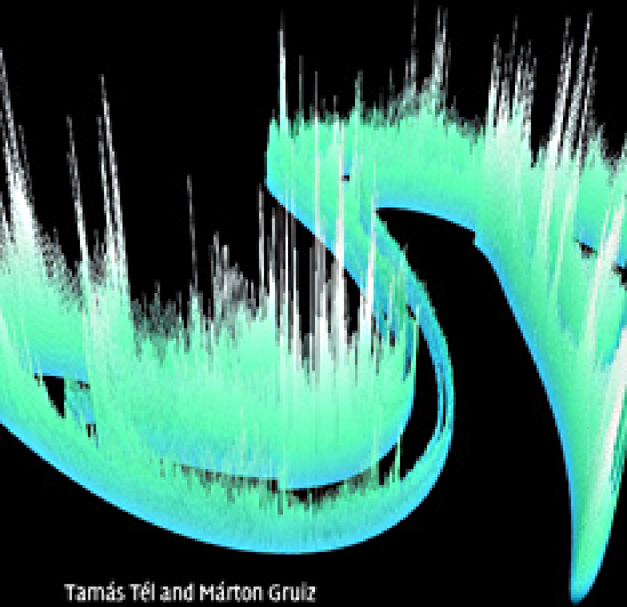

Tamás Tél and Márton Gruiz

Chaotic Dynamics

An Introduction Based on Classical Mechanics

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Chaotic Dynamics

An Introduction Based on Classical Mechanics

Since Newton, a basic principle of natural philosophy has been determinism, the possibility of predicting evolution over time into the far future, given the governing equations and starting conditions. Our everyday experience often strongly contradicts this expectation. In the past few decades we have come to understand that even motion in simple systems can have complex and surprising properties.

Chaotic Dynamics provides a clear introduction to chaotic phenomena, based on geometrical interpretations and simple arguments, without in-depth scientific and mathematical knowledge. Examples are taken from classical mechanics whose elementary laws are familiar to the reader. In order to emphasise the general features of chaos, the most important relations are also given in simple mathematical forms, independent of any mechanical interpretation. A broad range of potential applications are presented, ranging from everyday phenomena through engineering and environmental problems to astronomical aspects. It is richly illustrated throughout, and includes striking colour plates of the probability distribution of chaotic attractors.

Chaos occurs in a variety of scientific disciplines, and proves to be the rule, not the exception. The book is primarily intended for undergraduate students in science, engineering and mathematics.

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Chaotic Dynamics

An Introduction Based on Classical Mechanics

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XXVIII (Back cover illustration) Natural distribution on the chaotic attractor of an oscillator kicked with a parabola amplitude (on the parabola attractor). Colouring is the same as in Plate XIX.

[1] Groop, J.-U., Konopka, P. and Müller, R. 'Ozone chemistry during the 2002 Antarctic vortex split', *J. Atmos. Sci.* **62**, 860 (2005).

Preface

> We have just seen that the complexities of things can so easily and dramatically escape the simplicity of the equations which describe them. Unaware of the scope of simple equations, man has often concluded that nothing short of God, not mere equations, is required to explain the complexities of the world.
>
> . . . The next great era of awakening of human intellect may well produce a method of understanding the *qualitative* content of equations.
>
> *Richard Feynman in 1963, the year of publication of the Lorenz model*[1]

The world around us is full of phenomena that seem irregular and random in both space and time. Exploring the origin of these phenomena is usually a hopeless task due to the large number of elements involved; therefore one settles for the consideration of the process as noise. A significant scientific discovery made over the past few decades has been that phenomena complicated *in time* can occur in simple systems, and are in fact quite common. In such *chaotic* cases the origin of the random-like behaviour is shown to be the strong and non-linear interaction of the few components. This is particularly surprising since these are systems whose future can be deduced from the knowledge of physical laws and the current state, in principle, with arbitrary accuracy. Our contemplation of nature should be reconsidered in view of the fact that such deterministic systems can exhibit random-like behaviour.

Chaos is the complicated temporal behaviour of simple systems. According to this definition, and contrary to everyday usage, chaos is not spatial and not a static disorder. Chaos is a type of *motion*, or more generally a type of temporal evolution, dynamics. Besides numerous everyday processes (the motion of a pinball or of a snooker ball, the auto-excitation of electric circuits, the mixing of dyes), chaos occurs in technical, chemical and biological phenomena, in the dynamics of illnesses, in

[1] R. P. Feynman, R. B. Leighton and M. Sands, *The Feynman Lectures on Physics, Vol. II.* New York: Addison-Wesley, 1963, Chap. 40, pp. 11, 12.

elementary economical processes, and on much larger scales, for example in the alternation of the Earth's magnetic axis or in the motion of the components of the Solar System.

There is an active scientific and social interest in this phenomenon and its unusual properties. The motion of chaotic systems is complex but understandable: it provides surprises and presents those who investigate it with the delight of discovery.

Although numerous books are available on this topic, most of them follow an interdisciplinary presentation. The aim of our book is to provide an introduction to the realm of chaos related phenomena within the scope of a single discipline: classical mechanics. This field has been chosen because the inevitable need for a probabilistic view is most surprising within the framework of Newtonian mechanics, whose determinism and basic laws are well known.

The material in the book has been compiled so as to be accessible to readers with only an elementary knowledge of physics and mathematics. It has been our priority to choose the simplest examples within each topic; some could even be presented at secondary school level. These examples clearly show that almost all the mechanical processes treated in basic physics become chaotic when slightly generalised, i.e. when freed of some of the original constraints: chaos is not an *exceptional*, rather it is a *typical* behaviour.

The book is primarily intended for undergraduate students of science, engineering, and computational mathematics, and we hope that it might also contribute to clarifying some misconceptions arising from everyday usage of the term 'chaos'.

The book is based on the material that one of us (T. T.) has been teaching for fifteen years to students of physics and meteorology at Eötvös University, Budapest, and that we have been lecturing together in the last few years.

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Special thanks are due to K. Kulacsy for the translation, to Sz. Hadobás for the aesthetic presentation of the three-dimensional pictures, to G. Károlyi, G. Maros and P. Hámori for the photographs, and to Knorr Bremse (L. Palkovics) and to the *Journal of Atmospheric Science*, as well as the authors, J.-U. Groß, P. Konopka and R. Müller, for their permission to use their figures. The authors would like to thank NASA for the use of Fig. 9.32 and Plates XXIII and XXIV. This work has been supported by Grant OTKA TSO44839.

We would like to express our thanks to the editorial staff of Cambridge University Press; in particular our editors, Simon Capelin and Vince Higgs, and the copy-editor, Irene Pizzie.

How to read the book

The first part of the book presents the basic phenomena of chaotic dynamics and fractals at an elementary level. Chapter 1 provides, at the same time, a preview of the five main topics to be treated in Part III.

Part II is devoted to the analysis of simple motion. The geometric representation of dynamics in phase space, as well as basic concepts related to instability (hyperbolic points and stable and unstable manifolds), are introduced here. Two-dimensional maps are deduced from the equations of motion for driven systems. Elementary knowledge of ordinary differential equations, of linear algebra, of the Newtonian equation of a single point mass and of related concepts (energy, friction and potential) is assumed.

Part III provides a detailed investigation of chaos. The dynamics occurring on chaotic attractors characteristic of frictional, dissipative systems is presented first (Chapter 5). No preliminary knowledge is required upon accepting that two-dimensional maps can also act as the law of motion. Next, the finite time appearance of chaos, so-called transient chaotic behaviour, is investigated (Chapter 6). Subsequently, chaos in frictionless, conservative systems is considered in Chapter 7, along with its transient variant in the form of chaotic scattering in Chapter 8. Chapter 9 covers different applications of chaos, ranging from engineering to environmental aspects.

Problems constructed from the material of each chapter (many also require computer-based experimentation) motivate the reader to carry out individual work. Some of the solutions are given at the end of the book; the remainder appear (in a password-protected format) on the following website: www.cambridge.org/9780521839129.

Topics only loosely related to the main train of ideas, but of historical or conceptual interest, are presented in Boxes. Some important technical matter (for example numerical algorithms, writing equations in dimensionless forms) are relegated to an Appendix. A bibliography is given at the end of the book, and it is broken down according to topics, chapters and Boxes.

In order to emphasize the general aspects of chaos, the most important relations are also given in a formulation independent of mechanics (see Sections 3.5, 4.7, 5.4, 6.3, 7.5 and 8.4). The description of motion occurs primarily in terms of ordinary differential equations, and we concentrate on chaos from such a mathematical background. Irregular dynamics generated by other mathematical structures, which do not represent real phenomena, are thus beyond the scope of the book. The case of one-dimensional maps is mentioned therefore as a special limit only. This approach might provide a useful introduction to chaos for all disciplines whose dynamical phenomena are described by ordinary differential equations.

The book is richly illustrated with computer-generated pictures (24 of which are in colour), not only to provide a better understanding, but also to exemplify the novel and aesthetically appealing world of the geometry of dynamics.

Part I
The phenomenon: complex motion, unusual geometry

Chapter 1
Chaotic motion

1.1 What is chaos?

Certain long-lasting, sustained motion repeats itself exactly, periodically. Examples from everyday life are the swinging of a pendulum clock or the Earth orbiting the Sun. According to the view suggested by conventional education, sustained motion is always regular, i.e. periodic (or at most superposition of periodic motion with different periods). Important characteristics of a periodic motion are: (1) it repeats itself; (2) its later state is accurately predictable (this is precisely why a pendulum clock is suitable for measuring time); (3) it always returns to a specific position with exactly the same velocity, i.e. a single point characterises the dynamics when the return velocity is plotted against the position.

Regular motion, however, forms only a *small part* of all possible sustained motion. It has become widely recognised that long-lasting motion, even of simple systems, is often *irregular* and does not repeat itself. The motion of a body fastened to the end of a rubber thread is a good example: for large amplitudes it is much more complex than the simple superposition of swinging and oscillation. No regularity of any sort can be recognised in the dynamics.

The irregular motion of simple systems, i.e. systems containing only a few components, *is called chaotic*. As will be seen later, the existence of such motion is due to the fact that even *simple* equations can have very *complicated* solutions. Contrary to the previously generally accepted view, the simplicity of the equations of motion does not determine whether or not the motion will be regular.

Understanding chaotic motion requires a non-traditional approach and specific tools. Traditional methods are unsuitable for the description

Table 1.1. *Comparison of regular and chaotic motion.*

Regular motion	Chaotic motion
self-repeating	irregular
predictable	unpredictable
of simple geometry	of complicated geometry

of such motion, and the discovery of the ubiquity of chaotic dynamics has become possible through *computer-based experimentation*. Detailed observations have led to the result that chaotic motion is characterised by the *opposite* of the three properties mentioned above: (1) it does not repeat itself, (2) it is unpredictable because of its sensitivity to the initial conditions that are never exactly known, (3) the return rule is complicated: a complex but regular structure appears in the position vs. velocity representation. The differences between the two types of dynamics are summarised in Table 1.1.

The properties of chaotic systems are unusual, either taken individually or together; the most efficient way to understand them is by considering particular cases. In the following, we present the chaotic motion of very simple systems on the basis of numerical simulations, which are unavoidable when studying chaos. It should be emphasised that all of our examples are discussed for a unique set of parameters, and that slightly different choices of the parameters could result in substantially different behaviour. These examples also serve to classify different types of chaos and help in developing the new concepts necessary for a detailed understanding of chaotic dynamics.

1.2 Examples of chaotic motion

1.2.1 Irregular oscillations, driven pendulum – the chaotic attractor

Objects mounted on spring suspensions (for example car wheels and spin-dryers) oscillate. Because of the losses that are always present due to friction or air drag, these oscillations, when left alone, are damped and ultimately vanish. Sustained motion can only develop if energy is supplied from an external source. The supplied energy can be a more or less periodic shaking, i.e. the application of a driving force (caused by interactions with pot-holes in the case of the car wheel and by the uneven distribution of clothes in the spin-dryer), as indicated schematically in Fig. 1.1.

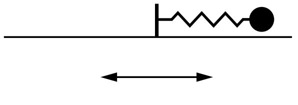

Fig. 1.1. Model of driven oscillations: a body of finite mass is fixed to one end of a weightless spring and the other end of the spring is moved sinusoidally with time.

As long as the displacement is small, the spring obeys a *linear* force law to a good approximation: the magnitude of the restoring force is

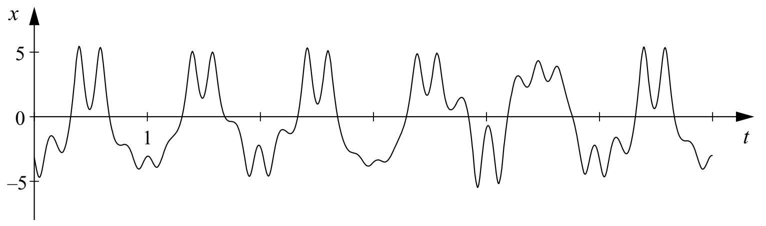

Fig. 1.2. Irregular sustained oscillations of a point mass fixed to the end of a stiffening spring (a driven non-linear oscillator), driven sinusoidally in the presence of friction.

proportional to the elongation. In this case the sustained motion is regular: it adopts the period of the driving force. If the natural period of the spring is close to that of the driving force, then the amplitude may become very large and the well known phenomenon of *resonance* develops. For large amplitudes, however, the force of the spring is usually no longer proportional to the elongation; i.e., the force law is *non-linear*. Resonance is therefore a characteristic example for the appearance of non-linearity.

For non-linear force laws, the restoring force increases more rapidly or more slowly than it would in linear proportion to the elongation: we can speak of stiffening or softening springs, respectively. Whichever type of non-linearity is involved, the sustained state of the driven oscillation may be chaotic. A qualitative explanation is that the spring is not able to adopt exactly the sinusoidal, harmonic motion of the forcing apparatus, since its own periodic behaviour is no longer harmonic. Thus, the sustained dynamics follows the driving force in an averaged sense only, but always differs from it in detail (instead of the uniform hum of the car or the spin-dryer, an irregular sound can be heard in such situations). Neither the amplitude nor the frequency is uniform: the sustained motion does not repeat itself regularly; it is chaotic.

Figure 1.2 shows the motion of a body fixed to the end of a stiffening spring and driven sinusoidally.[1] It can clearly be seen that there is no repetition in the displacement vs. time curve; i.e., the motion is *irregular*.

Slightly different initial conditions result in significant differences in the displacement after only a short time (Fig. 1.3): the dynamics is unpredictable. This figure also shows that the long-term behaviour is of a similar nature in both cases: the two motions are equivalent in a *statistical* sense.

[1] The precise equations of motion of the examples in this section can be found in Sections 5.6.2 and 5.6.3.

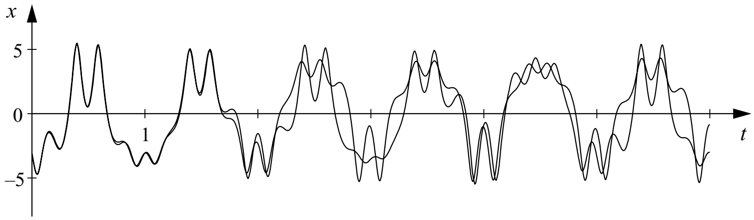

Fig. 1.3. Two sets of motion which started from nearly identical positions. The small initial difference increases rapidly: the motion is sensitive to the initial conditions and therefore it is unpredictable.

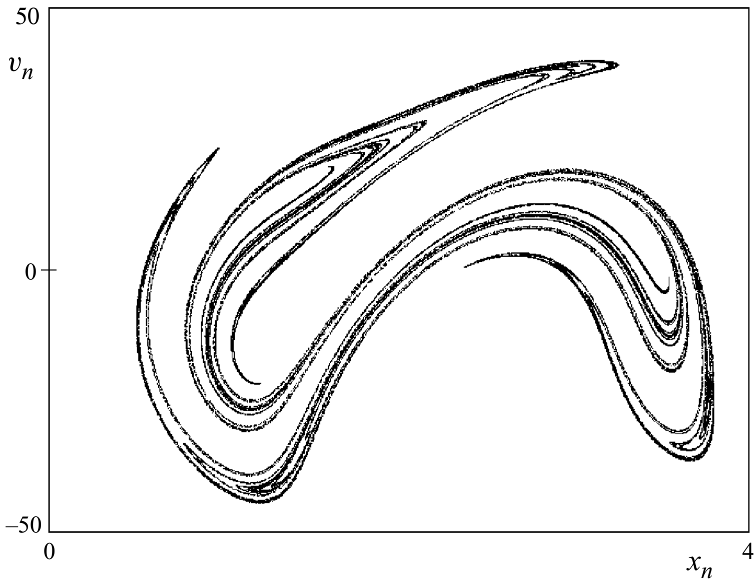

Fig. 1.4. Pattern resulting from a sustained non-linear oscillation in the velocity vs. position representation, using samples taken at time intervals corresponding to the period of the driving force. The position and velocity co-ordinates of the *n*th sample are x_n and v_n, respectively.

An interesting structure reveals itself when we do not follow the motion continuously, but only 'take samples' of it at equal time intervals. Figure 1.4 and Plate I have been generated by plotting the position and velocity co-ordinates (x_n, v_n) of the sustained motion at integer multiples, n, of the period of the driving force, through several thousands of periods.

It is surprising that there are numerous values of x_n to which many (according to detailed examinations, an *infinite* number of) different velocity values belong. Furthermore, the possible velocity values corresponding to a single position co-ordinate x_n do *not* form a continuous interval anywhere. The whole picture has a thready, filamentary pattern, indicating that chaos is associated with a definite structure. This pattern is much more complicated than those of traditional plane-geometrical objects: it is a structure called a *fractal* (a detailed definition of fractals will be given in Chapter 2). Remember that a *single point* would correspond to a periodic motion in this representation. Chaotic motion is therefore infinitely more complicated than periodic motion.

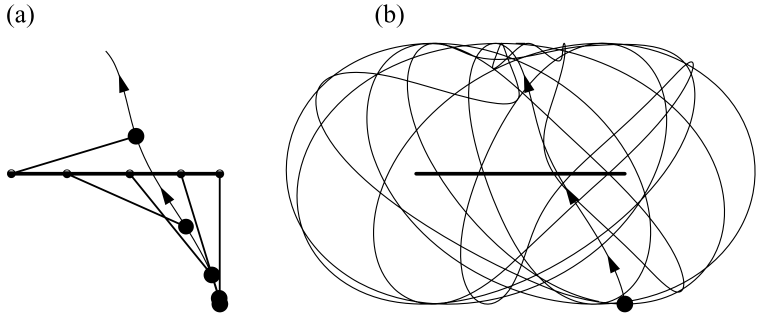

Fig. 1.6. Motion of a driven pendulum. (a) The pendulum a few moments after starting from a hanging state (over the first half period). (b) The path of the end-point of the pendulum for a longer time: the pendulum swings irregularly and often turns over. The horizontal bar indicates the interval over which the suspension point moves.

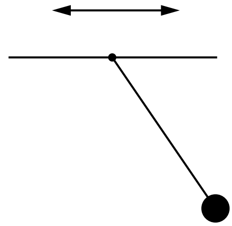

Fig. 1.5. Driven pendulum: the pendulum is driven by the periodic movement of its point of suspension in the horizontal plane.

Another example is the behaviour of a driven pendulum (Fig. 1.5). The large-amplitude swinging of a traditional simple pendulum is non-linear, since the restoring force is not proportional to the deflection angle but to the sine of this angle. Without any driving force, the swinging ceases because of friction or air drag: sustained motion is impossible. The pendulum can be driven in different ways. We examine the case when the point of suspension is moved horizontally, sinusoidally in time. In order to avoid the problem of the folding of the thread, the point mass is considered to be fixed to a very light, thin rod. With a sufficiently strong driving force, the motion may become chaotic. Figure 1.6 shows the path of the pendulum in the vertical plane.

Note that the pendulum turns over several times in the course of its motion. The 'upside down' state is especially unstable, just like that of a pencil standing on its point. Two paths of the pendulum starting from nearby initial positions remain close to each other only until an unstable state, an 'upside down' state, separates them. Then one of them turns over, while the other one falls back to the side it came from (Fig. 1.7). The reason for the unpredictability is that the motion passes through a series of *unstable states*.

The structure underlying the irregular motion can again be demonstrated by following the motion initiated in Fig. 1.6 for a long time and taking samples from it by plotting the position (angular deflection) and velocity (angular velocity) co-ordinates (x_n, v_n) at intervals corresponding to the period of the driving force (Fig. 1.8 and Plate II).

In a frictional (dissipative) system, sustained motion can only develop if some external energy supply (driving) is present. Regardless of the initial state, the dynamics converges to some sustained behaviour that will therefore be called an attracting object, or an *attractor* (for the

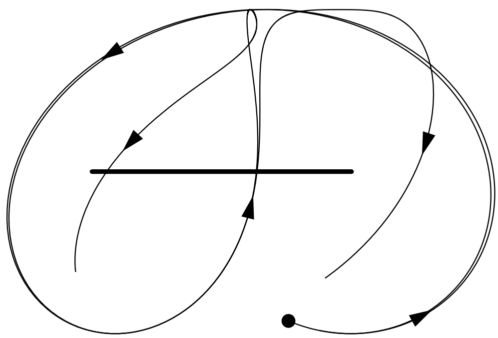

Fig. 1.7. Separation of the paths of two identical driven pendulums starting from nearby points while passing an unstable state. The notation is the same as in Fig. 1.6. The arrows show the direction in which the end-points of the pendulums move.

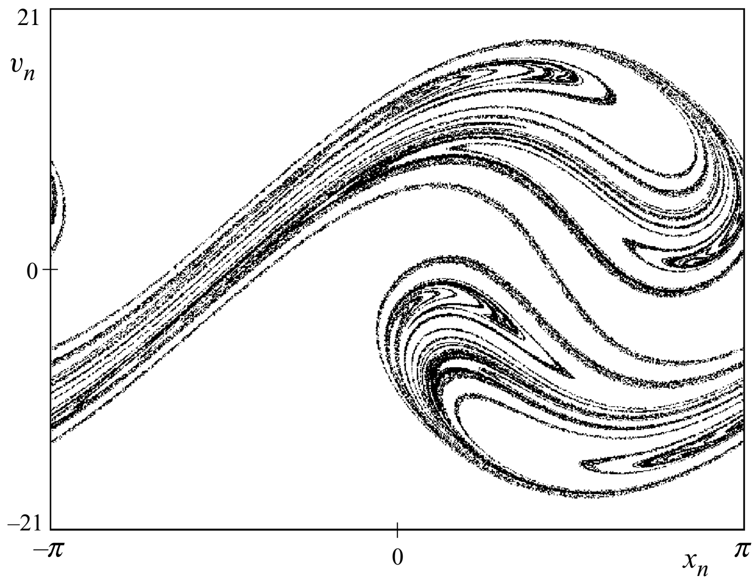

Fig. 1.8. Pattern resulting from a chaotic driven pendulum (chaotic attractor) obtained by plotting the state of the pendulum in the position–velocity co-ordinates at integer multiples of the driving period.

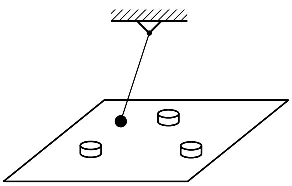

Fig. 1.9. The magnetic pendulum: magnets are fixed to the table and a point mass attracted by the magnets is fixed to the end of the thread. The pendulum ultimately settles in an equilibrium state pointing towards one of the magnets, but only after some irregular, chaotic motion.

exact definition, see Section 3.1.2). *Simple* attractors correspond either to regular or to ceasing motion. A sufficiently large supply of energy inevitably brings about the non-linearity of the system; the sustained dynamics is then usually irregular, i.e. chaotic. This is accompanied by the presence of a *chaotic attractor*, also called a *strange attractor* because of its peculiar structure. Figures 1.4 and 1.8 display examples of chaotic attractors.

1.2.2 Magnetic and driven pendulums, fractal basin boundary – transient chaos

Consider a pendulum, the end-point of which is a small magnetic body, moving above three identical magnets placed at the vertices of a horizontal equilateral triangle (Fig. 1.9). When the force between the

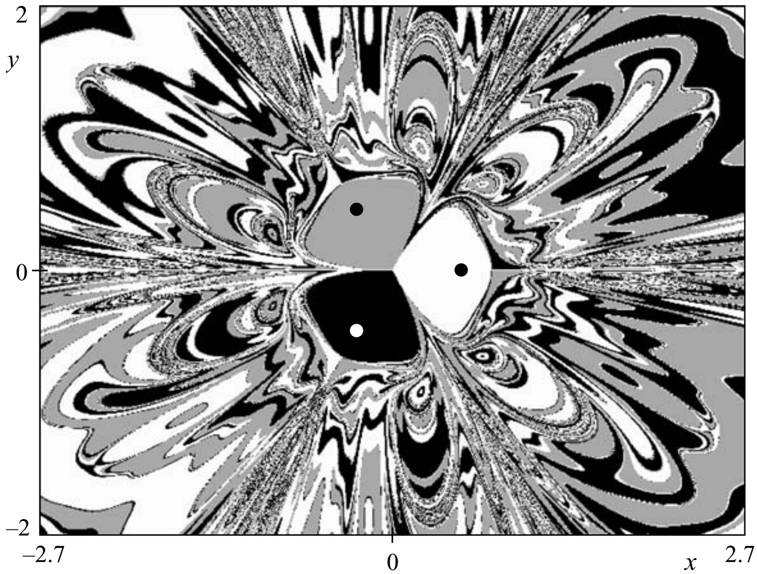

Fig. 1.10. Basin of attraction of the three equilibrium states of the magnetic pendulum (one white and two black dots). Each point on the horizontal plane is shaded according to the magnet in whose neighbourhood the pendulum comes to a rest when starting above that point with zero initial velocity.

end of the pendulum and the magnets is attracting, the pendulum can come to a halt, pointing towards any of the magnets. Thus there are three simple attractors in the system. Starting above any point of the plane, we can use a computer to calculate which magnet the pendulum will be closest to after coming to rest.[2] By assigning three different colours to the three attractors, and to the corresponding initial positions that converge towards them, the whole plane can be coloured. Each identically coloured area is a *basin of attraction*. Surprisingly, the basin boundaries are interwoven and entangled in a complicated manner (see Fig. 1.10 and Plates III–VI); these simple attractors have *fractal basin boundaries*. (Naturally, the close vicinity of each attractor appears in one colour only: the boundaries do not come close to the attractors.)

Motion starting near the fractal boundary remains irregular for a while, exhibiting *transient chaos*, i.e. chaos lasting for a finite period of time (Fig. 1.11), but ultimately it ends up on one of the attractors.

A driven pendulum (Fig. 1.5) may also exhibit transient chaos. When the friction is sufficiently large, the pendulum can exhibit regular sustained motion only. There are two options for the given parameters (see Fig. 1.12, which depicts the paths corresponding to these two simple attractors in the vertical plane). An overall view of the basins of attraction can again be obtained by representing the starting point in the position

[2] The equations of motion of the magnetic pendulum can be found in Section 6.8.3.

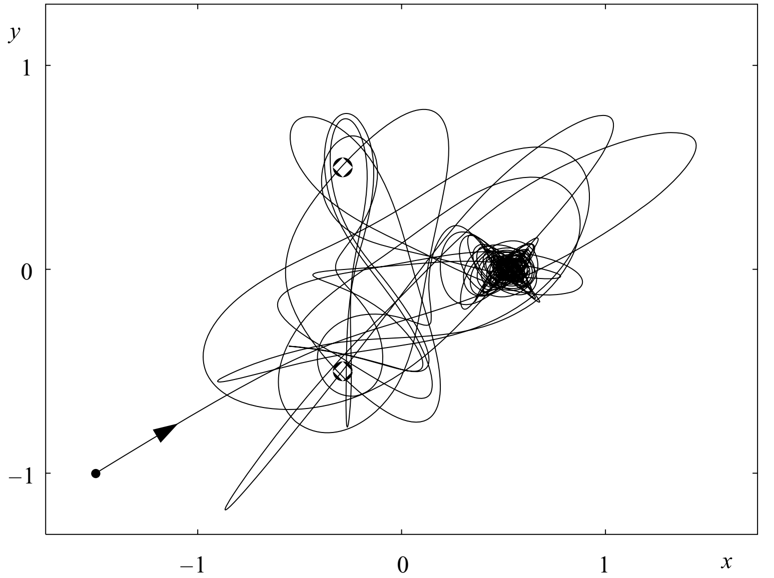

Fig. 1.11. Path of the end-point of the magnetic pendulum viewed from above. The motion is irregular before reaching one of the rest positions: it is transiently chaotic. (The fixed magnets are represented by solid black dots.)

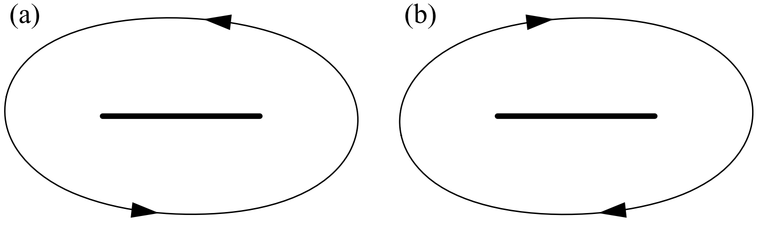

Fig. 1.12. Simple periodic attractors of the driven pendulum: for sufficiently strong friction only these two types of sustained motion exist. All the different initial conditions lead to one of these motions, corresponding to a simple attractor each.

(angular deflection) – velocity (angular velocity) plane in the colour of the attractor which the motion ultimately converges to (Fig. 1.13 and Plate VII).

Motion starting close to the boundary is similar initially to that seen in the case of the chaotic attractor, but it ultimately converges to one of the simple attractors. Irregular dynamics has a finite duration; it is transient. There exist, however, very exceptional initial conditions from which the dynamics never reaches any of the attractors, and is chaotic for any length of time. There exists an infinity of such motion (Fig. 1.14), but the initial conditions that describe these state do not form a compact domain in the plane, but rather a fractal cloud of isolated points called a *chaotic saddle*.

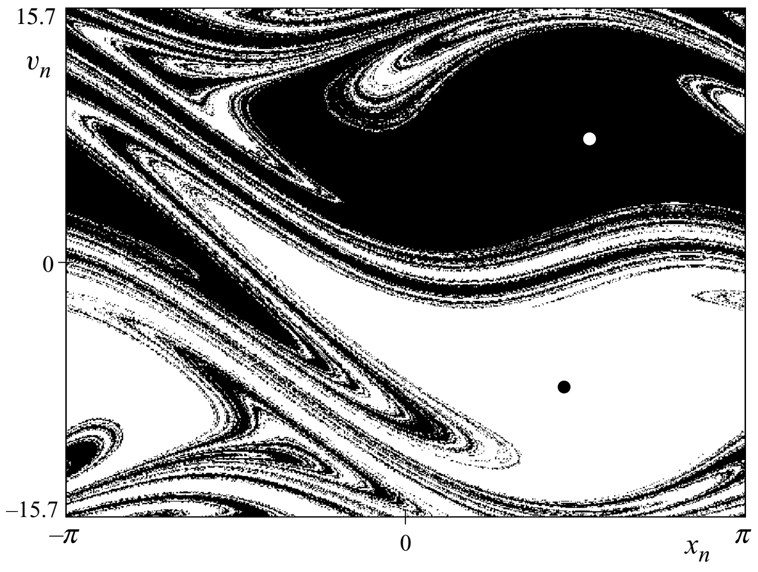

Fig. 1.13. Basins of attraction in the driven pendulum on the plane of initial conditions. The two simple attractors in Fig. 1.12 appear here as points (white and black dots), and the initial states converging towards them are marked in black and white, respectively.

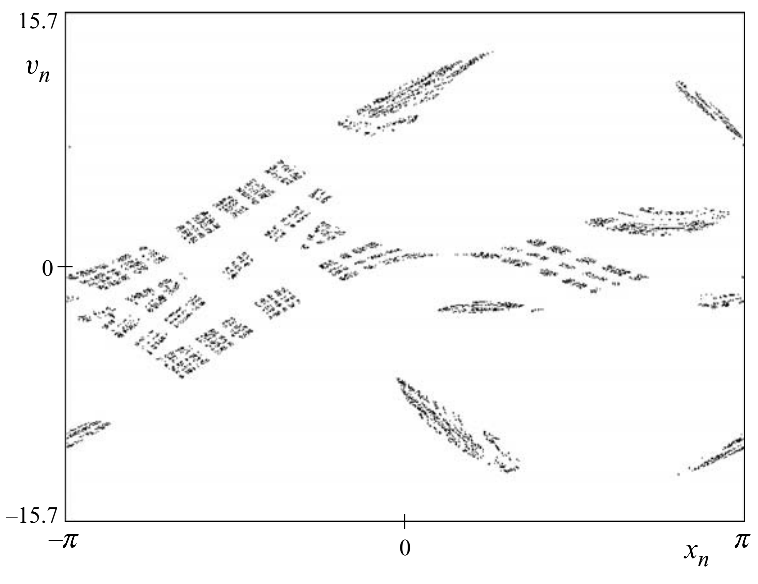

Fig. 1.14. Initial states of the driven pendulum of Fig. 1.13 that never reach either simple attractor: all points shown here are on the basin boundary and, if followed in time, they keep moving between themselves after every period of the driving force. This chaotic saddle is responsible for chaotic dynamics of transient type.

Thus, chaotic dynamics can also occur if the sustained forms of motion are regular, but there are many possible transient routes (chaotic transients) leading to them. In such cases several simple attractors coexist, each with its own basin of attraction defined by the set of initial conditions which converges to the given attractor. The basins of attraction often penetrate each other, and their boundaries can also be filamentary fractal curves. The motion starting from the vicinity of these fractal basin

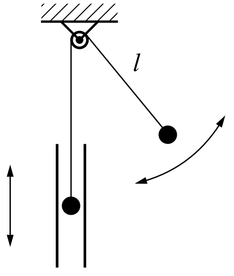

Fig. 1.15. Body swinging on a pulley: two point masses are joined by a thread wound around a pulley of negligible radius, one of them swinging freely in a vertical plane, the other moving vertically only.

boundaries behaves randomly along the boundary for a long time, as if it is difficult to decide which attractor to choose. During this period of uncertainty the motion is irregular and is bound to fractal structures.

1.2.3 Body swinging on a pulley, ball bouncing on slopes – chaotic bands

Let us examine what happens in frictionless (conservative) systems. Consider two point masses joined by a thread wound about a small pulley (see Fig. 1.15). The case when both points can only move vertically is a well known secondary school problem. Here, however, we let one of the point masses swing in a vertical plane (with the thread always stretched for the sake of simplicity). It will be shown that new types of chaotic motion develop under such conditions.[3]

The instantaneous length, l, of the thread of the point mass that can swing is one of the position co-ordinates; the other is the angle of deflection. In the traditional arrangement, where only vertical displacement is allowed, the heavier mass always pulls the other one up, but the situation is much more interesting now. If the swinging body is thrust horizontally with sufficient momentum while the other body moves downwards, then the swinging body turns over several times, the thread shortens, the body spins faster, and thus becomes able to pull the other body upward, even if the latter is the heavier. (It is assumed that the swinging body does not collide with anything and that the thread does not become unattached from the pulley when turning over.) Thus, a long-lasting, complicated, chaotic motion may develop. The path of the swinging body and the length of the thread vs. time are shown in Figs. 1.16(a) and (b), respectively. Again, the paths of the motion starting from nearby initial conditions soon branch off; the motion is unpredictable.

An overview of the motion corresponding to a given total energy can be presented with the help of some sampling technique. The system is not driven in this case, and therefore sampling will not take place at identical time intervals, rather at identical configurations: whenever the swinging body passes through the vertically hanging configuration, the instantaneous length, $l_n \equiv x_n$, of the swinging thread and the rate of change of this length, v_n, will be plotted as one point in the plane. Thus, chaotic motion is represented by a sequence of points jumping around in a disordered manner and dotting a finite region of the plane; This is called the *chaotic band* (Fig. 1.17). Other initial conditions outside of the

[3] The equations of motion of the examples in this section can be found in Sections 7.4.1 and 7.4.3.

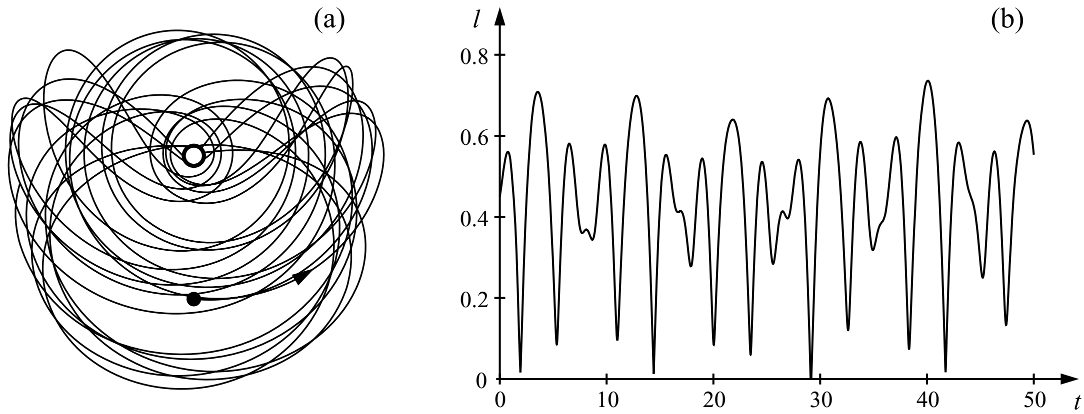

Fig. 1.16. Frictionless motion of a body swinging on a pulley. (a) The spatial path of the swinging body (the initial position is marked by a black dot, the pulley by the centre of an open circle); (b) the dependence of the length of the swinging thread on time within the same time interval.

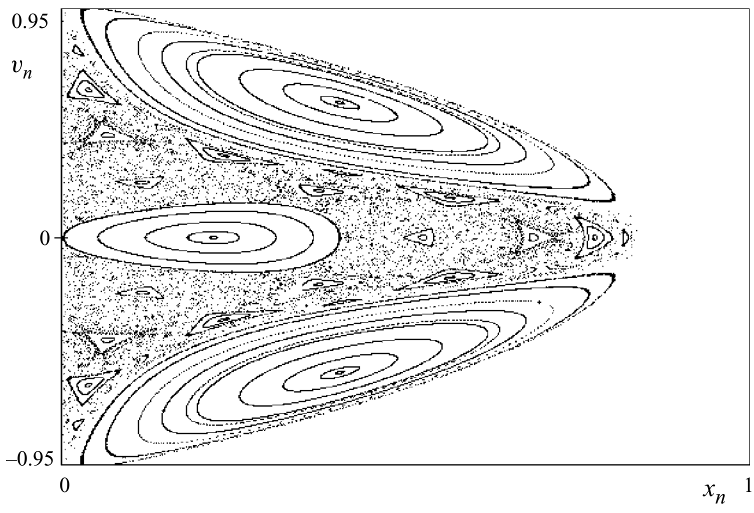

Fig. 1.17. Overview of the motion of a body swinging on a pulley without air drag and at a given total energy, on the basis of samples of length and velocity (x_n, v_n) taken when passing through the vertical position, from the left. The dotted region is a chaotic band, which can be traced out by motion starting from a single initial condition. The sets of closed curves form regular islands.

band may result in a single point, a few points or a continuous line, all of which correspond to regular motion. These objects together usually form closed domains that can be called regular islands. A frictionless chaotic system is characterised by a hierarchically nested pattern of chaotic bands and islands. Together they form a complicated structure of interesting texture, different from the fractals presented so far (see Fig. 1.17 and Plate VIII).

Our second example illustrates the fact that elastic collisions with flat surfaces can also lead to chaotic motion. Maybe the simplest situation

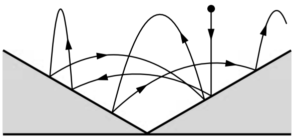

Fig. 1.18. Ball bouncing on two slopes of identical inclination that face each other in a gravitational field.

is the case of an elastic ball bouncing on two slopes that face each other (Fig. 1.18). (A motion very similar to this can be realised in experiments with atoms.) Chaotic behaviour arises because after bouncing back from the opposite slope the ball does not necessarily hit its original position. Non-linearity and inherent instability are caused by the break-point between the slopes. The chaotic motion of two balls dropped from identical heights but slightly different positions soon branches off (Fig. 1.19), just as in the previous examples.

A sampling technique providing a good overview of the dynamics is in this case to plot the two velocity components as points of a plane, at the instant of each bounce (Fig. 1.20).

There is no need to apply driving forces in order to sustain a motion in frictionless systems, since there is no dissipation and energy is conserved. On the other hand, this motion cannot converge to a well defined sustained motion because there are *no* attractors in frictionless, conservative

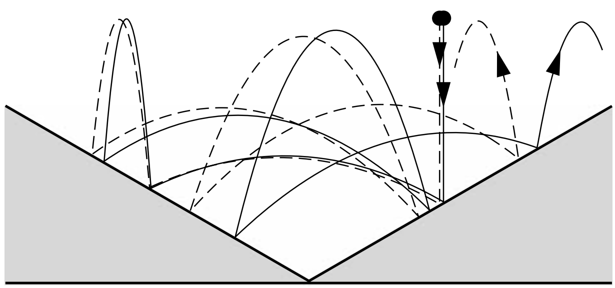

Fig. 1.19. Paths of two balls starting from nearly identical initial positions above the double slope (the continuous line is identical to that drawn in Fig. 1.18). The motion is sensitive to the initial conditions.

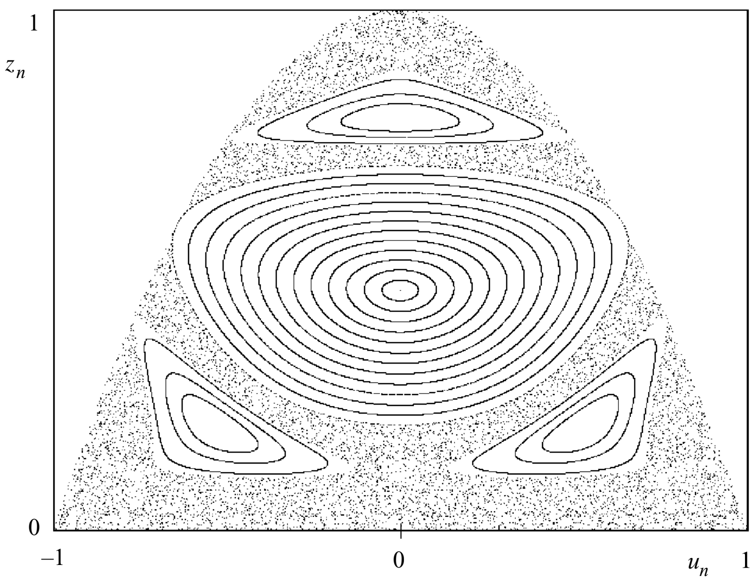

Fig. 1.20. Pattern generated by the possible motions of a ball bouncing on a double slope with given total energy in a representation where the abscissa is the velocity component parallel to the slope (u_n) and the ordinate is the square of the component perpendicular to the slope (z_n) taken at the instance of the nth bounce. The dotted region is a chaotic band. The angle of inclination of the slope is 50°.

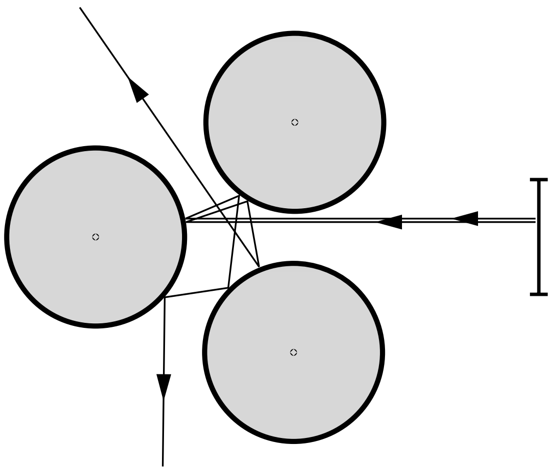

Fig. 1.21. The three-disc problem: particles bouncing perfectly elastically between identical discs fixed at the vertices of a regular triangle. Paths starting from nearby initial points soon diverge.

systems. As a result, the nature of all motion strongly depends on the initial conditions and the total energy. Regular motion corresponds to certain sets of initial conditions, while chaotic motion corresponds to other sets. The initial conditions that lead to chaotic motion form chaotic bands that, contrary to chaotic attractors, are plane-filling objects.

1.2.4 Ball bouncing between discs, mirroring Christmas-tree ornaments – chaotic scattering

Three identical discs are placed at the vertices of a regular triangle in the horizontal plane and a ball is bouncing among them – like in a pinball machine (see Fig. 1.21). The motion is considered to be frictionless; therefore the velocity of the particle is constant during the entire process. Starting from a given point, the motion depends on the initial direction of the velocity vector. Some initial conditions cause the particle to bounce for a *very* long time between the discs; during this time the dynamics of the particle is complicated and aperiodic.[4] Two slightly different initial conditions cause the paths to diverge rapidly (Fig. 1.21); therefore, this motion is also chaotic. The deviation of paths with nearby initial conditions is easy to explain, since the discs act as dispersing mirrors and the angle between the straight sections of the paths increases with each collision. The complicated structure related to the motion manifests itself in several ways. The number of bounces experienced by the particles that start along a segment in a given direction towards the discs strongly depends on the initial position. Some initial conditions lead to many collisions (Fig. 1.22). Moreover, there is an infinity of initial points

[4] A detailed investigation of this problem can be found in Section 8.2.3.

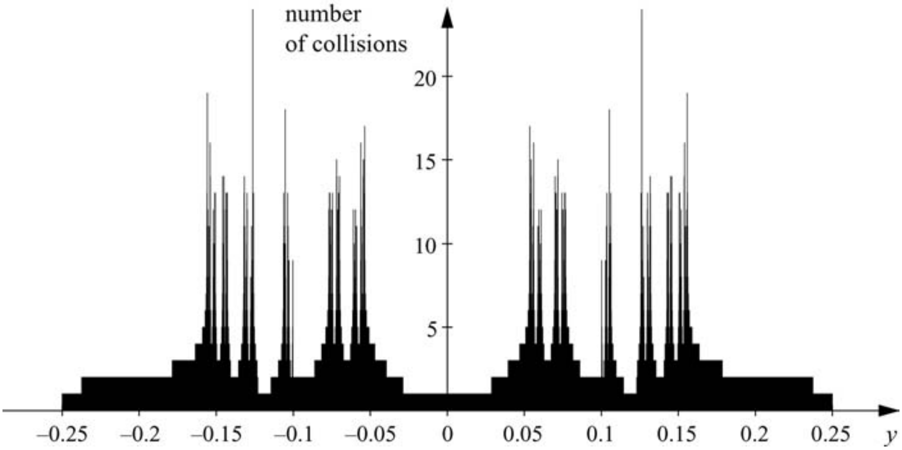

Fig. 1.22. Number of collisions of 20 000 particles starting with unit velocity at right angles to the line segment drawn in Fig. 1.21, as a function of the *y* co-ordinate. (The centres of the discs are at unit distance from each other.)

from which an arbitrary number of bounces can, in principle, occur (the particles then become trapped among the discs), but these do not form an interval: they form a scattered fractal cloud along the line segment.

Three or four Christmas-tree ornaments in contact with each other reflect light several times before light reaches our eyes. The interesting fractal images resulting from these reflective spheres (Plates IX and X) are examples of everyday consequences of chaotic motion.

The process whereby a significant force is only present in a finite region of a frictionless system is usually called scattering. Such a force can be tested via the motion of particles approaching from large distances. This motion is initially rectilinear, but the force causes the path to curve; then the particle leaves the scattering process and resumes its rectilinear motion, most probably in a new direction. The chaotic nature of the process arises because the motion may become long-lasting and irregular in the region where finite forces are in action. In these cases we speak of *chaotic scattering*. The average lifetime of chaos, similar to the dynamics around fractal basin boundaries, is finite. Even though there are no attractors in this case, the different outgoing states play a role similar to that of simple attractors. Chaotic scattering always involves transient chaos.

1.2.5 Spreading of pollutants – an application of chaos

Chaotic motion occurs in numerous phenomena related to practical applications. One of these is discussed here: the spreading of pollutants

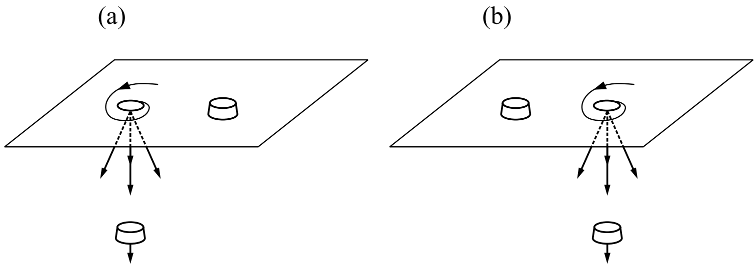

Fig. 1.23. Tank with two outlets. The outlets, when opened alternately, generate chaotic advection in a flat container. (a) and (b) illustrate the flow in the first and second half period, respectively. The flow itself is very simple; the advection of the particles is nevertheless chaotic.

in a flowing medium (air or water). The environmental significance of this matter is obvious.

Consider a large and flat container with two point-like outlets. Water whirls while flowing out. The two outlets are alternately open, each for half a period (Fig. 1.23), yielding a flow periodic in time. We want to know how a dye particle moves in this flow. For the sake of simplicity, it is assumed that the material properties of the dye are identical to that of the liquid; the only difference is the colour. In this case, the motion is determined by the condition that the instantaneous velocity of the particle is identical to that of the liquid. The path of the particle is then easy to follow.[5] Chaos arises because a particle moving towards the open outlet may not reach it within half a period; therefore it starts moving towards the other outlet, but again it may be too late to be drained, and so on. It may thus take a very long time before the particle flows out of the container. Figure 1.24 illustrates two complicated paths starting very close to each other, but leaving the tank via different outlets.

In the context of the spreading of pollution, it is especially important to follow the motion of a dye droplet. This corresponds to the examination of the dynamics of an ensemble of particles, each starting from a certain initial region, the initial shape of the droplet. A surprising discovery is that, despite the chaotic motion of each individual particle, the drop traces out a well defined thready fractal structure (after losing its original compact shape) within a short time (Fig. 1.25 and Plate XI).

The spreading of impurities in the form of filamentary patterns can be observed in numerous phenomena, ranging from oil stains on road surfaces through the mixing of cream in coffee to the propagation of

[5] The equation of motion for this example can be found in Section 9.4.1.

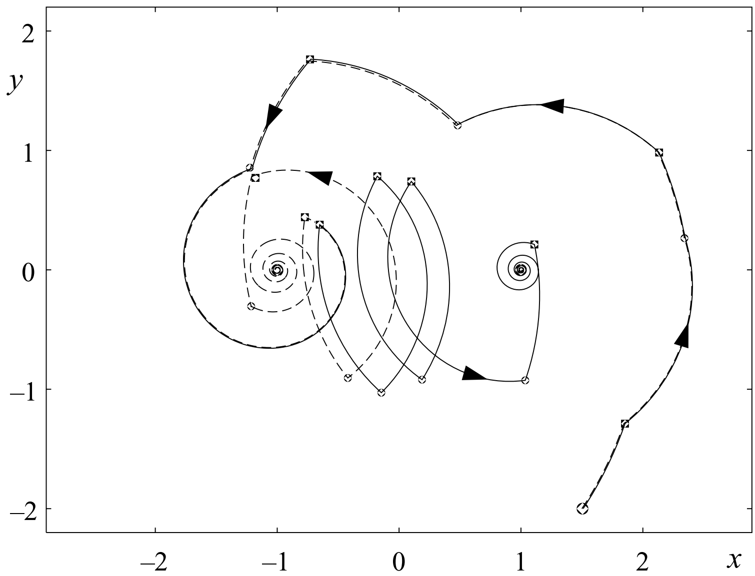

Fig. 1.24. Paths of two dye particles (continuous and broken lines) starting near each other in a tank with two outlets (situated at $(-1, 0)$ and $(1, 0)$). The consecutive black dots (squares) indicate the instants when the left (right) outlet is opened. In the initial instant the left outlet is opened. The time spent in the tank is very different for the two particles.

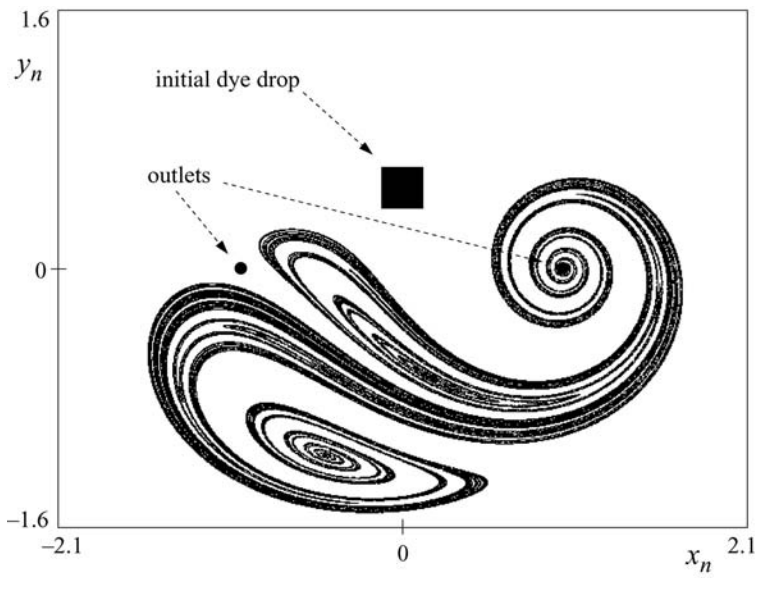

Fig. 1.25. Shape of a dye drop initially and after five periods in the tank with two outlets.

chemical pollution in the atmosphere. This thready structure unmistakably signals the chaotic motion of the individual pollutant particles.

The type of chaos found in the advection problem may depend on the parameters of the system. The problem of the tank with two outlets in the above arrangement is analogous to the problem of the fractal basin boundaries. If the outlets are closed but the alternating whirling motion is sustained by mixers, the so-called blinking vortex model is obtained. In this case there is no outflow that could be the analogue of the simple

Table 1.2. *Comparison of the traditional and phase space representations of dynamics.*

Traditional representation	Phase space representation
instantaneous co-ordinates	point in the phase space
time-dependence ($x(t)$, $v(t)$)	trajectory ($v(x)$)
structure in time	structure in phase space
individual	global

attractors for the advected particle; the chaotic behavior of the dye or impurity particles is therefore the same as that of conservative systems. It may be important to take into account that the density of the known pollutant may not be identical to that of the fluid and/or that the particle is of finite size (for example in the case of aerosols). Consequently, the velocity of the particles usually differs from that of the liquid. It can be shown that advection then corresponds to dissipative systems. The advection dynamics can then have attractors, often even chaotic ones. This implies that pollutant particles accumulate along a fractal pattern on the surface of the fluid. This phenomenon can indeed be observed in lakes, bays and harbours as a direct consequence of chaos!

1.3 Phase space

Our examples have shown that the traditional representation via the displacement or velocity vs. time graphs does not provide a suitable overview of the motion, since, however long the observation time may be, one can always expect some further novel behaviour. The order appearing in chaos does not manifest itself in the position vs. time representation, but rather in the position vs. velocity representation.

The instantaneous *state* of a mechanical system is given by its position and velocity co-ordinates, since the motion can be continued uniquely if one knows these co-ordinates and the dynamical equation. The position and velocity variables define the *phase space* of a system (for more details, see Section 3.5). For motion occurring along a straight line with position x and velocity v, the phase space is the (x, v) plane. The state of the system is represented by a single point in the phase space, and this point wanders, indicating the change of the state, as time passes. The path of the motion in phase space is called the *trajectory* (Fig. 1.26). The trajectory itself does not indicate directly how fast this change is in time. The arrow only shows the direction of the motion. A set of several trajectories, however, provides a global overview of the different possible types of motion of the system (see Table 1.2).

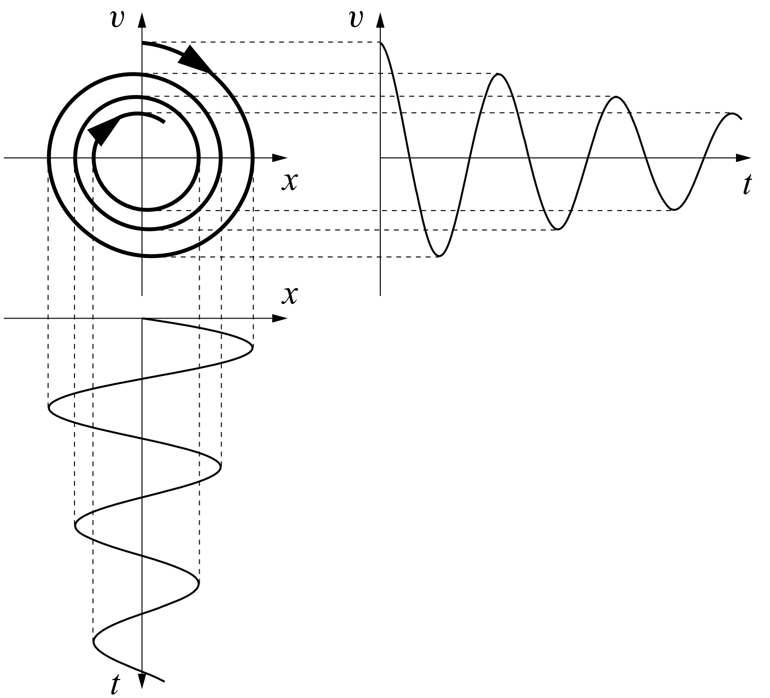

Fig. 1.26. Trajectory in phase space (thick line). The path described by the motion of a particle in phase space can be constructed from the respective projections of the $x(t)$ and $v(t)$ graphs. The direction of time is represented by the arrow on the trajectory.

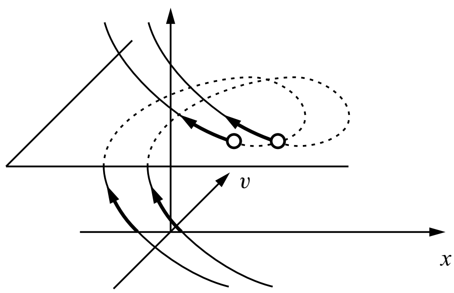

Fig. 1.27. Monitoring trajectories using maps. In higher-dimensional phase spaces, samples are taken on certain sections. The rule relating the co-ordinates of two consecutive intersects of a trajectory with this surface (or equivalent ones) is a map.

Two data points are often insufficient to define the state of a system uniquely; i.e., the phase space is three- or more dimensional (this is always the case with chaos). In such a situation it is useful to take samples from the higher dimensional phase space according to some rule. This is usually done by taking a 'section' of the phase space and recording the points of a trajectory on this section only, as illustrated by the schematic Fig. 1.27. In driven cases it is advisable to 'look at' the system at time instants corresponding to integer multiples of the driving period. This representation is called a *stroboscopic map*. Thus, Figs. 1.4 and 1.8 exhibit the results of stroboscopic mappings. In non-driven cases a section can be defined by the fulfilment of conditions corresponding to certain configurations. This defines a *Poincaré map*, like the one seen in Fig. 1.17.

Our examples have demonstrated that it is in such maps that the fractal structure of chaotic dynamics becomes plausible. Only in special cases (like those of the magnetic pendulum, the mirroring spheres and advection) can fractal structures be observed in real space. Therefore the use of phase space is inevitable as a means of understanding the structure accompanying chaos. (However, phase space is very useful in investigating regular motions also.)

1.4 Definition of chaos; summary

Chaos is a motion, a temporal dynamics of *simple* systems that can be described in terms of a few variables. Such motion is:

- irregular in time (it is not even the superposition of periodic motions, it is really aperiodic);
- unpredictable in the long term and sensitive to initial conditions;
- complex, but ordered, in the phase space: it is associated with a fractal structure.

These properties are so strongly and uniquely bound to chaotic dynamics that they may be used to define 'chaos'. We shall apply this definition throughout the book.

The listed characteristics are present simultaneously: when a simple system is aperiodic over a long time, its evolution must be unpredictable and representable by a fractal structure in suitable co-ordinates. From a traditional view, all three characteristics are novel and surprising. A single common feature underlying them is that the long-term behaviour is random-looking, irregular and therefore it can properly be described by using *probabilistic* concepts only.

On the other hand, not all complicated temporal behaviour can be considered to be chaotic, only those that derive from simple laws. *Noisy* motion is the random behaviour of some component of a system with a great number of constituents (for example the Brownian motion of a particle), which is the consequence of the complicated interaction with the environment (i.e. the other constituents). Chaos is a *bridge* between regular and noisy motion. It differs from regular motion in that it is probabilistic and differs from noise in that its randomness is due to the strong interaction (following from simple laws) of the few constituents, i.e. to the *inherent* dynamics. Noisy motion fills the phase space uniformly, thus fractal structures *cannot* develop.

Table 1.3. *Basic types of chaos and related phenomena and sets.*

	Permanent chaos	Transient chaos
Dissipative	motion on chaotic attractors	chaotic transients towards attractors, fractal basin boundaries (chaotic saddles)
Conservative	motion in chaotic bands	chaotic scattering (chaotic saddles)

The *traditional* investigation of motion concentrates on regular, periodic behaviour, since the applied classical mathematical tools are not suitable for describing chaos. These tools can only indicate chaos in as much they break down and yield meaningless results. The modern approach, supported by numerical investigations, makes it clear that it is regular motion that is exceptional.

Two important classes of chaotic dynamics (so far simply called chaos) are *permanent* and *transient* chaos. In the latter case, only exceptional initial conditions lead to steady chaotic motion; typical initial conditions result in finite time chaotic behaviour (which can last for an arbitrary long time, however). Both classes can occur in frictional (dissipative) systems as well as in frictionless (conservative) systems. The phase space sets underlying different kinds of chaos (chaotic attractors, bands and saddles) are collectively called chaotic sets. The main types of chaotic dynamics are summarised in Table 1.3, and will be studied in detail in Chapters 5–8.

It is also worth discussing the types of chaos from the point of view of the energy input. In non-driven frictional systems, motion ceases and chaos can only be present as a transient (often accompanied by a fractal basin boundary). Driven frictional motions may be related to chaotic attractors. In frictionless cases chaos (both in a chaotic band and in the form of chaotic scattering) might occur without forcing.

1.5 How should chaotic motion be examined?

Before turning to a detailed analysis of motion, we list some instructions worth keeping in mind in what follows, based on the lessons drawn from the examples of this chapter.

- You should understand unstable behaviour (considered to be uninteresting in traditional approaches), even in non-chaotic systems.
- Become acquainted with the phase space representation of and the geometric approach to dynamics and the use of the stroboscopic and Poincaré maps.

- It is pointless to hope that the long-term dynamics can be given analytically in terms of known functions (the infinite series constructed to describe the dynamics do not even converge).
- Solve the equations of motion numerically.
- Proper understanding requires the introduction of new concepts and the search for new theoretical relations.
- Do not forget about the measurement errors that inevitably accompany observation and simulation, and follow their temporal evolution.
- Accept the necessity of using particle ensembles and of describing them by means of probabilistic concepts (distribution, typical behaviour, average).
- Become acquainted with the geometry of fractals.

Box 1.1 Brief history of chaos

The possibility of chaotic motion was first formulated by the French mathematician Henri Poincaré in the 1890s (obviously in a terminology largely different from that used nowadays) in his paper on the stability of the Solar System. Some time later, the Russian mathematician, Sonia Kovalevskaia, proved that the motion of a heavy, asymmetric spinning top is usually chaotic (it is only regular at special values of the moment of inertia). These results were mostly forgotten and only lived on in the first half of the twentieth century due to the work of the American scientist George Birkhoff and his German colleague, Eberhard Hopf, on statistical mechanics and ergodic theory. Independently of these developments, chaotic behaviour was found in certain non-linear electrical circuits during World War II, but the results could not be properly interpreted. As a continuation of the Birkhoff–Hopf line, in the mid 1960s the Russians Andrey Kolmogorov and Vladimir Arnold and the German Jürgen Moser worked out the statement that has since been named after their initials, the KAM theorem, formulating the condition of weak chaotic motion in conservative systems. The investigation of strong chaos became possible due to the appearance of computers. The behaviour related to chaotic attractors occurring in dissipative systems was first described by the American meteorologist Edward Lorenz in 1963. He recognised the unpredictability of chaotic behaviour in connection with the numerical solution of a model named after him. The term ‘chaos’ itself was introduced by the American mathematician James Yorke for the random-looking dynamics of simple deterministic systems in a paper in 1975. The work of the American physicist Mitchell Feigenbaum helped the term become widespread. In 1978 he proved the system-independence, i.e. the so-called universality, of one of the possible routes towards chaos. In the investigation of the statistical properties of chaos, a major role was played by, among others, B. Chirikov, M. Berry, L. Bunimovich, J. P. Eckmann, H. Fujisaka, P. Grassberger, C. Grebogi, M. Hénon, P. Holmes, L. Kadanoff, E. Ott, O. Rössler, D. Ruelle, Y. Sinai, and S. Smale. The possibility of the occurrence of chaos has established a new way of thinking in widely different disciplines (see Box 9.3); this has been pioneered by H. Aref, P. Cvitanović, J. Gollub, A. Libchaber, R. May, C. Nicolis, H. Swinney, Y. Ueda, J. Wisdom, and others.