CAMBRIDGE TRACTS IN MATHEMATICS

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FORCING IDEALIZED

JINDŘICH ZAPLETAL

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Introduction

1.1 Welcome

This book reports on the state of a research program that I initiated in 1999. It connects the practice of proper forcing introduced by Shelah [64] with the study of various σ-ideals on Polish spaces from the point of view of abstract analysis, descriptive set theory, measure theory, etc. It turns out that the connection is far richer than I dared to imagine in the beginning. Its benefits include theorems about methodology of forcing as well as isolation of new concepts in measure theory or abstract analysis. It is my sincere hope that this presentation will help to draw attention from experts from these fields and to bring set theory and forcing closer to the more traditional parts of mathematics.

The book uses several theorems and proofs from my earlier papers; in several cases I coauthored these papers with others. The first treatment of the subject in [83] is superseded here on many accounts, but several basic theorems and proofs remain unchanged. The papers [18], [67], [82], [86], and [87] are incorporated into the text, in all cases reorganized and with significant improvements.

Many mathematicians helped to make this book what it is. Thanks should go in the first place to Bohuslav Balcar for his patient listening and enlightening perspective of the subject. Vladimir Kanovei introduced me to effective descriptive set theory. Ilijas Farah helped me with many discussions on measure theory. Joerg Brendle and Peter Koepke allowed me to present the subject matter in several courses, and that greatly helped organize my thoughts and results. Last but not least, the influence of the mathematicians I consider my teachers (Thomas Jech, Hugh Woodin, and Alexander Kechris) is certainly apparent in the text.

I enjoyed financial support through NSF grant DMS 0300201 and grant GA ČR 201-03-0933 of the Grant Agency of Czech Republic as I wrote this book.

1.2 Navigation

This is not a textbook. The complexity of the subject is such that it is impossible to avoid forward references and multiple statements of closely related results, and to keep the book organized in a logical structure at the same time. As a result, the linear reading of the book will be necessarily interspersed with some page flipping. This section should help the reader to find the subjects he is most interested in.

Chapter 2 provides the basic definitions, restatements of properness, and basic implications of properness, such as the reading of reals in the generic extension as images of the generic point under ground model coded Borel functions. Every reader should start with this chapter. A sample theorem:

Theorem 1.2.1. *Suppose that I is a σ-ideal on a Polish space X. The forcing P_I of I-positive Borel sets ordered by inclusion adds a single point $\dot{x}_{gen} \in X$ such that a set B belongs to the generic filter if and only if it contains the generic point $\dot{x}_{gen}$.*

Chapter 3 investigates the possible finer forcing properties of the forcings of the form P_I. These divide into three basic groups. The first group is that of Fubini forcing properties, introduced in Section 3.2. These correspond to the classical preservation properties such as the bounding property or preservation of outer Lebesgue measure. A sample theorem:

Theorem 1.2.2. *Suppose that I is a σ-ideal on a Polish space X such that the forcing P_I is proper. The following are equivalent:*

1. *P_I is bounding;*
2. *for every Polish topology τ on the space X that yields the same Borel structure as the original one, every Borel I-positive set contains a τ-compact I-positive subset.*

The second group of properties is entirely absent in the combinatorial treatment of forcings. These are the descriptive set theoretic properties of the ideals, represented by the various dichotomies of Section 3.9 and the $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ property. The dichotomies are constantly invoked in the proofs of absoluteness theorems and preservation theorems. The $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ property of ideals allows ZFC treatment of such operations as the countable support iteration, product, and illfounded iteration, with a more definite understanding of the underlying issues. A sample theorem:

Theorem 1.2.3. *(LC+CH) Suppose that I is a σ-ideal generated by a universally Baire collection of analytic sets such that every I-positive $\mathbf{\Sigma}^1_2$ set has an I-positive Borel subset. If the forcing P_I is ω-proper then every function $f \in 2^{\omega_1}$ in the extension either is in the ground model or has a countable initial segment which is not in the ground model.*

Here, LC denotes a suitable large cardinal assumptions, as explained in the next section.

The third group of properties is connected with determinacy of games on Boolean algebras. A number of forcing properties can be expressed in terms of infinitary games of the poset P_I which are determined in the definable context. The games are usually variations on standard fusion arguments, and the winning strategies are a necessary tool in the treatment of product forcing, illfounded iteration, and other subjects. A sample application:

Theorem 1.2.4. *(LC) Suppose that I is a universally Baire σ-ideal on a Polish space X such that the forcing P_I is proper. The following are equivalent:*

1. *P_I preserves Baire category;*
2. *there is a collection T of Polish topologies on the space X such that I is the collection of all sets which are τ-meager for every topology $\tau \in T$.*

Chapter 4 gives a number of classes of σ-ideals I for which I can prove that the forcing P_I is proper. While the presentation is based on a joint paper with Ilijas Farah [18], it is nevertheless greatly expanded. There are two very distinct groups of ideals in this respect: the ideals satisfying the first dichotomy, whose treatment occupies almost the whole chapter, and the ideals that do not satisfy the first dichotomy, treated in Section 4.7. It seems that the former group is much larger. Its treatment is divided into several very populous subgroups, each treated in its own section. These subgroups are typically connected with a basic underlying idea from abstract analysis, such as capacities or Hausdorff measures. The sections are all very much alike: first comes the definition of the class of ideals, then the properness theorem, then the dichotomy theorem (which, mysteriously, is always proved in the same way as properness), then several general theorems regarding the finer forcing properties of the ideals. The section closes with a list of examples. A sample result:

Theorem 1.2.5. *Suppose that ϕ is an outer regular subadditive capacity on a Polish space X. Let $I = \{A \subset X : \phi(A) = 0\}$. Then:*

1. *if the capacity is stable then the forcing P_I is proper;*
2. *if the forcing P_I is proper and the capacity is strongly subadditive then the forcing P_I preserves outer Lebesgue measure;*
3. *if the forcing P_I is proper and the capacity is Ramsey then the forcing does not add splitting reals;*
4. *every capacity used in potential theory is stable.*

My original hope that the idealization of forcings would closely relate to the creature forcing technology [58] proved to be naive; the symmetric difference of the two approaches turned out to be quite large. Nevertheless, in several cases I could identify a precise correspondence between a class of ideals and a class of creature forcings.

Chapter 5 relates operations on ideals with operations on forcings. The key case here is that of the countable support iteration which corresponds to a transfinite Fubini product of ideals, Section 5.1. The other operations I can handle are side-by-side product with a great help from determinacy of games on Boolean algebras, the illfounded iteration, which provides a treatment dual to and more general than that of [43], the towers of ideals which is a method of obtaining forcings adding objects more complex than just reals, and the union of ideals, which forcingwise is an entirely mysterious operation. A sample theorem:

Theorem 1.2.6. *(LC) Suppose that $I_\alpha : \alpha \in \kappa$ is a collection of universally Baire σ-ideals on some Polish spaces such that the forcings P_{I_α} are all proper and preserve Baire category bases. Then the countable support side-by-side product of these forcings is proper as well and preserves Baire category bases. In addition, the ideals satisfy a rectangular Ramsey property.*

Chapter 6 is probably the primary reason why a forcing practitioner may want to read this book; however its methods are entirely incomprehensible without the reading of the previous chapters. There are several separate sections.

Section 6.1 contains the absoluteness results which originally motivated the work on the subject of this book. There are many theorems varying in the exact large cardinal strength necessary and in the class of problems they can handle, but on the heuristic level they all say the same thing. If $\mathfrak{x}$ is a simply definable cardinal invariant and I is a σ-ideal such that the forcing P_I is proper, then if the inequality $\mathfrak{x} < \mathtt{cov}^*(I)$ holds in some extension then it holds in the iterated P_I extension. Moreover, there is a forcing axiom CPA(I) which holds in the iterated P_I extension and which then must directly imply the inequality $\mathfrak{x} < \mathtt{cov}^*(I)$. The CPA-type axioms have been defined independently in the work of Ciesielski and Pawlikowski [9] in an effort to axiomatize the iterated Sacks model. A sample theorem:

Theorem 1.2.7. *(LC) Suppose that $\mathfrak{x}$ is a tame cardinal invariant and $\mathfrak{x} < \mathfrak{c}$ holds in some forcing extension. Then $\aleph_1 = \mathfrak{x} < \mathfrak{c}$ holds in every forcing extension satisfying CPA; in particular it holds in the iterated Sacks model.*

Section 6.2 considers the duality theorems. These are theorems that partially confirm the old duality heuristic: if I, J are σ-ideals and the inequality $\mathtt{cov}(I) \leq \mathtt{add}(J)$ is provable in ZFC, then so should be its dual inequality $\mathtt{non}(I) \geq \mathtt{cof}(J)$. This is really completely false, but several theorems can be proved that rescue nontrivial pieces of this unrealistic expectation. This is the one part of this book where the combinatorics of uncountable cardinals actually enters the computation of inequalities between cardinal invariants, with considerations involving various pcf and club guessing structures. A sample theorem:

Theorem 1.2.8. *Suppose that J is a σ-ideal on a Polish space generated by a universally Baire collection of analytic sets. If ZFC+LC proves* $\mathtt{cov}(I) = \mathfrak{c}$ *then ZFC+LC proves* $\mathtt{non}(I) \leq \aleph_2$.

Section 6.3 gives a long list of preservation theorems for the countable support iteration of definable forcings. Compared to the combinatorial approach of Shelah [64], these theorems have several advantages: they connect well with the motivating problems in abstract analysis, and they have an optimal statement. Among their disadvantages I must mention the restriction to definable forcings and the necessity of large cardinal assumptions for a full strength version. Many of the preservation theorems of this section have no combinatorial counterpart. A sample result:

Theorem 1.2.9. *(LC) Suppose that I is a universally Baire σ-ideal on a Polish space X such that the forcing P_I is proper. Suppose that ϕ is a strongly subadditive capacity. If P_I forces every set to have the same ϕ-mass in the ground model as it has in the extension, then even the countable support iterations of the forcing P_I have the same property.*

1.3 Notation

My notation follows the set theoretic standard of [29]. If T is a tree of finite sequences ordered by extension then $[T]$ denotes the set of all infinite paths through that tree; if $T \subset 2^{<\omega}$ then $[T]$ is a closed subset of the space 2^ω. If X, Y are Polish spaces and $A \subset X \times Y$ is a set then the expression $\mathrm{proj}(A)$ denotes the set $\{x \in X : \exists y \in Y \ \langle x, y\rangle \in A\}$, for a point $x \in X$ the expression A_x stands for the vertical section $\{y \in Y : \langle x, y\rangle \in A\}$, and for a point $y \in Y$ the expression A^y stands for the horizontal section $\{x \in X : \langle x, y\rangle \in A\}$. For a Polish space X, $K(X)$ is the hyperspace of its compact subsets with the Vietoris topology and $P(X)$ is the space of probability Borel measures on X. The expression $\mathcal{B}(X)$ denotes the collection of all Borel subsets of the space X. The word “measure” refers to a σ-additive Borel measure. If a set function is σ-subadditive rather than σ-additive then I use the word “submeasure.” The value of a measure (submeasure, capacity) ϕ at a set B is referred to as the ϕ-mass of the set B. A tower of models is a sequence $\langle M_\alpha : \alpha \in \beta\rangle$ where β is an ordinal and M_α’s are elementary submodels of some large structure (typically $\langle H_\theta, \in\rangle$ for a suitable large cardinal θ) such that

$\alpha' \in \alpha \in \beta$ implies $M_{\alpha'} \in M_\alpha$. The tower is continuous if for limit ordinals $\alpha \in \beta$, $M_\alpha = \bigcup_{\gamma\in\alpha} M_\gamma$.

One important deviation from the standard set theoretical usage is the liberal use of large cardinal assumptions. In order to prove suitably general theorems of a statement that is easy to understand and refer to, I frequently have to resort to a large cardinal assumption of this or that kind. There are only three classes of applications of large cardinal assumptions in this book–absoluteness, determinacy of (long and complex) games, and definable uniformization. The minimum large cardinal necessary for each of these applications is different, sometimes difficult to state, sometimes unknown, and invariably completely irrelevant for the goals of this book; the existence of a supercompact cardinal is always sufficient. As a result, I decided to denote the use of large cardinal assumptions by a simple (LC) preceding the statement of the theorems. For most but not all *specific* applications of the general theorems in this book the large cardinal assumption can be eliminated by manual construction of all the winning strategies and uniformization functions necessary. At least in one case (the countable support iteration of Laver forcing) I made an effort to show that the key dichotomy requires a large cardinal assumption, and in the rather restrictive case of $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ ideals almost all general theorems in this book are proved in ZFC.

The labeling of the various claims in this book is indicative of their position and function. Facts are statements that are proved elsewhere, and I will not restate their proofs. Theorems are quotable self-standing statements, ready for use in the reader's work. Propositions are self-standing statements referred to at some other, possibly quite distant, place in the book. Finally, claims and lemmas appear in the proofs of theorems and propositions, and they are not referred to in any other place.

1.4 Background

The subject of this book demands the reader to be proficient in several areas of set theory and willing to ask at least the basic questions about several other fields of mathematics. This section sums up the basic definitions and results which are taken for granted in the text.

1.4.1 Polish spaces

A *Polish space* is a separable completely metrizable topological space. Many Polish spaces occur in this book. If T is a countably branching tree without endnodes, then the set $[T]$ of all infinite branches through the tree T equipped with the topology generated by the sets $O_t = \{x \in [T] : t \subset x\}$ is a Polish space, with important special cases the Cantor space 2^ω and the Baire space ω^ω.

I will make use of basic theory of Polish spaces as exposed in [40]. Every uncountable Polish space X is a Borel bijective image of the Cantor space and it is a continuous bijective image of a closed subset of the Baire space. A G_δ subset of a Polish space is again Polish in the inherited topology. Every Polish space is homeomorphic to a G_δ subset of the Hilbert cube.

There are several useful operations on Polish spaces. If X, Y are Polish spaces then their product is again Polish; even a product of countably many Polish spaces is still Polish. If X is a Polish space then $K(X)$ denotes the space of all compact subsets of X equipped with *Vietoris topology* generated by sets of the form $\{K \in K(X) : K \subset O\}$ and $\{K \in K(X) : K \cap O \neq 0\}$ for open sets $O \subset X$. The space $K(X)$ is referred to as the *hyperspace* of X; it is Polish and if X is compact then $K(X)$ is compact as well.

It is possible to change the topology on a Polish space to a new, more convenient one. Whenever X is Polish with topology τ and $B_n : n \in \omega$ are τ-Borel subsets of X then there is a Polish topology η extending τ such that the sets $B_n : n \in \omega$ are η-clopen and the η-Borel sets are exactly the τ-Borel sets.

1.4.2 Definable subsets of Polish spaces

Definability of subsets of Polish spaces plays a critical role. Let X be a Polish space, with a countable topology basis $\mathcal{O}$. *Borel sets* are those sets which can be obtained from the basic open sets by a repeated application of countable union, countable intersection, and taking a complement. This is a class of sets closed under continuous preimages and continuous one-to-one images, but not under arbitrary continuous images. *Analytic sets* are those that can be obtained as continuous images of Borel sets. This is a class of sets containing the Borel sets, closed under continuous images, countable unions and intersections, but not under complements. Every analytic set $A \subset X$ is a projection of a closed subset $C \subset X \times \omega^\omega$, $A = \mathrm{proj}(C)$. Every analytic subset of the Baire space is of the form $\mathrm{proj}[T]$. Every analytic set whose complement is analytic is in fact Borel.

The paper [20] isolated an important and very practical broad definability class of subsets of Polish spaces. A set $A \subset 2^\omega$ is *universally Baire* if there are class trees $S, T \subset (2 \times \mathrm{Ord})^{<\omega}$ which in all set generic extensions project into complementary subsets of 2^ω and $A = \mathrm{proj}[T]$. A subset of another Polish space is universally Baire if it is in Borel bijective correspondence with a universally Baire subset of the Cantor space. Equivalently, a set is universally Baire if all of its continuous preimages have the property of Baire.

In ZFC, analytic sets and coanalytic sets are universally Baire, and consistently the class of universally Baire sets does not reach far beyond that. However, under large cardinal assumptions the class of universally Baire sets expands considerably. If there is a proper class of Woodin cardinals then the class of universally Baire sets is closed under complementation and continuous images and preimages, and every set of reals in the model $L(\mathbb{R})$ is universally Baire.

1.4.3 Measure theory

Let X be a Polish space. A *submeasure* on X is a map $\phi : \mathcal{P}(X) \to \mathbb{R}^+$ such that $\phi(0) = 0$, $A \subset B \to \phi(A) \leq \phi(B)$ and $\phi(\bigcup_n A_n) \leq \Sigma_n \phi(A_n)$ whenever $A_n : n \in \omega$ is a countable collection of subsets of the space X. The submeasures on uncountable Polish spaces in this book will always be countably subadditive in this sense. The submeasure ϕ is *outer regular* if $\phi(A) = \inf\{\phi(O) : A \subset O, O \text{ open}\}$ and it is *outer* if $\phi(A) = \inf\{\phi(B) : A \subset B : B \text{ Borel}\}$.

A Borel measure (or *measure*) is a map $\phi : \mathcal{B}(X) \to \mathbb{R}^+$ such that $\phi(0) = 0$, $A \subset B \to \phi(A) \leq \phi(B)$ and $\phi(\bigcup_n A_n) = \Sigma_n \phi(A_n)$ if $A_n : n \in \omega$ is a countable collection of pairwise disjoint Borel sets. Finite Borel measures on Polish spaces are outer regular and tight: $\phi(A) = \inf\{\phi(O) : A \subset O, O \text{ open}\} = \sup\{\phi(K) : K \subset A, K \text{ compact}\}$. I will need a criterion for the restriction of a submeasure ϕ on X to the Borel subsets of X to be a measure. If d is a complete separable metric on X and for every pair of closed sets $C_0, C_1 \subset X$ which are nonzero distance apart, $\phi(C_0 \cup C_1) = \phi(C_0) + \phi(C_1)$ then indeed $\phi \restriction \mathcal{B}(X)$ is a measure. In this situation I will say that ϕ is a *metric measure*.

A capacity on a Polish space X is a map $\phi : \mathcal{P}(X) \to \mathbb{R}^+$ such that $\phi(0) = 0$, $A \subset B \to \phi(A) \leq \phi(B)$, $\phi(\bigcup_n A_n) = \sup_n \phi(A_n)$ whenever $A_n : n \in \omega$ is a countable inclusion-increasing sequence of subsets of the space X, and $\phi(K) = \inf\{\phi(O) : K \subset O, O \text{ open}\}$ for compact sets $K \subset X$. Capacities are tight on analytic sets: if $A \subset X$ is analytic then $\phi(A) = \sup\{\phi(K) : K \subset A : K \text{ compact}\}$.

1.4.4 Determinacy

Infinitary games of all kinds, lengths, and complexities are a basic feature of this book. The key problem always is whether one of the players must have a winning strategy, an issue referred to as the *determinacy* of the game in question.

An *integer game* of length ω is specified by the *payoff set* $A \subset \omega^\omega$. In the game, Players I and II alternate infinitely many times, each playing an integer in his turn. Player I wins if the infinite sequence they obtained belongs to the set A, otherwise Player II wins. Insignificant variations of this concept, which are nevertheless much more intuitive and easier to use, obtain when Players I and II can use moves from some other countable set in place of ω.

Fact 1.4.1. *[49] Games with Borel payoff set are determined. [20] If large cardinals exist then games with universally Baire payoff set are determined.*

A significant variation occurs if the players are allowed to choose their moves from a set larger than countable. Let U be an arbitrary set, and let $A \subset U^\omega$ be a set. The associated game with payoff A of length ω is played just as in the previous paragraph. To state the determinacy theorems, consider U^ω as a topological space with basic open neighborhoods of the form $O_t = \{\vec{u} \in U^\omega : t \subset \vec{u}\}$ as t varies over all finite sequences of elements of the set U.

Fact 1.4.2. *[48] Games with Borel payoff set are determined. Suppose that large cardinals exist, $A \subset U^\omega$ is a Borel set, $f : A \to X$ is a continuous function into a Polish space, and $B \subset X$ is a universally Baire set. The game with payoff set $f^{-1}B$ is determined, and moreover there is a winning strategy which remains winning in all set generic extensions.*

Still another significant variation occurs if the moves of the two players come from some fixed Polish space X and the game has α many rounds for some countable ordinal α. Consider the space X^α equipped with the standard Polish product topology.

Fact 1.4.3. *[55] (LC) Games with real entries, countable length, and universally Baire payoff set are determined.*

The games of longer than countable length are important and interesting, and in this book they appear in Section 6.1. However, I will never be concerned with their determinacy.

In numerous places I will refer to the Axiom of Determinacy (AD) and its variations, such as AD+, and the natural models for these axioms.

Definition 1.4.4. *The Axiom of Determinacy (AD) is the statement that integer games with arbitrary payoff set are determined. AD+ is the statement: every set of reals is ∞-Borel and games with ordinal entries, length ω, and payoff sets which are preimages of subsets of ω^ω under continuous maps $\mathrm{Ord}^\omega \to \omega^\omega$ are determined.*

Happily, I will never have to delve into the subtleties of AD+. Let me just state that it is an open question whether AD is in fact equivalent to AD+. In this book, I will need the following two pieces of information about the axiom AD+:

Fact 1.4.5. *Suppose that suitable large cardinals exist. Then $L(\mathbb{R}) \models$AD+. If Γ is a class of universally Baire sets closed under continuous preimages then $L(\mathbb{R})(\Gamma) \models$AD+.*

Fact 1.4.6. *[28] (ZF+DC+AD+) If $\kappa \in \Theta$ is a regular uncountable cardinal then there is a set $A \subset \omega^\omega$ and a prewellordering $\leq$ on A of length κ such that every analytic subset of A meets fewer than κ many classes.*

Here as usual Θ is the supremum of lengths of prewellorderings of the real numbers.

1.4.5 Forcing

The standard reference book for forcing terminology and basic facts is [29]. Suppose that $P, \leq$ is a partially ordered set, a *poset* for short. P is *separative* if for every $p, q \in P$, if every $r \leq p$ is compatible with q then $p \leq q$. The separative quotient of P is the partially ordered set of E-equivalence classes on P where pEq if every extension of p is compatible with q and vice versa, every extension of q is compatible with p, with the ordering inherited from the poset P. The separative quotient of P is separative. The posets considered in this book are generally not separative, and no effort is wasted on considering their separative quotients instead. Every separative poset P is isomorphic to a dense subset of a unique complete Boolean algebra denoted by $RO(P)$.

There is a historically and mathematically important forcing model mentioned in many places in the book, the *choiceless Solovay model*. Let me briefly outline its construction and basic features. Let κ be an inaccessible cardinal and $G \subset \text{Coll}(\omega, < \kappa)$ be a generic filter. Consider the submodel $M \subset V[G]$ consisting of those sets hereditarily definable in $V[G]$ from real parameters and parameters in the ground model. This is the definition of the choiceless Solovay model.

Fact 1.4.7. *The basic features of the Solovay model include*

1. *for every real number $r \in M$ the model M is a choiceless Solovay model over the model $V[r]$;*
2. *every set of reals is a wellordered union of length $\kappa = \omega_1^M$ of Borel sets.*

The book contains several isolated references to the *nonstationary tower forcing* Q_δ discovered by Woodin [79], recently exposed in [45]. If δ is a Woodin cardinal and $G \subset Q_\delta$ is a generic filter, then in $V[G]$ there is an elementary embedding $j: V \to M$ such that the model M is transitive, contains the same countable sequences of ordinals as $V[G]$, and $\omega_1^M = \delta$.

On several occasions I will refer to the Gandy–Harrington forcing [47]. This is the countable forcing of all nonempty lightface Σ^1_1 subsets of some fixed Polish space. As a countable forcing, this is similar to Cohen forcing; its worth derives from its particular representation. The forcing adds a single point in the Polish space which belongs to all sets in the generic filter. Note that there are some atoms in the forcing–the Σ^1_1 singletons, but there are nonatomic parts too. I will also consider the obvious relativized variations of the Gandy–Harrington forcing.

Throughout the book, I will use a trick commonplace in the literature. Let P be a partial ordering, M a countable elementary submodel of some large structure (the structure is typically H_θ for some large ordinal θ, never to be exactly spelled out) containing all the necessary information (the objects previously named in the argument, including the poset P). An *M-generic filter* $g \subset P$ is a filter on $P \cap M$ which intersects every dense subset of P which happens to be an element of the model M. The expression $M[g]$ describes the generic extension of the transitive collapse of the model M by the collapsed image of the filter g. If $\dot{x}$ is a P-name for an element of ω^ω then $\dot{x}/g$ is the element of ω^ω defined by $\dot{x}/g(n) = m \leftrightarrow \exists p \in g\ p \Vdash \dot{x}(\check{n}) = \check{m}$. The complexity of this operation is recorded in the following fact.

Fact 1.4.8. *Suppose that P is a forcing, $\dot{x}$ a P-name for an element of ω^ω, and M is a countable elementary submodel of a large enough structure. The set $A = \{y \in \omega^\omega : \exists g \subset M \cap P\ g$ is M-generic and $y = \dot{x}/g\}$ is Borel.*

Proof. Let $Q \subset r.o.(P)$ be the complete Boolean algebra generated by the name $\dot{x}$. Then $A = \{y \in \omega^\omega : \exists g \subset M \cap Q\ g$ is M-generic and $y = \dot{x}/g\}$. Let N by the transitive collapse of the model M, and consider the Polish space X of all N-generic filters on $\pi(Q)$ with the usual topology. Then A is the image of the space X under the continuous injection $g \mapsto \dot{x}/g$, and so A is Borel by a classical theorem of Lusin [40], 15.1. $\square$

1.4.6 Absoluteness

The universally Baire sets (in particular, the analytic and coanalytic sets) have a natural interpretation in forcing extensions. Suppose $A \subset 2^\omega$ is universally Baire, as witnessed by trees $T, S \subset (2 \times \mathrm{Ord})^{<\omega}$ which project to complements in all set forcing extensions and $A = \mathrm{proj}[T]$. If $V[G], G \subset P$ is an arbitrary set forcing extension then $A^{V[G]}$, the interpretation of the set A in the model $V[G]$, is defined as $(\mathrm{proj}[T])^{V[G]}$. A wellfoundedness argument shows that the interpretation does not depend on the choice of the witness trees T, S. I will use this feature to denote by $\dot{A}$ the P-name for the interpretation of the set A in the extension, and when speaking about this extended interpretation, I will frequently omit the superscript in the expression $A^{V[G]}$. This usage is commonplace throughout the book.

The following facts connecting the validity of certain sentences in generic extensions and the ground model are indispensable throughout the book.

Fact 1.4.9. *(Analytic absoluteness) Suppose that M is a transitive model of set theory, $\vec{x} \in M \cap \omega^\omega$ is a sequence of parameters, and ϕ is a Σ^1_1 formula with free variables. Then $\phi(\vec{x})$ holds if and only if $M \models \phi(\vec{x})$ holds.*

This is typically used in a situation where M is a generic extension of the transitive collapse of some countable elementary submodel of a large enough structure.

Fact 1.4.10. *(Shoenfield absoluteness) Suppose that M is a transitive model of set theory containing all countable ordinals, $\vec{x} \in M \cap \omega^\omega$ is a sequence of parameters, and ϕ is a Σ^1_2 formula with free variables. Then $\phi(\vec{x})$ holds if and only if $M \models \phi(\vec{x})$ holds.*

This is typically used in a generic extension with M equal to the ground model.

Fact 1.4.11. *[79] (Universally Baire absoluteness) (LC) Suppose that $\vec{A}$ is a finite sequence of universally Baire sets and M is a countable elementary submodel of some large structure containing $\vec{A}$. Suppose that $M[g]$ is a generic extension of the transitive collapse of the model M and $\vec{x} \in \omega$ is a finite sequence of parameters in the model $M[g]$. Suppose that ϕ is a formula quantifying over reals only. Then $\phi(\vec{x}, \vec{A})$ holds if and only if $M[g] \models \phi(\vec{x}, \vec{A})$ holds.*

Fact 1.4.12. *[45] (Σ^2_1 absoluteness) (LC+CH) Suppose that $\vec{A}$ is a finite sequence of universally Baire sets and ϕ is a formula of the form $\exists B \subset \omega^\omega\ \psi$ where ψ quantifies only over real numbers. If $\phi(\vec{A})$ holds in some generic extension, then $\phi(\vec{A})$ holds.*

1.4.7 Cardinal invariants of the continuum

The original motivation for the work contained in this book were the problems associated with comparison of cardinals defined in various ways from Polish spaces. I use [2] as a canonical reference.

Among the cardinal invariants that frequently occur in this book, let me quote $\mathfrak{a}$ = the least size of a maximal almost disjoint family of subsets of ω, $\mathfrak{b}$ = the least size of modulo finite unbounded subset of ω^ω, $\mathfrak{c}$ = the size of the continuum, $\mathfrak{d}$ = the least size of modulo finite dominating subset of ω^ω.

Given a σ-ideal I on a Polish space X, I will consider the cardinals $\mathtt{cov}(I)$ = the least number of sets in the ideal I necessary to cover the whole space, $\mathtt{non}(I)$ = the smallest possible size of an I-positive set, $\mathtt{add}(I)$ = the smallest size of a family of I-small sets whose union is not I-small, and $\mathtt{cof}(I)$ = the smallest possible size of a basis for the ideal I. It will be of advantage to consider starred variations of these cardinals: $\mathtt{cov}^*(I) =$ the least number of sets in the ideal I necessary to cover some Borel I-positive set, $\mathtt{non}^*(I) =$ the least cardinal such that every Borel I-positive set contains an I-positive subset of this size, and similarly for $\mathtt{add}^*$ and $\mathtt{cof}^*$.

2
Basics

2.1 Forcing with ideals

2.1.1 The key definition

Definition 2.1.1. *Suppose that X is a Polish space and I is a σ-ideal on the space X. The symbol P_I denotes the partial order of I-positive Borel sets ordered by inclusion.*

I will always tacitly assume that the Polish space X is uncountable and the ideal I contains all singletons. There are several cases in which this will not hold, and they will be pointed out explicitly. Note that the poset P_I depends only on the membership of Borel sets in the ideal I, but it will frequently be of interest to look at the membership of non-Borel sets in I.

It is clear that the partial order P_I is not separative, and its separative quotient is the σ-algebra $\mathcal{B}(X) \mod I$. There is exactly one property all partial orders of this kind share.

Proposition 2.1.2. *The poset P_I adds an element $\dot{x}_{gen}$ of the Polish space X such that for every Borel set $B \subset X$ coded in the ground model, $B \in G$ iff $\dot{x}_{gen} \in B$.*

Proof. It is easy to see that the closed sets contained in the generic filter form a collection closed under intersection which contains sets of arbitrarily small diameter. A completeness argument shows that such a collection has a nonempty intersection containing a single point, and $\dot{x}_{gen}$ is a name for the single point in the intersection. Another way to describe the generic point is to say that it is the unique element in all basic open sets in the generic filter.

By induction on the complexity of the Borel set B prove that $B \Vdash \dot{x}_{gen} \in \dot{B}$. For closed sets this follows from the definition of the name $\dot{x}_{gen}$. Suppose that $B = \bigcup_n C_n$ and we already know that each set C_n forces $\dot{x}_{gen} \in \dot{C}_n$. Whenever $D \subset B$ is an I-positive Borel set then for some number n, $D \cap C_n$ is I-positive, $D \cap C_n \subset C_n$

and $D \cap C_n \Vdash \dot{x}_{gen} \in \dot{C}_n \subset \dot{B}$. By the genericity, $B \Vdash \dot{x}_{gen} \in \dot{B}$. Now suppose that $B = \bigcap_n C_n$ and we already know that each set C_n forces $\dot{x}_{gen} \in \dot{C}_n$. Then for every number n, $B \Vdash \dot{x}_{gen} \in \dot{C}_n$ since $B \subset C_n$. In other words, $B \Vdash \dot{x}_{gen} \in \bigcap_n C_n = \dot{B}$ as desired. Since the Borel sets in Polish spaces are obtained from closed sets by a repeated application of countable union and intersection, the induction is complete.

Now it is not difficult to prove that $C \Vdash \dot{x}_{gen} \in \dot{B}$ iff $C \setminus B \in I$. On one hand, if $C \setminus B \in I$ then every strengthening of the condition C is compatible with B and the previous paragraph applies to show that $C \Vdash \dot{x}_{gen} \in \dot{B}$. On the other hand, if $C \setminus B \notin I$, then $C \setminus B \subset C$ is a condition strengthening C which forces $\dot{x}_{gen} \in \dot{C} \setminus \dot{B}$ by the previous paragraph again, in particular $\dot{x}_{gen} \notin \dot{B}$.

The proposition follows. $\square$

Note the key role played by the closure of the ideal I under countable unions in the argument. An important observation is that the forcings of the form P_I can be presented in various forms.

Definition 2.1.3. *Suppose I is a σ-ideal on a Polish space X. A* different presentation *of the poset P_I is a Borel bijection $f : X \to Y$ between X and another Polish space Y, the σ-ideal J on the space Y given by $A \in J \leftrightarrow f^{-1}A \in I$, and the resulting poset P_J.*

If f, J constitute a different presentation of the forcing P_I then the function f extends to a bijection $\hat{f} : P_I \to P_J$ given by $\hat{f}(A) = f''A$. Note that one-to-one Borel images of Borel sets are Borel by a theorem of Lusin [40], 15.1, and therefore the image of the function $\hat{f}$ indeed consists of Borel sets.

While a given forcing P_I can have many presentations, it is true that some presentations are more natural than others. In fact, I will frequently derive some forcing properties of the poset P_I from the topological features of a certain natural presentation. The forcing properties of P_I then persist through different presentations while the topological features may not. Note that there is a Borel bijection between any two uncountable Polish spaces, and so the nature of the Polish space does not restrict the kind of partial orders that can live on it. It may be occasionally difficult to decide whether a given presentation is the simplest possible one or the one most suitable to study.

2.1.2 Representation theorems

The study of the partial orders of the form P_I does entail a certain restriction in generality, but not too great a restriction. The following results show that many forcings encountered in practice can be presented as P_I for a suitable σ-ideal I on a Polish space.

Fact 2.1.4. *[68] Suppose that B is a σ-complete countably σ-generated Boolean algebra. Then there is a σ-ideal I on the Cantor space such that B is isomorphic to $\mathcal{B}(2^\omega) \mod I$.*

Corollary 2.1.5. *Suppose that P is a partially ordered set consisting of binary trees ordered by inclusion, such that for every tree $T \in P$ and every node $t \in T$ the tree $T \restriction t$ is in P as well. Then P is in the forcing sense equivalent to a forcing of the form P_I.*

Proof. If $\dot{G}$ is a name for the generic filter write $\dot{x}_{gen}$ for the generic real: $\dot{x}_{gen} = \bigcup \bigcap \dot{G} \in 2^\omega$. Let $P \subset B$ be the complete Boolean algebra generated by the poset P. I will show that the σ-algebra $C \subset B$ σ-generated by the elements $b_t = |\check{t} \subset \dot{x}_{gen}| : t \in 2^{<\omega}$ is dense. By the previous fact the algebra C is isomorphic to some P_I and at the same time poset P is equivalent to it.

It is enough to show that for every tree $T \in P$ it is the case that $T = c_T$ where $c_T = \bigwedge_n \bigvee_{t \in 2^n \cap T} b_t$. It is clear that $T \leq c_T$. And if $c_T \not\leq T$ then there would be a tree $S \not\subset T$ such that $S \leq c_T$ and a node $s \in S \setminus T$ of length $n \in \omega$. Then $S \restriction s \in P$ and clearly $S \restriction s \Vdash \dot{x}_{gen} \restriction n \notin \check{T}$, contradicting the assumption that $S \leq c_T$. □

There is frequently a more direct way of deriving the σ-ideal from the tree forcing in question.

Proposition 2.1.6. *Suppose that P is a partially ordered set consisting of binary trees ordered by inclusion such that for every tree $T \in P$ and every node $t \in T$ the tree $T \restriction t$ is in P as well. Suppose moreover that P has the continuous reading of names. Then the collection $I = \{A \subset 2^\omega :$ A analytic and for no condition $T \in P$ it is the case that $[T] \subset A\}$ is a σ-ideal and the forcing P is in the forcing sense equivalent to P_I.*

Here the *continuous reading of names* is the statement that for every condition $T \in P$ and every name $\dot{f} \in \omega^\omega$ there is a condition $S \subset P$, natural numbers $n_0 \in n_1 \in \dots$ and a function $g : \bigcup_m (S \cap 2^{n_m}) \to \omega$ such that for every number m and every sequence $t \in S \cap 2^{n_m}$ it is the case that $S \restriction t \Vdash \dot{f}(\check{m}) = \check{g}(\check{t})$. This is a property frequently found in practice; consult Section 3.1 for a topological restatement of it.

Proof. Suppose that $A = \bigcup A_n : n \in \omega$ are analytic sets such that A contains all branches of some tree $T \in P$. I will produce a tree $S \subset T$ and a number $n \in \omega$ such that all branches of the tree S belong to the set A_n. This will prove the proposition.

Note that the forcing P adds a canonical generic point $\dot{x}_{gen} \in 2^{<\omega}$ which is a branch of all trees in the generic filter. Use a Shoenfield absoluteness argument to show that $T \Vdash \dot{x}_{gen} \in \dot{A}$ and therefore there is a condition $T' \subset T$ and a number n such that $T' \Vdash \dot{x}_{gen} \in \dot{A}_n$.

Let $U \subset (2 \times \omega)^{<\omega}$ be a tree such that $A_n = \text{proj}[U]$. There is a name $\dot{f}$ for an element of the Baire space ω^ω such that $T' \Vdash \langle \dot{x}_{gen}, \dot{f} \rangle$ forms a branch through the tree $\check{U}$. Use the continuous reading of names to find a tree $S \subset T'$, natural numbers $n_0 \in n_1 \in \ldots$ and a function $g : \bigcup_m (S \cap 2^{n_m}) \to \omega$ such that for every number m and every node $t \in S \cap 2^{n_m}$ the condition $S \restriction t$ forces $\dot{f}\check{m}) = \check{g}(\check{t})$. Then for every branch b through the tree S it must be the case that b together with the function $m \mapsto g(b \restriction n_m)$ forms a branch through the tree U and therefore $b \in A_n$. I have just proved that $[S] \subset A_n$ as desired. $\square$

Partial orders for adding a real which do not consist of trees and the previous proposition cannot be applied to them are fairly rare in the practice of definable forcing. Nevertheless, many of them can be obtained through the methods of this book. The following is a characterization theorem which does not depend on the specific combinatorial form of the forcing.

Definition 2.1.7. *A forcing P is a* universally Baire real forcing *if*

1. *its conditions are elements of some Polish space Y;*
2. *there is a name $\dot{x}_{gen}$ for an element of some Polish space X;*
3. *there is a universally Baire set $A \subset X \times Y$ such that for every condition $p \in P$ $P \Vdash \check{p} \in \dot{G} \leftrightarrow \langle \dot{x}_{gen}, \check{p} \rangle \in \dot{A}$;*
4. *for every basic open set $O \subset X$ there is a condition $p \in P$ such that $P \Vdash \dot{x}_{gen} \in \dot{O} \leftrightarrow \check{p} \in \dot{G}$.*

Proposition 2.1.8. *[83] (LC) Every proper universally Baire real forcing is in the forcing sense equivalent to one of the form P_I.*

Proof. I claim that $I = \{B \subset X : B$ universally Baire and $P \Vdash \dot{x}_{gen} \notin \dot{B}\}$ is the σ-ideal with the required properties. It is clear that I is closed under countable unions. Write $\dot{y}_{gen}$ for the P_I-name for its generic point in the space X, and let $\dot{G}$ be the P_I-name for the set $\{p \in \check{P} : \langle \dot{y}_{gen}, \check{p} \rangle \in \dot{A}\}$. It will be enough to show that $P_I \Vdash \dot{G} \subset \check{P}$ is a V-generic filter; the proposition then follows by standard abstract forcing considerations. Suppose that $B \in P_I$ is a condition, $p, q \in P$ are conditions such that $B \Vdash \check{p}, \check{q} \in \dot{G}$ and $D \subset P$ is open dense. I must find a condition $B' \in P_I$ and a condition $r \in P$ such that $B' \subset B$, $r \leq p, q$, $r \in D$, and $B' \Vdash \dot{r} \in \dot{G}$.

Let M be a countable elementary submodel of a large enough structure, let Z be the Polish space of all M-generic filters on P with the usual topology, let $f : Z \to X$ be a map defined by $f(g) = \dot{x}_{gen}/g$. This map is continuous by (4) and injective by (3) of the definition of universally Baire real forcing. Thus the range $f''Z$ is Borel by a classical theorem of Luzin [40], 15.1. Write $C = B \cap \text{rng}(f)$ and for every condition $r \in P \cap M$ write $C_r = C \cap f''O_r$ and $\bar{C}_r = C \setminus f''O_r$, where O_r is the open set of all filters in the space Z containing the condition r. Now,

- $C \notin I$. To see this, note that as $B \notin I$, there must be a condition $r \in P$ such that $r \Vdash \dot{x}_{gen} \in \dot{B}$. By elementarity, there must be such a condition in the model M. Any M-master condition below r forces $\dot{x}_{gen} \in \dot{C}$, and so $C \notin I$ as required.
- For every condition $r \in P \cap M$, $C_r \Vdash \check{r} \in \dot{G}$ and $\bar{C}_r \Vdash \check{r} \notin \dot{G}$ if these sets are I-positive. To see this, note $\forall x \in C_r \; M[x] \models \langle x, r \rangle \in A$, by an absoluteness argument $\forall x \in C_r \; \langle x, r \rangle \in A$, and by the universally Baire absoluteness this statement will still be true in the P_I extension, in particular $C_r \Vdash \dot{y}_{gen} \in \dot{C}_r$ and $\langle \dot{y}_{gen}, \check{r} \rangle \in \dot{A}$.
- $\bar{C}_p, \bar{C}_q \in I$ and so $C_p \cap C_q \notin I$. This follows from the previous item.
- The sets $C_r : r \in D \cap M$ is a lower bound of p, q cover the I-positive set $C_p \cap C_q$, therefore one of them is I-positive, and $C_r = B' \subset B$ is the required condition.

This completes the proof. □

Example 2.1.9. Consider the Sacks forcing P of all perfect binary trees ordered by inclusion. Corollary 2.1.5, Proposition 2.1.6 and Proposition 2.1.8 all can be used to show that $P = P_I$ for some σ-ideal I. None of this abstract reasoning can replace the information obtained from the perfect set theorem: the σ-ideal I is the ideal of countable subsets of 2^ω.

2.1.3 Generalizations

There are several ways in which the previous ideas can be generalized, each of them important and deserving a thorough discussion.

First, one can consider forcing with analytic (projective, universally Baire, etc.) sets positive with respect to a given σ-ideal I. For most of the forcings considered in this book it will be the case that every I-positive universally Baire set has an I-positive Borel subset, and so the poset P_I is dense in all of these variations, and under large cardinal assumptions it is dense in the poset $(\mathcal{P}(X) \mod I)^{L(\mathbb{R})}$ – refer to Section 3.9 for a thorough discussion. Nevertheless, I will have to enter situations in which this property has not been verified yet, and then the following definition and proposition will be important.

Definition 2.1.10. *Suppose that I is a σ-ideal on a Polish space X. The symbol Q_I stands for the poset of I-positive analytic sets ordered by inclusion.*

Proposition 2.1.11. *Suppose that the σ-ideal I is generated by coanalytic sets. There is a Q_I-name $\dot{x}_{gen}$ for an element of the Polish space X such that an analytic set belongs to the generic filter if and only if it contains the point $\dot{x}_{gen}$ in the extension.*

Proof. I will handle the case of $X = 2^\omega$, the other spaces being Borel bijective images of 2^ω. As in the P_I case, let $\dot{x}_{gen}$ be the unique point in the intersection of all basic open sets in the generic filter.

First note that any set $A \in Q_I$ forces $\dot{x}_{gen} \in \dot{A}$. To see this, let $T \subset (2 \times \omega)^{<\omega}$ be a tree such that $A = \text{proj}[T]$, let $G \subset Q_I$ be a generic filter containing the condition A and in the generic extension let $S \subset \omega^{<\omega}$ be the tree consisting of all nodes $t \in \omega^{<\omega}$ such that $\text{proj}[T \restriction \langle \dot{x}_{gen} \restriction |t|, t\rangle] \in \dot{G}$. Clearly $0 \in S$ and it will be enough to show that S contains no terminal nodes. Well, if $t \in S$ is a node and $B \subset A$ is a condition forcing $\check{t} \in \dot{S}$ then strengthening the condition B if necessary I may assume that there is a binary sequence s such that $B \subset \text{proj}[T \restriction \langle s, t\rangle]$. By the σ-additivity of the ideal I there must be a number $n \in \omega$ and a bit $b \in 2$ such that $C = B \cap \text{proj}[T \restriction \langle s^\frown b, t^\frown n\rangle] \notin I$. Clearly, $C \Vdash \check{t}^\frown \check{n} \in \dot{S}$ as required.

Second, if $A, B \in Q_I$ are sets and $A \Vdash \dot{x}_{gen} \in \dot{B}$ then $A \cap B \notin I$: if $A \cap B \in I$ then let $C \in I$ be a coanalytic set including it as a subset, and $A \setminus C \in Q_I$ is a condition which forces the point $\dot{x}_{gen}$ into itself by the previous paragraph, and by the analytic absoluteness it forces $\dot{x}_{gen} \notin B$, contradicting the assumption. But now $A \cap B \in Q_I$ is a common lower bound of A, B, forcing $B \in \dot{G}$. □

It is remarkable that in all cases when I need to use the forcing Q_I it is only to show that in fact $P_I \subset Q_I$ is dense. However, the statement that every I-positive analytic set has an I-positive Borel subset seems to be interesting in its own right. See Section 3.9 on this and similar dichotomies.

The second way to generalize the forcings of the form P_I is to consider spaces of the form Y^ω with an uncountable set Y and the standard tree topology instead of Polish spaces X. For a sequence $t \in Y^{<\omega}$ let O_t be the basic open set determined by t, $O_t = \{x \in Y^\omega : t \subset x\}$.

Proposition 2.1.12. *Suppose that Y is a set and I is a σ-ideal on the space Y^ω with the following closure property:*

(*) *if $A_t : t \in Y^{<\omega}$ are sets in the ideal with $A_t \subset O_t$, then $\bigcup_t A_t \in I$.*

There is a name $\dot{x}_{gen}$ for an element of the space Y^ω such that in the generic extension by the poset P_I, a Borel set $B \subset Y^\omega$ belongs to the generic ultrafilter if an only if $\dot{x}_{gen} \in \dot{B}$.

Proof. Let $\dot{x}_{gen} = \bigcup\{t \in Y^{<\omega} : O_t \in G\}$ where G is the P_I-generic filter. It is clear that the sequences in the union are linearly ordered. Moreover if $n \in \omega$ is a number and $B \in P_I$ is a condition the one of the sets $B \cap O_t : t \in Y^n$ is I-positive by the property (*) and forces the corresponding sequence into the union defining the sequence $\dot{x}_{gen}$. Thus $P_I \Vdash \dot{x}_{gen} \in Y^\omega$.

Before the remainder of the proof note that similarly to Borel subsets of 2^ω the Borel subsets of Y^ω have natural interpretations in every generic extension which does not depend on the particular Borel definition of the set.

By induction on the complexity of the Borel set $B \in P_I$ I will show that $B \Vdash \dot{x}_{gen} \in \dot{B}$, where $\dot{B}$ denotes the interpretation of the Borel set B in the extension. Suppose first that B is open. If $C \subset B$ is any condition then by the property (*) there must be a sequence $t \in Y^{<\omega}$ such that $O_t \subset B$ and $C \cap O_t \notin I$. Clearly $C \cap O_t \subset C$ is a condition forcing $\dot{x}_{gen} \in \dot{B}$ and so $B \Vdash \dot{x}_{gen} \in \dot{B}$. The remaining steps in the induction are the same as in Proposition 2.1.2.

Now suppose that $B, C \in P_I$ are sets such that $B \Vdash \dot{x}_{gen} \in \dot{C}$. I must show that $B \cap C \notin I$; then $B \cap C$ is the required lower bound of the conditions B, C. Suppose $B \cap C \in I$. Then $B \setminus C$ is a condition in P_I which by the previous paragraph forces $\dot{x}_{gen}$ into $B \setminus C$ and outside of the set C, contradicting the choice of the set B. $\square$

Example 2.1.13. Namba forcing [54]. Let $Y = \omega_2$ and let I be the ideal of sets $B \subset Y^\omega$ such that there is a map $f : Y^{<\omega} \to \omega_2$ such that $B \subset B_f = \{y \in Y^\omega : \exists^\infty n \ y(n) \in f(y \restriction n)\}$. It is not difficult to see that the ideal I has the closure property (*) from the previous proposition. I will show that a Borel set $B \subset Y^\omega$ is I-positive if and only if it contains all branches of some Namba tree, that is an infinite tree $T \subset Y^{<\omega}$ such that all but finitely many of its nodes have $\aleph_2$ many immediate successors. This means that the Namba forcing is in a natural sense isomorphic to a dense subset of the poset R_I.

Let $B \subset Y^\omega$ be a Borel set, and consider the game G between Player I and II. Player I produces a sequence of ordinals $\alpha_n \in \omega_2 : n \in \omega$ and Player II in response produces a sequence of ordinals $\beta_n \in \omega_2 : n \in \omega$. Moreover Player II must raise a flag at some round m and for all $n > m$ it must be the case that $\alpha_n \in \beta_n$. Player II wins if his sequence of answers belongs to the Borel set B. The payoff set of the game G is Borel and therefore determined by Fact 1.4.2. I will be finished if I show that Player I has a winning strategy iff $B \in I$ and Player II has a winning strategy iff B contains all branches of some Namba tree.

Suppose first that Player I has a winning strategy σ. For every sequence $t \in Y^< \omega$ there are at most $|t|$ many ways how the play could reach a position in which Player I followed his strategy σ and Player II produced the sequence t, depending on where and if Player II decided to raise the flag. Let $f(t) =$ maximum of all the possible answers by the strategy σ in that position. It is easy to see that $B \subset B_f$.

On the other hand, suppose that Player II has a winning strategy σ, and let t be some sequence for which there is a position in which Player II followed his strategy

σ, produced the sequence t and raised the flag at that point. It is easy to see that then $[T] \subset B$ for some Namba tree T with trunk t.

Another generalization is to consider spaces X^Y for a Polish space X and an uncountable set Y with the standard product topology, a σ-ideal I on it and a partial order R_I of I-positive Baire sets ordered by inclusion. Here the Baire sets are those subsets of the space X^Y obtained from basic open sets by countable repetition of countable unions, countable intersections, and complementation. The basic open sets are those of the form $\{\vec{x} \in X^Y : \vec{x}(y) \in O\}$ for some basic open set $O \subset X$ and an index $y \in Y$. Such partial orders are the results of the countable support iterations or products or the tower technology of Section 5.5. Let me include the basic property here, and defer the detailed treatment to that section.

Proposition 2.1.14. *There is a R_I-name $\vec{x}_{gen}$ for a function from Y to X such that a Baire set $A \subset X^Y$ belongs to the generic filter if and only if it contains the function $\vec{x}_{gen}$.*

Still another generalization is to consider partial orders $\mathcal{P}(Y) \mod I$ for a suitable set Y and an ideal I on it. These partial orders lack the basic feature of the previously considered cases: the canonical generic object as an element of some ground model coded simple space. The case $Y = \omega$ has been extensively studied [15], [74], [87]. The case $Y = \omega_1$ and $I =$ the nonstationary ideal has been the subject of the precipitousness and saturation considerations. The general case of a σ-ideal I has been studied by Gitik and Shelah [24], [25] who showed that the resulting partial orders cannot be in the forcing sense equivalent to most of the forcings of the form P_J, where J is a σ-ideal on a Polish space.

2.1.4 Basic definability issues

This book deals with *suitably definable* σ-ideals on Polish spaces, with very few exceptions. The demands on definability vary depending on the large cardinal axioms one is willing to use. This section spells out several definitions used throughout the book.

In the presence of large cardinal axioms such as the existence of a supercompact cardinal, the following definability restriction is used.

Definition 2.1.15. *A σ-ideal I on a Polish space X is* universally Baire *if for every universally Baire set $A \subset 2^\omega \times X$ the set $\{y \in 2^\omega : A_y \in I\}$ is universally Baire.*

Without large cardinals more sophisticated notions of definability and absoluteness are needed.

Definition 2.1.16. *A σ-ideal I on a Polish space X is* ZFC-correct *if it is defined by a formula ϕ with a possible real parameter r (so that $I = \{A \subset X : \phi(A, r)\}$) and every transitive model M of a large fragment of ZFC containing r is correct about I on its analytic sets (so that if $s \in M$ is a code for an analytic set A_s then $\phi(A_s, r) \leftrightarrow M \models \phi(A_s, r)$).*

Note that this definition speaks really about the formula defining the ideal rather than the ideal itself. It turns out that nearly all definitions of σ-ideals considered in this book are ZFC-correct in this sense. This assertion is never completely trivial though and its proof is surprisingly close to the determinacy dichotomy and properness arguments used for other purposes.

Example 2.1.17. The ideals associated with Hausdorff submeasures as in Definition 4.4.1 are ZFC-correct. To see this, fix a Hausdorff submeasure ψ on a Polish space X with the associated σ-ideal I generated by sets of finite ψ-mass. Given an analytic set $A \subset X$ let $C \subset X \times \omega^\omega$ be a closed set which projects to A, and consider the integer game $G(C)$ as in the proof of Theorem 4.4.5. Player I has a winning strategy in the game $G(C)$ if and only if $A \in I$. Now given a transitive model M containing the set C, $M \models G(C)$ is determined. The winning strategy the model M finds is still a winning strategy in V since the nonexistence of a successful counterplay is a wellfoundedness statement. Thus the statement $A \in I$ is absolute between M and V.

A measure-theoretic counterpart of the above definition is the following.

Definition 2.1.18. *A submeasure ψ on a Polish space X is* ZFC-correct *if it is defined by a formula ϕ with a possible real parameter r (so that $\psi(A) < q \leftrightarrow \phi(A, q, r)$ for every set $A \subset X$) and every transitive model M of a large fragment of ZFC evaluates ψ-mass correctly (so that if $s \in M$ is a code for an analytic set A_s then $\psi(A_s, r, q) \leftrightarrow M \models \psi(A_s, r, q)$ for every rational number q).*

Example 2.1.19. Every pavement submeasure defined from a countable set of Borel pavers is ZFC-correct. Let ψ be the pavement submeasure on a Polish space X, let $A \subset X$ be an analytic set, let $C \subset X \times \omega^\omega$ be a closed set projecting to A, and let q be a rational number. $\psi(A) < q$ if and only if there is a rational number $q' < q$ such that Player I has a winning strategy in the game $G(C, q')$ as in the proof of Theorem 4.5.6. As in the previous arguments, whenever M is a transitive model containing the set C then it finds a winning strategy for one of the players in the games $G(C, q')$ for all rationals q', and these winning strategies of the model M stay winning in V. Thus M evaluates the ψ-mass of the set A correctly.

Example 2.1.20. Every outer regular strongly subadditive capacity is ZFC-correct. It is possible to supply the same argument as above using the integer game from

Theorem 4.3.6, however here I can use an argument which at least on the surface has no game theoretic content. Let ψ be a strongly subadditive capacity on a Polish space X, determined by its values on the sets from some fixed countable basis $\mathcal{O}$ closed under finite unions. The key fact: Fact 4.3.5, showing that the capacity ψ is simply derivable from its values on basic open sets. Now let $A \subset X$ be an analytic set, a projection of a closed subset $C \subset X \times \omega^\omega$. Let M be a transitive model containing the code for the set C, and let $q > 0$ be a rational number. By the definitions, if $M \models \psi(A) < q$ then $M \models \exists O \subset X\ O$ is open, $\psi(O) < q$ and $A \subset O$, this set O maintains these properties in V by a wellfoundedness argument, and therefore even in V, $\phi(A) < q$. What happens though if $M \models \psi(A) > q$? The key fact mentioned above implies that $M \models \psi$ is a capacity, and by the Choquet's capacitability theorem $M \models \exists K \subset X \times \omega^\omega\ K$ compact, $K \subset C$ and $\psi(\mathrm{proj}(K)) > q$. Now the set K maintains these properties in V by a wellfoundedness argument. Note that $p(K) \subset X$ is a compact set, and therefore its ϕ-mass is the infimum of $\phi(O) : O \in \mathcal{O}, K \subset O$, a computation which works the same in the model M as in V by a wellfoundedness argument again.

The ZFC-correctness is a useful tool in a number of situations such as in the statement of ZFC-provable preservation theorems. Nevertheless, I will need a more sophisticated and more restrictive notion as well. Unlike the ZFC-correctness, it can be stated without a reference to models of ZFC and it has been studied in descriptive set theory for at least a century.

Definition 2.1.21. *A σ-ideal I on a Polish space X is $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ if for every analytic set $A \subset 2^\omega \times X$ the set $\{y \in 2^\omega : A_y \in I\}$ is coanalytic.*

Unlike the ZFC-correctness which places no significant restrictions on the forcing properties of the poset P_I, the $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ condition does have important forcing consequences – its associated forcing can never add dominating reals. This notion is studied in detail in Section 3.8. Here, let me just include two connections with ZFC-correctness.

Proposition 2.1.22. *If a σ-ideal I on a Polish space X is provably $\mathbf{\Pi}^1_1$ on $\mathbf{\Sigma}^1_1$ then it has a ZFC-correct definition.*

Proof. Let $A \subset 2^\omega \times X$ be a universal analytic set and $C \subset 2^\omega$ be a coanalytic set such that ZFC proves $\forall y \in 2^\omega\ A_y \in I \leftrightarrow y \in C$. Every transitive model M evaluates the membership of a point $y \in 2^\omega$ in the set C correctly by a wellfoundedness argument. Thus M evaluates the membership in the ideal I correctly as well. $\square$

Proposition 2.1.23. *Every ZFC-correct ideal is $\mathbf{\Delta}^1_2$ on $\mathbf{\Sigma}^1_1$. Every ZFC-correct submeasure is $\mathbf{\Delta}^1_2$ on $\mathbf{\Sigma}^1_1$.*

Proof. Let I be a ZFC-correct σ-ideal on a Polish space X and let $A \subset 2^\omega \times X$ be an analytic set. I must show that the set $\{y \in 2^\omega : A_y \in I\}$ is $\mathbf{\Delta}^1_2$ on $\mathbf{\Sigma}^1_1$. To see this note that $A_y \in I \leftrightarrow$for every countable model M containing the real y, either M is illfounded or $M \models A_y \in I$, and $A_y \notin I \leftrightarrow$for every countable model M containing the real y, either M is illfounded or $M \models A_y \notin I$.

Let ϕ be a ZFC-correct submeasure on a Polish space X, let $\varepsilon \in \mathbb{R}^+$ be a real number, and let $A \subset 2^\omega \times X$ be an analytic set. I must show that the set $\{y \in 2^\omega : \phi(A_y) < \varepsilon\}$ is $\mathbf{\Delta}^1_2$ on $\mathbf{\Sigma}^1_1$. This is proved in the same way as in the previous paragraph. $\square$

2.2 Properness

The following definition has been central to the development of the forcing theory in the last several decades.

Definition 2.2.1. *[64] A forcing notion P is* proper *if for every set X and every stationary set $S \subset [X]^{\aleph_0}$ it is the case that $P \Vdash \check{S}$ is stationary. Another equivalent restatement is the following. The forcing P is proper if for every large enough cardinal θ, every countable elementary submodel $M \prec H_\theta$ containing P and every condition $p \in P \cap M$ there is an M-*master *condition $q \leq p$; that is, a condition q forcing $\dot{G} \cap \check{M}$ meets every dense subset of P which is an element of M, where $\dot{G}$ is the name for the P-generic filter.*

It turns out that in the context of definable forcing this is exactly the right notion. In its presence there is a rich structure and extensive theory, in its absence there is collapse. I will first restate it in the terms of σ-ideals:

Proposition 2.2.2. *Suppose that I is a σ-ideal on a Polish space X. The following are equivalent:*

1. *the forcing P_I is proper;*
2. *for every countable elementary submodel M of a large enough structure and every condition $B \in M \cap P_I$ the set $C = \{x \in B : x$ is M-generic$\}$ is not in the ideal I.*

Here, a point $x \in X$ is M-generic if the collection $\{A \in P_I \cap M : x \in A\}$ is a filter on $P_I \cap M$ which meets all open dense subsets of the poset P_I that are elements of the model M.

Proof. This is just a restatement of the definitions. First note that the set C is Borel: $C = B \cap \bigcap\{\bigcup(D \cap M) : D \in M$ is an open dense subset of the poset $P_I\}$. If $C \in I$