |f_M(x') - f_M(x)| + |f_M(x) - f(x)| \\
&< \lim_{k\to\infty} |f_k(x') - f_M(x')| + \frac{\epsilon}{3} + \lim_{k\to\infty} |f_M(x) - f_k(x)| \\
&\leq \limsup_{k\to\infty} \|f_k - f_M\|_\infty + \frac{\epsilon}{3} + \limsup_{k\to\infty} \|f_M - f_k\|_\infty \\
&< \epsilon
\end{aligned}$$

for all $k > M$ and x in U, whence f is continuous.

The last step is to show that the sequence $\{\mathsf{A}_k\}$ converges in the supremum norm to A. Fix $\epsilon > 0$ and choose a natural number N so large that $\|\mathsf{A}_k - \mathsf{A}_l\|_\infty < \epsilon/2$ whenever k and l are greater than N. Then

$$\begin{aligned}
\|\mathsf{A}_k(x) - \mathsf{A}(x)\|_{op} &= \lim_{l\to\infty} \|\mathsf{A}_k(x) - \mathsf{A}_l(x)\|_{op} \\
&\leq \limsup_{l\to\infty} \|\mathsf{A}_k - \mathsf{A}_l\|_\infty \\
&\leq \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

for $k > N$ and x in X. This inequality holds for each x in X and therefore

$$\lim_{k\to\infty} \|\mathsf{A}_k - \mathsf{A}\|_\infty = 0.$$

□

In this book we will work almost exclusively with square matrices.

This will allow us to endow our Banach spaces $\mathrm{M}(n, C(X))$ with an additional operation that gives us an *algebra*:

Definition 1.2.5 *An* algebra *is a vector space $\mathcal{V}$ equipped with a multiplication $\mathcal{V} \times \mathcal{V} \longrightarrow \mathcal{V}$ that makes $\mathcal{V}$ into a ring, possibly without unit, and satisfies $\alpha(vv') = (\alpha v)v' = v(\alpha v')$ for all v and v' in $\mathcal{V}$ and α in $\mathbb{C}$. If in addition $\mathcal{V}$ is a Banach space such that $\|vv'\| \leq \|v\| \, \|v'\|$ for all v and v' in $\mathcal{V}$, we call $\mathcal{V}$ a* Banach algebra.

Proposition 1.2.6 *Let X be a compact Hausdorff space and let n be a natural number. Then $\mathrm{M}(n, C(X))$ is a Banach algebra with unit under matrix multiplication.*

Proof Proposition 1.2.4 tells us that $\mathrm{M}(n, C(X))$ is a Banach space, and the reader can check that $\mathrm{M}(n, C(X))$ is an algebra under pointwise matrix multiplication. To complete the proof, observe that

$$\begin{aligned}
\|\mathsf{AB}\|_\infty &= \sup\{\|\mathsf{A}(x)\mathsf{B}(x)\|_{op} : x \in X\} \\
&\leq \sup\{\|\mathsf{A}(x)\|_{op} : x \in X\} \sup\{\|\mathsf{B}(x)\|_{op} : x \in X\} \\
&= \|\mathsf{A}\|_\infty \, \|\mathsf{B}\|_\infty
\end{aligned}$$

for all A and B in $\mathrm{M}(n, C(X))$. $\square$

Before we leave this section, we establish some notation. We will write the zero matrix and the identity matrix in $\mathrm{M}(n, C(X))$ as 0_n and I_n respectively when we want to highlight the matrix size. Next, suppose that B is an element of $\mathrm{M}(n, C(X))$ and that A is a subspace of X. Then B restricts to define an element of $\mathrm{M}(n, C(A))$; we will use the notation $\mathsf{B}|A$ for this restricted matrix.

Finally, we will often be working with matrices that have block diagonal form, and it will be convenient to have a more compact notation for such matrices. Given matrices A and B in $\mathrm{M}(m, C(X))$ and $\mathrm{M}(n, C(X))$ respectively we set

$$\operatorname{diag}(\mathsf{A}, \mathsf{B}) = \begin{pmatrix} \mathsf{A} & 0 \\ 0 & \mathsf{B} \end{pmatrix} \in \mathrm{M}(m + n, C(X)).$$

We will use the obvious extension of this notation for matrices that are comprised of more than two blocks.

1.3 Invertibles

Invertible matrices play several important roles in defining K-theory groups of a topological space. In this section we will prove various results about such matrices.

Definition 1.3.1 *Let X be compact Hausdorff. For each natural number n, the group of invertible elements of* $\mathrm{M}(n, C(X))$ *under multiplication is denoted* $\mathrm{GL}(n, C(X))$.

We begin by defining an important family of invertible matrices.

Definition 1.3.2 *Let n be a natural number. For every $0 \le t \le 1$, define the matrix*

$$\mathsf{Rot}(t) = \begin{pmatrix} \cos(\frac{\pi t}{2})I_n & -\sin(\frac{\pi t}{2})I_n \\ \sin(\frac{\pi t}{2})I_n & \cos(\frac{\pi t}{2})I_n \end{pmatrix}.$$

Note that for each t, the matrix $\mathsf{Rot}(t)$ is invertible with inverse

$$\mathsf{Rot}^{-1}(t) = \begin{pmatrix} \cos(\frac{\pi t}{2})I_n & \sin(\frac{\pi t}{2})I_n \\ -\sin(\frac{\pi t}{2})I_n & \cos(\frac{\pi t}{2})I_n \end{pmatrix}.$$

Proposition 1.3.3 *Let X be a compact Hausdorff space, let n be a natural number, and suppose S and T are elements of* $\mathrm{GL}(n, C(X))$. *Then*

$$\mathrm{diag}(\mathsf{S}, I_n)\mathsf{Rot}(t)\,\mathrm{diag}(\mathsf{T}, I_n)\mathsf{Rot}^{-1}(t)$$

is a homotopy in $\mathrm{GL}(2n, C(X))$ *from* $\mathrm{diag}(\mathsf{ST}, I_n)$ *to* $\mathrm{diag}(\mathsf{S}, \mathsf{T})$.

Proof Compute. □

Proposition 1.3.4 *Let X be a compact Hausdorff space, let n be a natural number, and suppose that S in* $\mathrm{M}(n, C(X))$ *has the property that* $\|I_n - \mathsf{S}\|_\infty < 1$. *Then S is in* $\mathrm{GL}(n, C(X))$ *and*

$$\|\mathsf{S}^{-1}\|_\infty \le \frac{1}{1 - \|I_n - \mathsf{S}\|_\infty}.$$