Functional Integration

Action and Symmetries

PIERRE CARTIER
AND CÉCILLE DEWITT-MORETTE

CAMBRIDGE MONOGRAPHS
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FUNCTIONAL INTEGRATION

Functional integration successfully entered physics as path integrals in the 1942 Ph.D. dissertation of Richard P. Feynman, but it made no sense at all as a mathematical definition. Cartier and DeWitt-Morette have created, in this book, a new approach to functional integration. The close collaboration between a mathematician and a physicist brings a unique perspective to this topic. The book is self-contained: mathematical ideas are introduced, developed, generalized, and applied. In the authors' hands, functional integration is shown to be a robust, user-friendly, and multi-purpose tool that can be applied to a great variety of situations, for example systems of indistinguishable particles, caustics-analysis, superanalysis, and non-gaussian integrals. Problems in quantum field theory are also considered. In the final part the authors outline topics that can profitably be pursued using material already presented.

Pierre Cartier is a mathematician with an extraordinarily wide range of interests and expertise. He has been called "un homme de la Renaissance." He is Emeritus Director of Research at the Centre National de la Recherche Scientifique, France, and a long-term visitor of the Institut des Hautes Etudes Scientifiques. From 1981 to 1989, he was a senior researcher at the Ecole Polytechnique de Paris, and, between 1988 and 1997, held a professorship at the Ecole Normale Supérieure. He is a member of the Société Mathématique de France, the American Mathematical Society, and the Vietnamese Mathematical Society.

Cécile DeWitt-Morette is the Jane and Roland Blumberg Centennial Professor in Physics, Emerita, at the University of Texas at Austin. She is a member of the American and European Physical Societies, and a Membre d'Honneur de la Société Française de Physique. DeWitt-Morette's interest in functional integration began in 1948. In F. J. Dyson's words, "she was the first of the younger generation to grasp the full scope and power of the Feynman path integral approach in physics." She is co-author with Yvonne Choquet-Bruhat of the two-volume book *Analysis, Manifolds and Physics*, a standard text first published in 1977, which is now in its seventh edition. She is the author of 100 publications in various areas of theoretical physics and has edited 28 books. She has lectured, worldwide, in many institutions and summer schools on topics related to functional integration.

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PART VI PROJECTS

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[1] An expression of L. Rosenfeld.

C. DeW. in all aspects of the preparation from critical comments to typing the final version.

Cécile thanks her graduate students

My career began on October 1, 1944. My gratitude encompasses many teachers and colleagues. The list would be an exercise in name-dropping. For this book I wish to bring forth the names of those who have been my graduate students. Working with graduate students has been the most rewarding experience of my professional life. In a few years the relationship evolves from guiding a student to being guided by a promising young colleague.

Dissertations often begin with a challenging statement. When completed, a good dissertation is a wonderful document, understandable, carefully crafted, well referenced, presenting new results in a broad context.

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List of symbols, conventions, and formulary

Symbols

$A := B$	A is defined by B
$A \overset{\int}{=} B$	the two sides are equal only after they have been integrated
θ	step function
$B \gg A$	B is inside the lightcone of A
$\mathrm{d}^\times l = \mathrm{d}l/l$	multiplicative differential
$\partial^\times/\partial l = l\partial/\partial l$	multiplicative derivative
$\mathbb{R}^D, \mathbb{R}_D$	are dual of each other; $\mathbb{R}^D$ is a space of contravariant vectors, $\mathbb{R}_D$ is a space of covariant vectors
$\mathbb{R}^{D\times D}$	space of D by D matrices
$\mathbb{X}, \mathbb{X}'$	$\mathbb{X}'$ is dual to $\mathbb{X}$
$\langle x', x\rangle$	dual product of $x \in \mathbb{X}$ and $x' \in \mathbb{X}'$
$(x\|y)$	scalar product of $x, y \in \mathbb{X}$, assuming a metric
$(\mathbb{M}^D, g)$	D-dimensional riemannian space with metric g
$T\mathbb{M}$	tangent bundle over $\mathbb{M}$
$T^*\mathbb{M}$	cotangent bundle over $\mathbb{M}$
$\mathcal{L}_X$	Lie derivative in the X-direction
$U^{2D}(S), U^{2D}$	space of critical points of the action functional S (Chapter 4)
$\mathcal{P}_{\mu,\nu}(\mathbb{M}^D)$	space of paths with values in $\mathbb{M}^D$, satisfying μ initial conditions and ν final conditions
$U_{\mu,\nu} := U^{2D}(S) \cap \mathcal{P}_{\mu,\nu}(\mathbb{M}^D)$	arena for WKB

$\hbar = h/(2\pi)$	Planck's constant
$[h] = \mathrm{ML}^2\mathrm{T}^{-1}$	physical (engineering) dimension of h
$\omega = 2\pi\nu$	ν frequency, ω pulsation
$t_\mathrm{B} = -\mathrm{i}\hbar\beta = -\mathrm{i}\hbar/(k_\mathrm{B}T)$	(1.70)
$\tau = \mathrm{i}t$	(1.100)

Superanalysis
(Chapter 9)

$\tilde{A}$	parity of $A \in \{0, 1\}$
$AB = (-1)^{\tilde{A}\tilde{B}}BA$	graded commutativity
$[A, B]$	graded commutator (9.5)
$\{A, B\}$	graded anticommutator (9.6)
$A \wedge B = -(-1)^{\tilde{A}\tilde{B}}B \wedge A$	graded exterior algebra
$\xi^\mu\xi^\sigma = -\xi^\sigma\xi^\mu$	Grassmann generators (9.11)
$z = u + v$	supernumber, u even $\in \mathbb{C}_c$, v odd $\in \mathbb{C}_a$ (9.12)
$\mathbb{R}_c \subset \mathbb{C}_c$	real elements of $\mathbb{C}_c$ (9.16)
$\mathbb{R}_a \subset \mathbb{C}_a$	real elements of $\mathbb{C}_a$ (9.16)
$z = z_\mathrm{B} + z_\mathrm{S}$	supernumber; z_B body, z_S soul (9.12)
$x^A = (x^a, \xi^\alpha) \in \mathbb{R}^{n\|\nu}$	superpoints (9.17)
$z = c_0 + c_i\xi^i + \frac{1}{2!}c_{ij}\xi^i\xi^j + \cdots = \rho + \mathrm{i}\sigma,$	where both ρ and σ have real coefficients
$z^* := \rho - \mathrm{i}\sigma,$	imaginary conjugate, $(zz')^* = z^*z'^*$ (9.13)
$z^\dagger$	hermitian conjugate, $(zz')^\dagger = z'^\dagger z^\dagger$

Conventions

We use traditional conventions unless there is a compelling reason for using a different one. If a sign is hard to remember, we recall its origin.

Metric signature on pseudoriemannian spaces

$$\begin{aligned} \eta_{\mu\nu} &= \mathrm{diag}(+,-,-,-) \\ p_\mu p^\mu &= (p^0)^2 - |\vec{p}|^2 = m^2c^2,\ p^0 = E/c \\ p_\mu x^\mu &= Et - \vec{p}\cdot\vec{x},\ x^0 = ct \\ E &= \hbar\omega = h\nu,\ \vec{p} = \hbar\,\vec{k},\ \text{plane wave } \omega = \vec{v}\cdot\vec{k} \end{aligned}$$

Positive-energy plane wave $\exp(-\mathrm{i}p_\mu x^\mu/\hbar)$

Clifford algebra

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu}$$

Quantum operators

$$[p_\mu, x^\nu] = -\mathrm{i}\hbar\delta_\mu^\nu \qquad \Rightarrow p_\mu = -\mathrm{i}\hbar\partial_\mu$$

Quantum physics (time t) and statistical mechanics (parameter τ)

$$\tau = \mathrm{i}t \qquad \text{(see (1.100))}$$

Physical dimension

$$[\hbar] = ML^2T^{-1}$$
$$\hbar^{-1}\langle p, x\rangle = \frac{2\pi}{h}\langle p, x\rangle \text{ is dimensionless}$$

Fourier transforms

$$(\mathcal{F}f)(x') := \int_{\mathbb{R}^D} \mathrm{d}^D x \ \exp(-2\pi\mathrm{i}\langle x', x\rangle) f(x) \qquad x \in \mathbb{R}^D, x' \in \mathbb{R}_D$$

For Grassmann variables

$$(\mathcal{F}f)(\kappa) := \int \delta\xi \ \exp(-2\pi\mathrm{i}\kappa\xi) f(\xi)$$

In both cases

$$\langle \delta, f\rangle = f(0) \qquad \text{i.e. } \delta(\xi) = C^{-1}\xi$$
$$\mathcal{F}\delta = 1 \qquad \text{i.e. } C^2 = (2\pi\mathrm{i})^{-1}$$
$$\int \delta\xi \ \xi = C, \qquad \text{here } C^2 = (2\pi\mathrm{i})^{-1}$$

Formulary (giving a context to symbols)

- Wiener integral

$$\mathbb{E}\left[\exp\left(-\int_{\tau_a}^{\tau_b} \mathrm{d}\tau \ V(q(\tau)\right)\right] \tag{1.1}$$

- Peierls bracket

$$(A, B) := \mathcal{D}_A^- B - (-1)^{\tilde{A}\tilde{B}}\mathcal{D}_B^- A \tag{1.9}$$

- Schwinger variational principle

$$\delta\langle A|B\rangle = \mathrm{i}\langle A|\delta\mathbf{S}/\hbar|B\rangle \tag{1.11}$$

- Quantum partition function

$$Z(\beta) = \mathrm{Tr}(\mathrm{e}^{-\beta\hat{H}}) \tag{1.71}$$

- Schrödinger equation

$$\begin{cases} \mathrm{i}\partial_t\psi(x,t) = \big(-\tfrac{1}{2}\,\mu^2\Delta_x + \hbar^{-1}V(x)\big)\psi(x,t) \\ \quad \psi(x,t_a) = \phi(x) \end{cases} \tag{1.77}$$

$$\mu^2 = \hbar/m$$

- Gaussian integral

$$\int_{\mathbb{X}} \mathrm{d}\Gamma_{s,Q}(x)\exp(-2\pi\mathrm{i}\langle x',x\rangle) := \exp(-s\pi W(x')) \tag{2.29}_s$$

$$\mathrm{d}\Gamma_{s,Q}x \stackrel{\int}{=} \mathcal{D}_{s,Q}(x)\ \exp\Big(-\frac{\pi}{s}Q(x)\Big) \tag{2.30}_s$$

$$Q(x) = \langle Dx, x\rangle, \qquad W(x') = \langle x', Gx'\rangle \tag{2.28}$$

$$\int_{\mathbb{X}} \mathrm{d}\Gamma_{s,Q}(x)\ \langle x'_1,x\rangle\dots\langle x'_{2n},x\rangle = \Big(\frac{s}{2\pi}\Big)^n {\sum}' W(x'_{i_1},x'_{i_2})\dots W(x'_{i_{2n-1}},x'_{i_{2n}})$$

sum without repetition

- Linear maps

$$\langle \tilde{L}y', x\rangle = \langle y', Lx\rangle \tag{2.58}$$

$$W_{\mathbb{Y}'} = W_{\mathbb{X}'}\circ\tilde{L}, \qquad Q_{\mathbb{X}} = Q_{\mathbb{Y}}\circ L \tag{Chapter 3, box}$$

- Scaling and coarse graining (Section 2.5)

$$\begin{aligned} S_l u(x) &= l^{[u]}u\Big(\frac{x}{l}\Big) \\ S_l[a,b[\ &= \Big[\frac{a}{l},\frac{b}{l}\Big[\\ P_l &:= S_{l/l_0}\cdot\mu_{[l_0,l[}* \end{aligned} \tag{2.94}$$

- Jacobi operator

$$S''(q)\cdot\xi\xi = \langle \mathcal{J}(q)\cdot\xi,\xi\rangle \tag{5.7}$$

- Operator formalism

$$\langle b|\hat{O}|a\rangle = \int_{\mathcal{P}_{a,b}} O(\gamma)\exp(\mathrm{i}S(\gamma)/\hbar)\mu(\gamma)\mathcal{D}\gamma \tag{Chapter 6, box}$$

- Time-ordered exponential

$$T\exp\Big(\int_{t_0}^{t}\mathrm{d}s\,A(s)\Big) \tag{6.38}$$

- Dynamical vector fields

$$\mathrm{d}x(t,z) = X_{(A)}(x(t,z))\mathrm{d}z^A(t) + Y(x(t,z))\mathrm{d}t \tag{7.14}$$

$$\Psi(t,\mathbf{x}_0) := \int_{\mathcal{P}_0\mathbb{R}^D} \mathcal{D}_{s,Q_0} z \cdot \exp\Big(-\frac{\pi}{s}Q_0(z)\Big)\phi(\mathbf{x}_0 \cdot \Sigma(t,z)) \tag{7.12}$$

$$Q_0(z) := \int_{\mathbb{T}} \mathrm{d}t\; h_{AB}\dot{z}^A(t)\dot{z}^B(t) \tag{7.8}$$

$$\begin{cases} \dfrac{\partial\Psi}{\partial t} = \dfrac{s}{4\pi}h^{AB}\mathcal{L}_{X_{(A)}}\mathcal{L}_{X_{(B)}}\Psi + \mathcal{L}_Y\Psi \\ \Psi(t_0,\mathbf{x}) = \phi(\mathbf{x}) \end{cases} \tag{7.15}$$

- Homotopy

$$|K(b,t_b;a,t_a)| = \left|\sum_\alpha \chi(\alpha)K^\alpha(b,t_b;a,t_a)\right| \qquad \text{(Chapter 8, box)}$$

- Koszul formula

$$\mathcal{L}_X\omega = \mathrm{Div}_\omega(X)\cdot\omega \tag{11.1}$$

- Miscellaneous

$$\det\exp A = \exp\mathrm{tr}\; A \tag{11.48}$$
$$\mathrm{d}\ln\det A = \mathrm{tr}(A^{-1}\,\mathrm{d}A) \tag{11.47}$$
$$\nabla^i f \equiv \nabla^i_{g^{-1}} f := g^{ij}\,\partial f/\partial x^j \qquad \text{gradient} \tag{11.73}$$
$$(\nabla_{g^{-1}}|V)_g = V^j{}_{,j} \qquad \text{divergence} \tag{11.74}$$
$$(V|\nabla f) = -\,(\mathrm{div}\,V|f) \qquad \text{gradient/divergence} \tag{11.79}$$

- Poisson processes

$$N(t) := \sum_{k=1}^{\infty}\theta(t-T_k) \qquad \text{counting process} \tag{13.17}$$

- Density of energy states

$$\rho(E) = \sum_n \delta(E-E_n), \qquad H\psi_n = E_n\psi_n$$

- Time ordering

$$T(\phi(x_j)\phi(x_i)) = \begin{cases} \phi(x_j)\phi(x_i) & \text{for } j \succcurlyeq i \\ \phi(x_i)\phi(x_j) & \text{for } i \succcurlyeq j \end{cases} \tag{15.7}$$

- The "measure" (Chapter 18)

$$\mu[\phi] \approx \left(\operatorname{sdet} G^{+}[\phi]\right)^{-1/2} \tag{18.3}$$

$${}_{i,}S_{,k}[\phi]G^{+kj}[\phi] = -{}_{i}\delta^{j} \tag{18.4}$$

$$G^{+ij}[\phi] = 0 \qquad \text{when } i \succcurlyeq j \tag{18.5}$$

$$\boldsymbol{\phi}^{i} = \begin{cases} u^{i}_{\text{in}\,A}\mathbf{a}^{A}_{\text{in}} + u^{i\,*}_{\text{in}\,A}\mathbf{a}^{A\,*}_{\text{in}} \\ u^{i}_{\text{out}\,X}\mathbf{a}^{X}_{\text{out}} + u^{i\,*}_{\text{out}\,X}\mathbf{a}^{X\,*}_{\text{out}} \end{cases} \tag{18.18}$$

- Wick (normal ordering)
  operator normal ordering

$$(a + a^{\dagger})(a + a^{\dagger}) = :(a + a^{\dagger})^{2}: + 1 \tag{D.1}$$

  functional normal ordering

$$:F(\phi):_{G} := \exp\left(-\frac{1}{2}\Delta_{G}\right)F(\phi) \tag{D.4}$$

  functional laplacian defined by the covariance G

$$\Delta_{G} := \int_{\mathbb{M}^{D}} \mathrm{d}^{D}x \int_{\mathbb{M}^{D}} \mathrm{d}^{D}y\, G(x,y)\frac{\delta^{2}}{\delta\phi(x)\delta\phi(y)} \tag{2.63}$$

Part I

The physical and mathematical environment

1
The physical and mathematical environment

> A physicist needs that his equations should be mathematically sound[1]
> Dirac [1]

A: An inheritance from physics

1.1 The beginning

In 1933, Dirac [2] laid the foundation stone for what was destined to become in the hands of Feynman a new formulation of quantum mechanics. Dirac introduced the action principle in quantum physics [3], and Feynman, in his doctoral thesis [4] "The principle of least action in quantum physics," introduced path integrals driven by the classical action *functional*, the integral of the lagrangian. The power of this formalism was vindicated [5] when Feynman developed techniques for computing functional integrals and applied them to the relativistic computation of the Lamb shift.

In 1923, after some preliminary work by Paul Lévy [6], Norbert Wiener published "Differential space" [7], a landmark article in the development of functional integration. Wiener uses the term "differential space" to emphasize the advantage of working not with the values of a function but with the difference of two consecutive values. He constructed a measure in "differential space." Mark Kac remarked that the Wiener integral

$$\mathbb{E}\left[\exp\left(-\int_{\tau_a}^{\tau_b} \mathrm{d}\tau\; V(q(\tau))\right)\right], \tag{1.1}$$

[1] Because, says N. G. Van Kampen, "When dealing with less simple and concrete equations, physical intuition is less reliable and often borders on wishful thinking."

where $\mathbb{E}$ denotes the expectation value relative to the Wiener measure, becomes the Feynman integral

$$\int \mathcal{D}q \cdot \exp(\mathrm{i}S(q)) \qquad \text{for } S(q) = \int_{t_a}^{t_b} \mathrm{d}t \Big(\frac{1}{2}\dot{q}(t)^2 - V(q(t)) \Big) \tag{1.2}$$

if one sets $\tau = \mathrm{i}t$. Kac concluded that, because of i in the exponent, Feynman's theory is not easily made rigorous. Indeed, one needs an integration theory more general than Lebesgue theory for making sense of Kac's integral for $\tau = \mathrm{i}t$, and such a theory has been proposed in [8, 9]. The usefulness of this theory and its further developments can be seen in this book. For a thorough discussion of the relation $\tau = \mathrm{i}t$, see Sections 1.9–1.11. Feynman, however, objected to a Kac-type integral because "it spoils the physical unification of kinetic and potential parts of the action" [10]. The kinetic contribution is hidden from sight.

The arguments of the functionals considered above are functions on a line. The line need not be a time line; it can be a scale line, a one-parameter subgroup, etc. In all cases, the line can be used to "time" order products of values of the function at different times.[2] Given a product of factors $U_1, \ldots, U_N$, each attached to a point t_i on an oriented line, one denotes by $T(U_1 \ldots U_N)$ the product $U_{i_N} \ldots U_{i_1}$, where the permutation $i_1 \ldots i_N$ of $1 \ldots N$ is such that $t_{i_1} < \ldots < t_{i_N}$. Hence in the rearranged product the times attached to the factors increase from right to left.

The evolution of a system is dictated by the "time" line. Thus Dirac and Feynman expressed the time evolution of a system by the following N-tuple integral over the variables of integration $\{q_i'\}$, where q_i' is the "position" of the system at "time" t_i; in Feynman's notation the probability amplitude $(q_t'|q_0')$ for finding at time t in position q_t' a system known to be in position q_0' at time t_0 is given by

$$(q_t'|q_0') = \iint \ldots \int (q_t'|q_N')\mathrm{d}q_N'(q_N'|q_{N-1}')\mathrm{d}q_{N-1}' \ldots (q_2'|q_1')\mathrm{d}q_1'(q_1'|q_0'). \tag{1.3}$$

The continuum limit, if it exists, is a "path integral" with its domain of integration consisting of functions on $[t_0, t]$. Dirac showed that $(q_t'|q_0')$ defines the exponential of a function $\mathcal{S}$,

$$\exp\big(\mathrm{i}\mathcal{S}(q_t', q_0', t)/\hbar\big) := (q_t'|q_0'). \tag{1.4}$$

The function $\mathcal{S}$ is called by Dirac [3] the "quantum analogue of the classical action function (a.k.a. Hamilton's principal function)" because its real

[2] There are many presentations of time-ordering. A convenient one for our purpose can be found in "Mathemagics" [11]. In early publications, there are sometimes two different time-ordering symbols: T^*, which commutes with both differentiation and integration, and T, which does not. T^* is the good time-ordering and simply called T nowadays. The reverse time-ordering defined by (1.3) is sometimes labeled $\overleftarrow{T}$.

part[3] is equal to the classical action function $\mathcal{S}$ and its imaginary part is of order $\hbar$. Feynman remarked that $(q'_{t+\delta t}|q'_t)$ is "often equal to

$$\exp \frac{\mathrm{i}}{\hbar} L\left(\frac{q'_{t+\delta t} - q'_t}{\delta t}, q'_{t+\delta t}\right)\delta t \tag{1.5}$$

within a normalization constant in the limit as δt approaches zero"[4]. Feynman expressed the finite probability amplitude $(q'_t|q'_0)$ as a path integral

$$(q'_t|q'_0) = \int \mathcal{D}q \cdot \exp\left(\frac{\mathrm{i}}{\hbar} S(q)\right), \tag{1.6}$$

where the action functional $S(q)$ is

$$S(q) = \int_{t_o}^{t} \mathrm{d}s\, L(\dot{q}(s), q(s)). \tag{1.7}$$

The path integral (1.6) constructed from the infinitesimals (1.5) is a product integral (see Appendix B for the definition and properties of, and references on, product integrals). *The action functional, broken up into time steps, is a key building block of the path integral.*

Notation. The Dirac quantum analog of the classical action, labeled $\mathcal{S}$, will not be used. The action function, namely the solution of the Hamilton–Jacobi equation, is labeled $\mathcal{S}$, and the action functional, namely the integral of the lagrangian, is labeled S. The letters $\mathcal{S}$ and S are not strikingly different but are clearly identified by the context. See Appendix E.

Two phrases give a foretaste of the rich and complex issue of path integrals: "the imaginary part [of $\mathcal{S}$] is of order $\hbar$," says Dirac; "within a normalization constant,"[4] says Feynman, who summarizes the issue in the symbol $\mathcal{D}q$.

Feynman rules for computing asymptotic expansions of path integrals, order by order in perturbation theory, are stated in terms of graphs, which are also called diagrams [12]. The Feynman-diagram technique is widely used because the diagrams are not only a powerful mathematical aid to the calculation but also provide easy bookkeeping for the physical process of interest. Moreover, the diagram expansion of functional integrals in quantum field theory proceeds like the diagram expansion of path integrals in quantum mechanics. The time-ordering (1.3) becomes a chronological

[3] More precisely, the real part is the classical action, up to order $\hbar$.
[4] See the remark at the end of Section 2.2 for a brief comment on the early calculations of the normalization constant.

ordering, dictated by lightcones: if the point x_j is in the future lightcone of x_i, one writes $j \succcurlyeq i$, and defines the chronological ordering T by the symmetric function

$$T(U_j U_i) = T(U_i U_j) := U_j U_i, \tag{1.8}$$

where $U_i := U(x_i)$, $U_j := U(x_j)$, and $j \succcurlyeq i$.

The Feynman diagrams are a graphic expression of gaussian integrals of polynomials. The first step for computing the diagram expansion of a given functional integral is the expansion into polynomials of the exponential in the integrand. We shall give an explicit diagram expansion as an application of gaussian path integrals in Section 2.4.

1.2 Integrals over function spaces

Wiener and Feynman introduced path integrals as the limit for $N = \infty$ of an N-tuple integral. Feynman noted that the limit of N-tuple integrals is at best a crude way of defining path integrals. Indeed, the drawbacks are several.

- How does one choose the short-time probability amplitude $(q'_{t+\delta t}|q'_t)$ and the undefined normalization constant?
- How does one compute the N-tuple integral?
- How does one know whether the N-tuple integral has a unique limit for $N = \infty$?

The answer is to do away with N-tuple integrals and to identify the function spaces which serve as domains of integration for functional integrals. The theory of promeasures [13] (projective systems of measures on topological vector spaces, which are locally convex, but not necessarily locally compact), combined with Schwartz distributions, yields a practical method for integrating on function spaces. The step from promeasures to prodistributions (which were first introduced as pseudomeasures [8]) is straightforward [14]. It is presented in Section 1.7. Already in their original form, prodistributions have been used for computing nontrivial examples, e.g. the explicit cross section for glory scattering of waves by Schwarzschild black holes [15]. A useful byproduct of prodistributions is the definition of complex gaussians in Banach spaces presented in Section 2.3.

1.3 The operator formalism

A functional integral is a mathematical object, but historically its use in physics is intimately connected with quantum physics. Matrix elements of an operator on Hilbert spaces or on Fock spaces have been used for defining their functional integral representation.

Bryce DeWitt [16] constructs the operator formalism of quantum physics from the Peierls bracket. This formalism leads to the Schwinger variational principle and to functional integral representations.

The bracket invented by Peierls [17] in 1952 is a beautiful, but often neglected, covariant replacement for the canonical Poisson bracket, or its generalizations, used in canonical quantization. Let A and B be any two physical observables. Their *Peierls bracket* (A, B) is by definition

$$(A, B) := \mathcal{D}_A^- B - (-1)^{\tilde{A}\tilde{B}} \mathcal{D}_B^- A, \tag{1.9}$$

where the symbol $\tilde{A} \in \{0, 1\}$ is the Grassmann parity of A and $\mathcal{D}_A^- B$ ($\mathcal{D}_A^+ B$) is known as the retarded (advanced) effect of A on B. The precise definition follows from the theory of measurement, and can be found in [16].

The *operator quantization rule* associates an operator $\mathbf{A}$ with an observable A; the supercommutator $[\mathbf{A}, \mathbf{B}]$ is given by the Peierls bracket:

$$[\mathbf{A}, \mathbf{B}] = -\mathrm{i}\hbar(A, B) + O(\hbar^2). \tag{1.10}$$

Let $|A\rangle$ be an eigenvector of the operator $\mathbf{A}$ for the eigenvalue A. The *Schwinger variational principle* states that the variation of the transition amplitude $\langle A|B\rangle$ generated by the variation $\delta\mathbf{S}$ of an action $\mathbf{S}$, which is a functional of field operators, acting on a space of state vectors is

$$\delta\langle A|B\rangle = \mathrm{i}\langle A|\delta\mathbf{S}/\hbar|B\rangle. \tag{1.11}$$

The variations of matrix elements have led to their functional integral representation. The solution of this equation, obtained by Bryce DeWitt, is the Feynman functional integral representation of $\langle A|B\rangle$. It brings out explicitly the exponential of the classical action functional in the integrand, and the "measure" on the space of paths, or the space of histories, as the case may be. The domain of validity of this solution encompasses many different functional integrals needed in quantum field theory and quantum mechanics. The measure, called $\mu(\phi)$, is an important contribution in the applications of functional integrals over fields ϕ. (See Chapter 18.)

1.4 A few titles

By now functional integration has proved itself useful. It is no longer a "secret weapon used by a small group of mathematical physicists"[5] but

[5] Namely "an extremely powerful tool used as a kind of secret weapon by a small group of mathematical physicists," B. Simon (1979).

is still not infrequently a "sacramental formula."[6] Compiling a bibliography of functional integration in physics, other than a computer-generated list of references, would be an enormous task, best undertaken by a historian of science. We shall mention only a few books, which together give an idea of the scope of the subject. In chronological order:

- R. P. Feynman and A. R. Hibbs (1965). *Quantum Mechanics and Path Integrals* (New York, McGraw-Hill);
- S. A. Albeverio and R. J. Høegh-Krohn (1976). *Mathematical Theory of Feynman Path Integrals* (Berlin, Springer);
- B. Simon (1979). *Functional Integration and Quantum Physics* (New York, Academic Press);
- J. Glimm and A. Jaffe (1981). *Quantum Physics*, 2nd edn. (New York, Springer);
- L. S. Schulman (1981). *Techniques and Applications of Path Integration* (New York, John Wiley);
- K. D. Elworthy (1982). *Stochastic Differential Equations on Manifolds* (Cambridge, Cambridge University Press);
- A. Das (1993). *Field Theory, A Path Integral Approach* (Singapore, World Scientific);
- G. Roepstorff (1994). *Path Integral Approach to Quantum Physics: An Introduction* (Berlin, Springer); (original German edition: *Pfadintegrale in der Quantenphysik*, Braunschweig, Friedrich Vieweg & Sohn, 1991);
- C. Grosche and F. Steiner (1998). *Handbook of Feynman Path Integrals* (Berlin, Springer);
- A *Festschrift* dedicated to Hagen Kleinert (2001): *Fluctuating Paths and Fields*, eds. W. Janke, A. Pelster, H.-J. Schmidt, and M. Bachmann (Singapore, World Scientific);
- M. Chaichian and A. Demichev (2001). *Path Integrals in Physics*, Vols. I and II (Bristol, Institute of Physics);
- G. W. Johnson and M. L. Lapidus (2000). *The Feynman Integral and Feynman's Operational Calculus* (Oxford, Oxford University Press; paperback 2002);
- B. DeWitt (2003). *The Global Approach to Quantum Field Theory* (Oxford, Oxford University Press; with corrections, 2004).
- J. Zinn-Justin (2003). *Intégrale de chemin en mécanique quantique: introduction* (Les Ulis, EDP Sciences and Paris, CNRS);

[6] "A starting point of many modern works in various areas of theoretical physics is the path integral $\int \mathcal{D}q \exp \mathrm{i}S(q)/\hbar$. What is the meaning of this sacramental formula?" M. Marinov [18].

- J. Zinn-Justin (2004). *Path Integrals in Quantum Mechanics* (Oxford, Oxford University Press); and
- H. Kleinert (2004). *Path Integrals in Quantum Mechanics, Statistics, and Polymer Physics*, 3rd edn. (Singapore, World Scientific).

Many books on quantum field theory include several chapters on functional integration. See also [19].

These few books, together with their bibliographies, give a good picture of functional integration in physics at the beginning of the twenty-first century. We apologize for having given an incomplete list of our inheritance.

B: A toolkit from analysis

1.5 A tutorial in Lebesgue integration

It is now fully understood that Feynman's path integrals are *not* integrals in the sense of Lebesgue. Nevertheless, Lebesgue integration is a useful device, and the Kac variant of Feynman's path integrals is a genuine Lebesgue integral (see Part C). Lebesgue integration introduces concepts useful in the study of stochastic processes presented in Section 1.6.

Polish spaces

This class of spaces is named after Bourbaki [13]. A Polish space is a *metric space*, hence a set $\mathbb{X}$ endowed with a *distance* $d(x, y)$ defined for the pairs of points in $\mathbb{X}$, satisfying the following axioms ($d(x, y)$ is a real number):

- $d(x, y) > 0$ for $x \neq y$ and $d(x, y) = 0$ for $x = y$;
- *symmetry* $d(x, y) = d(y, x)$; and
- *triangle inequality* $d(x, z) \leq d(x, y) + d(y, z)$.

Furthermore, a Polish space should be *complete*: if $x_1, x_2, \ldots$ is a sequence of points in $\mathbb{X}$ such that $\lim_{m=\infty, n=\infty} d(x_m, x_n) = 0$, then there exists a (unique) point x in $\mathbb{X}$ such that $\lim_{n=\infty} d(x_n, x) = 0$. Finally, we assume that *separability* holds: there exists a countable subset D in $\mathbb{X}$ such that any point in $\mathbb{X}$ is a limit of a sequence of points of D.

Note. Any separable Banach space is a Polish space.

We give the basic examples:

- a countable set D, with $d(x, y) = 1$ for $x \neq y$;
- the real line $\mathbb{R}$, with $d(x, y) = |x - y|$;

- the euclidean n-space $\mathbb{R}^n$ with

$$d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2} \tag{1.12}$$

  for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$;
- the Hilbert space ℓ^2 of infinite sequences $x = (x_1, x_2, \ldots)$ of real numbers such that $\sum_{n=1}^{\infty} x_n^2 < +\infty$, with

$$d(x,y) = \left(\sum_{n=1}^{\infty} (x_n - y_n)^2 \right)^{1/2}; \tag{1.13}$$

- the space $\mathbb{R}^\infty$ of all unrestricted sequences $x = (x_1, x_2, \ldots)$ of real numbers with

$$d(x,y) = \sum_{n=1}^{\infty} \min(2^{-n}, |x_n - y_n|); \tag{1.14}$$

- let $\mathbb{T} = [t_a, t_b]$ be a real interval. The space $\mathcal{P}(\mathbb{T})$ (also denoted by $C^0(\mathbb{T}; \mathbb{R})$) of paths consists of the continuous functions $f : \mathbb{T} \to \mathbb{R}$; the distance is defined by

$$d(f,g) = \sup_{t \in \mathbb{T}} |f(t) - g(t)|. \tag{1.15}$$

In a Polish space $\mathbb{X}$, a subset U is *open*[7] if, for every x_0 in U, there exists an $\epsilon > 0$ such that U contains the ϵ-neighborhood of x_0, namely the set of points x with $d(x_0, x) < \epsilon$. The complement $F = \mathbb{X} \setminus U$ of an open set is *closed*. A set $A \subset \mathbb{X}$ is called a G_δ if it is of the form $\cap_{n \geq 1} U_n$, where $U_1 \supset U_2 \supset \ldots \supset U_n \supset \ldots$ are open. Dually, we define an F_σ set $A = \cup_{n \geq 1} F_n$ with $F_1 \subset F_2 \subset \ldots$ closed. Any open set is an F_σ; any closed set is a G_δ.

Next we define the *Baire hierarchy* of subsets of $\mathbb{X}$:

- class 1: all open or closed sets;
- class 2: all limits of an increasing or decreasing sequence of sets of class 1 (in particular the F_σ sets and the G_δ sets).

In general, if α is any ordinal with predecessor β, the class α consists of the limits of monotonic sequences of sets of class β; if α is a limit ordinal, a set of class α is any set belonging to a class β with $\beta < \alpha$.

We stop at the first uncountable ordinal ϵ_0. The corresponding class ϵ_0 consists of the *Borel subsets* of $\mathbb{X}$. Hence, if $B_1, B_2, \ldots, B_n, \ldots$ are Borel subsets, so are $\cap_{n \geq 1} B_n$ and $\cup_{n \geq 1} B_n$ as well as the differences $\mathbb{X} \setminus B_n$. Any open (or closed) set is a Borel set.

[7] A closed (or open) set is itself a Polish space.

Measures in a Polish space

A *measure* in a Polish space $\mathbb{X}$ associates a real number $\mu(B)$ with any Borel subset B of $\mathbb{X}$ in such a way that

- $0 \leq \mu(B) \leq +\infty$;
- if B is a disjoint union of Borel sets $B_1, B_2, \ldots, B_n, \ldots$ then $\mu(B) = \sum_{n=1}^{\infty} \mu(B_n)$; and
- the space $\mathbb{X}$ can be covered[8] by a family of open sets U_n (for $n = 1, 2, \ldots$) such that $\mu(U_n) < +\infty$ for all n.

The measure μ is *finite* (or bounded) if $\mu(\mathbb{X})$ is finite. The previous definition defines the so-called positive measures. *A complex measure* μ assigns to any Borel set B a complex number, such that $\mu(B) = \sum_{n=1}^{\infty} \mu(B_n)$ for a disjoint union B of $B_1, B_2, \ldots$ (the series is then absolutely convergent). Such a complex measure is of the form $\mu(B) = c_1\mu_1(B) + \cdots + c_p\mu_p(B)$ where $c_1, \ldots, c_p$ are complex constants, and $\mu_1, \ldots, \mu_p$ are bounded (positive) measures. The *variation* of μ is the bounded (positive) measure $|\mu|$ defined by

$$|\mu|(B) = \text{l.u.b.} \left\{ \sum_{i=1}^{q} |\mu(B_i)| : B = B_1 \cup \ldots \cup B_q \quad \text{disjoint union} \right\}. \tag{1.16}$$

The number $|\mu|(\mathbb{X})$ is called the *total variation* of μ, and l.u.b stands for least upper bound.

A measure μ on a Polish space is *regular*. That is,

(a) if K is a compact[9] set in $\mathbb{X}$, then $\mu(K)$ is finite; and
(b) if B is a Borel set with $\mu(B)$ finite, and $\epsilon > 0$ is arbitrary, then there exist two sets K and U, with K compact, U open, $K \subset B \subset U$ and $\mu(U \setminus K) < \epsilon$.

As a consequence, if B is a Borel set with $\mu(B) = +\infty$, there exists,[10] for each $n \geq 1$, a compact subset K_n of B such that $\mu(K_n) > n$. Since the measure μ is regular, the knowledge of the numbers $\mu(K)$ for K compact enables us to reconstruct the measure $\mu(B)$ of the Borel sets. It is possible

[8] This condition can be expressed by saying that the measures are "locally bounded."
[9] A subset K in $\mathbb{X}$ is compact if it is closed and if, from any sequence of points in K, we can extract a convergent subsequence.
[10] From the regularity of the measures, we can deduce the following statement: the Polish space $\mathbb{X}$ can be decomposed as a disjoint union

$$\mathbb{X} = N \cup K_1 \cup K_2 \cup \ldots,$$

where each K_n is compact, and $\mu(N) = 0$. After discarding the null set N, the space $\mathbb{X}$ is replaced by a locally compact space.

to characterize the functionals $\mu(K)$ of compact subsets giving rise to a measure [13].

Construction of measures

To construct a measure, we can use *Caratheodory's extension theorem*, whose statement follows:

let $\mathcal{C}$ be a class of Borel subsets, stable under finite union, finite intersection and set difference;
assume that all Borel sets can be obtained via a Baire hierarchy, starting with the sets in $\mathcal{C}$ as class 1; and
let $I : \mathcal{C} \to \mathbb{R}$ be a functional with (finite) positive values, which is *additive*,

$$I(C \cup C') = I(C) + I(C') \qquad \text{if } C \cap C' = \emptyset, \tag{1.17}$$

and *continuous at* $\emptyset$,

$$\lim_{n=\infty} I(C_n) = 0 \qquad \text{for } C_1 \supset C_2 \supset \dots \text{ and } \cap_n C_n = \emptyset. \tag{1.18}$$

We can then extend uniquely the functional $I(C)$ to a functional $\mu(B)$ of Borel subsets such that $\mu(B) = \sum_{n\geq 1} \mu(B_n)$ when B is the disjoint union of $B_1, B_2, \dots, B_n, \dots$

As an application, consider the case in which $\mathbb{X} = \mathbb{R}$ and $\mathcal{C}$ consists of the sets of the form

$$C =]a_1, b_1] \ \cup \ \cdots \ \cup \ ]a_n, b_n].$$

Let $F : \mathbb{R} \to \mathbb{R}$ be a function that is increasing[11] and right-continuous.[12] Then there exists a unique measure μ on $\mathbb{R}$ such that $\mu(]a, b]) = F(b) - F(a)$. The special case $F(x) = x$ leads to the so-called Lebesgue measure λ on $\mathbb{R}$ satisfying $\lambda(]a, b]) = b - a$.

Product measures

Another corollary of Caratheodory's result is about the construction of a *product measure*. Suppose that $\mathbb{X}$ is the cartesian product $\mathbb{X}_1 \times \mathbb{X}_2$ of two Polish spaces; we can define on $\mathbb{X}$ the distance

$$d(x, y) = d(x_1, y_1) + d(x_2, y_2) \tag{1.19}$$

for $x = (x_1, x_2)$ and $y = (y_1, y_2)$ and $\mathbb{X}$ becomes a Polish space. Let μ_i be

[11] That is $F(a) \leq F(b)$ for $a \leq b$.
[12] That is $F(a) = \lim_{n=\infty} F(a + 1/n)$ for every a.

a measure on $\mathbb{X}_i$ for $i = 1, 2$. Then there exists a unique measure μ on $\mathbb{X}$ such that

$$\mu(B_1 \times B_2) = \mu_1(B_1) \cdot \mu_2(B_2), \tag{1.20}$$

where $B_1 \subset \mathbb{X}_1$, $B_2 \subset \mathbb{X}_2$ are Borel subsets. This measure is usually denoted $\mu_1 \otimes \mu_2$. This construction can be easily generalized to $\mu_1 \otimes \cdots \otimes \mu_p$, where μ_i is a measure on $\mathbb{X}_i$ for $i = 1, \ldots, p$. The same construction applies to complex measures.

For probabilistic applications, it is useful to consider infinite products $\bigotimes_{i=1}^{\infty} \mu_i$ of measures. For simplicity, we consider only the case of $\mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots$. Suppose given a sequence of measures $\mu_1, \mu_2, \mu_3, \ldots$ on $\mathbb{R}$; assume that each μ_i is positive and normalized $\mu_i(\mathbb{R}) = 1$, or else that each μ_i is a complex measure and that the infinite product $\prod_{i=1}^{\infty} |\mu_i|(\mathbb{R})$ is convergent. Then there exists a unique measure $\mu = \bigotimes_{i=1}^{\infty} \mu_i$ on $\mathbb{R}^\infty$ such that

$$\mu(B_1 \times \cdots \times B_p \times \mathbb{R} \times \mathbb{R} \times \cdots) = \mu_1(B_1) \cdots \mu_p(B_p) \tag{1.21}$$

for $p \geq 1$ arbitrary and Borel sets $B_1, \ldots, B_p$ in $\mathbb{R}$.

Integration in a Polish space

We fix a Polish space $\mathbb{X}$ and a positive measure μ, possibly unbounded. A function $f : \mathbb{X} \to \overline{\mathbb{R}} = [-\infty, +\infty]$ is called Borel if the set

$$\{a < f < b\} = \{x \in \mathbb{X} : a < f(x) < b\} \tag{1.22}$$

is a Borel set in $\mathbb{X}$ for arbitrary numbers a, b with $a < b$. The standard operations (sum, product, pointwise limit, series, ...) applied to Borel functions yield Borel functions. Note also that continuous functions are Borel. We denote by $\mathcal{F}_+(\mathbb{X})$ the set of Borel functions on $\mathbb{X}$ with values in $\overline{\mathbb{R}}_+ = [0, +\infty]$.

An *integral on* $\mathbb{X}$ is a functional $I : \mathcal{F}_+(\mathbb{X}) \to \overline{\mathbb{R}}_+$ with the property

$$I(f) = \sum_{n=1}^{\infty} I(f_n)$$

if

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad \text{for all } x \text{ in } \mathbb{X}.$$

The integral is *locally bounded* if there exists an increasing sequence of continuous functions f_n such that

- $0 \leq f_n(x)$, $\lim_{n=\infty} f_n(x) = 1$ for all x in $\mathbb{X}$; and
- $I(f_n) < +\infty$ for every $n \geq 1$.

Given a locally bounded integral on $\mathbb{X}$, we define as follows a measure μ on $\mathbb{X}$: if B is any Borel set in $\mathbb{X}$, let χ_B be the corresponding characteristic function[13] and let $\mu(B) = I(\chi_B)$. It is trivial that μ is a measure, but the converse is also true – and a little more difficult to prove – that is, given any measure μ on $\mathbb{X}$, there exists a unique locally bounded integral I such that $I(\chi_B) = \mu(B)$ for every Borel set B in $\mathbb{X}$. In this case, we use the notation[14] $\int f \, \mathrm{d}\mu$ instead of $I(f)$ for f in $\mathcal{F}_+(\mathbb{X})$.

We denote by $L^1 = L^1(\mathbb{X}, \mu)$ the class of Borel functions f on $\mathbb{X}$ such that $\int |f| \, \mathrm{d}\mu < +\infty$. The integral $\int f \, \mathrm{d}\mu$ for the nonnegative elements f of L^1 extends to a linear form on the vector space L^1. Then L^1 is a separable Banach space[15] for the norm

$$\|f\|_1 := \int |f| \mathrm{d}\mu. \tag{1.23}$$

The familiar convergence theorems are valid. We quote only the "dominated convergence theorem."

Theorem. If $f_1, f_2, \ldots, f_n, \ldots$ are functions in L^1, and if there exists a Borel function $\phi \geq 0$ with $\int \phi \, \mathrm{d}\mu < \infty$ and $|f_n(x)| \leq \phi(x)$ for all x and all n, then the pointwise convergence

$$\lim_{n=\infty} f_n(x) = f(x) \tag{1.24}$$

for all x entails that f is in L^1 and that

$$\lim_{n=\infty} \|f_n - f\|_1 = 0\,, \qquad \lim_{n=\infty} \int f_n \, \mathrm{d}\mu = \int f \, \mathrm{d}\mu. \tag{1.25}$$

For $p \geq 1$, the space L^p consists of the Borel functions f such that $\int |f|^p \, \mathrm{d}\mu < \infty$, and the norm $\|f\|_p$ is then defined as

$$\|f\|_p := \left(\int |f|^p \, \mathrm{d}\mu \right)^{1/p}. \tag{1.26}$$

Then L^p is a separable Banach space. The two more important cases are $p = 1$ and $p = 2$. The space L^2 is a (real) Hilbert space with scalar product $(f|g) = \int fg \, \mathrm{d}\mu$. We leave to the reader the extension to complex-valued functions.

Assume that $\mathbb{X}$ is a compact metric space. Then it is complete and separable, and hence a Polish space. Denote by $C^0(\mathbb{X})$ the Banach space

[13] That is $\chi_B(x) = 1$ for x in B and $\chi_B(x) = 0$ for x in $\mathbb{X} \setminus B$.

[14] Or any of the standard aliases $\int_{\mathbb{X}} f \, \mathrm{d}\mu$, $\int_{\mathbb{X}} f(x) \mathrm{d}\mu(x), \ldots$

[15] This norm $\|f\|_1$ is zero iff the set $\{f \neq 0\}$ is of measure 0. We therefore have to identify, as usual, two functions f and g such that the set $\{f \neq g\}$ be of measure 0.

of continuous functions $f : \mathbb{X} \to \mathbb{R}$ with the norm

$$\|f\|_\infty = \text{l.u.b.}\{|f(x)| : x \in \mathbb{X}\}. \tag{1.27}$$

If μ is a (positive) measure on $\mathbb{X}$, then it is bounded, and one defines a linear form I on $C^0(\mathbb{X})$ by the formula[16]

$$I(f) = \int_{\mathbb{X}} f \, \mathrm{d}\mu. \tag{1.28}$$

Then I is positive, that is $f(x) \geq 0$ for all x in $\mathbb{X}$ entails $I(f) \geq 0$. Conversely, any positive linear form on $C^0(\mathbb{X})$ comes from a unique[17] positive[18] measure μ on $\mathbb{X}$ (the "Riesz–Markoff theorem").

A variation on the previous theme: $\mathbb{X}$ is a locally compact Polish space (that is, every point has a compact neighborhood), $C^0_{\mathrm{c}}(\mathbb{X})$ is the space of continuous functions $f : \mathbb{X} \to \mathbb{R}$ vanishing off a compact subset K of $\mathbb{X}$. Then the measures on $\mathbb{X}$ can be identified with the positive linear forms on the vector space $C^0_{\mathrm{c}}(\mathbb{X})$. Again $C^0_{\mathrm{c}}(\mathbb{X})$ is dense in the Banach space $L^1(\mathbb{X}, \mu)$ for every measure μ on $\mathbb{X}$.

1.6 Stochastic processes and promeasures

The language of probability

Suppose that we are measuring the blood parameters of a sample: numbers of leukocytes, platelets... The record is a sequence of numbers (after choosing proper units) subject eventually to certain restrictions. The record of a sample is then a point in a certain region Ω of a numerical space $\mathbb{R}^D$ (D is the number of parameters measured). We could imagine an infinite sequence of measurements, and we should replace Ω by a suitable subset of $\mathbb{R}^\infty$.

In such situations, the record of a sample is a point in a certain space, the *sample space* Ω. We make the mathematical assumption that Ω is a Polish space, for instance the space $C^0(\mathbb{T}; \mathbb{R})$ of continuous paths $(q(t))_{t \in \mathbb{T}}$, for the observation of a sample path of the brownian motion during the time interval $\mathbb{T} = [t_a, t_b]$. The outcome is subject to random fluctuations, which are modeled by specifying a measure $\mathbb{P}$ on Ω, normalized by $\mathbb{P}[\Omega] = 1$ (a so-called *probability measure*, or *probability law*). Any Borel subset B of Ω corresponds to a (random) event, whose *probability* is the measure

[16] Notice the inclusion $C^0(\mathbb{X}) \subset L^1(\mathbb{X}, \mu)$.

[17] The uniqueness comes from the fact that $C^0(\mathbb{X})$ is dense in $L^1(\mathbb{X}, \mu)$ for every positive measure μ on $\mathbb{X}$.

[18] More generally, the elements of the dual space $C^0(\mathbb{X})'$ of the Banach space $C^0(\mathbb{X})$ can be identified with the (real) measures on $\mathbb{X}$.

$\mathbb{P}[B]$; hence $0 \leq \mathbb{P}[B] \leq 1$. Any numerical quantity that can be measured on the sample corresponds to a Borel function $F : \Omega \to \mathbb{R}$, called a *random variable*. The mean value of F is by definition

$$\mathbb{E}[F] := \int_\Omega F \, \mathrm{d}\mathbb{P}.$$

A basic notion is that of (stochastic) *independence*. Suppose that we perform independently the selection of samples ω_1 in Ω_1 and ω_2 in Ω_2. The joint outcome is a point $\omega = (\omega_1, \omega_2)$ in the Polish space $\Omega = \Omega_1 \times \Omega_2$. Let $\mathbb{P}_1$ ($\mathbb{P}_2$) be the probability law of Ω_1 (Ω_2). Then $\mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2$ is a probability law on Ω characterized by the rule

$$\mathbb{P}[A_1 \times A_2] = \mathbb{P}_1[A_1] \cdot \mathbb{P}_2[A_2]. \tag{1.29}$$

Stated otherwise, the probability of finding jointly ω_1 in A_1 and ω_2 in A_2 is the product of the probabilities of these two events E_1 and E_2,

$$\mathbb{P}[E_1 \cap E_2] = \mathbb{P}[E_1] \cdot \mathbb{P}[E_2]. \tag{1.30}$$

This is the definition of the *stochastic independence of the events* E_1 *and* E_2.

A general measurement on the sample modeled by $(\Omega, \mathbb{P})$ is a function $F : \Omega \to \mathbb{X}$, where $\mathbb{X}$ is a Polish space, such that the inverse image $F^{-1}(B)$ of a Borel set B in $\mathbb{X}$ is a Borel set in Ω. Two measurements are stochastically independent if the corresponding maps $F_i : \Omega \to \mathbb{X}_i$ satisfy

$$\mathbb{P}[E_1 \cap E_2] = \mathbb{P}[E_1] \cdot \mathbb{P}[E_2], \tag{1.31a}$$

where E_i is the event that $F_i(\omega)$ belongs to B_i, that is $E_i = F_i^{-1}(B_i)$, where B_i is a Borel set in $\mathbb{X}_i$ for $i = 1, 2$. In terms of mean values, we obtain the equivalent condition

$$\mathbb{E}[\xi_1 \cdot \xi_2] = \mathbb{E}[\xi_1] \cdot \mathbb{E}[\xi_2], \tag{1.31b}$$

where ξ_i is of the form $u_i \circ F_i$ for $i = 1, 2$, and u_i is a (bounded) Borel function on $\mathbb{X}_i$.

Marginals

To be specific, let the sample space Ω be $C^0(\mathbb{T}; \mathbb{R})$; that is we record continuous curves depicting the evolution of a particle on a line during an interval of time $\mathbb{T} = [t_a, t_b]$ (figure 1.1).

In general, we don't record the whole curve, but we make measurements at certain selected times $t_1, \ldots, t_n$ such that

$$t_a \leq t_1 < \cdots < t_n \leq t_b. \tag{1.32}$$

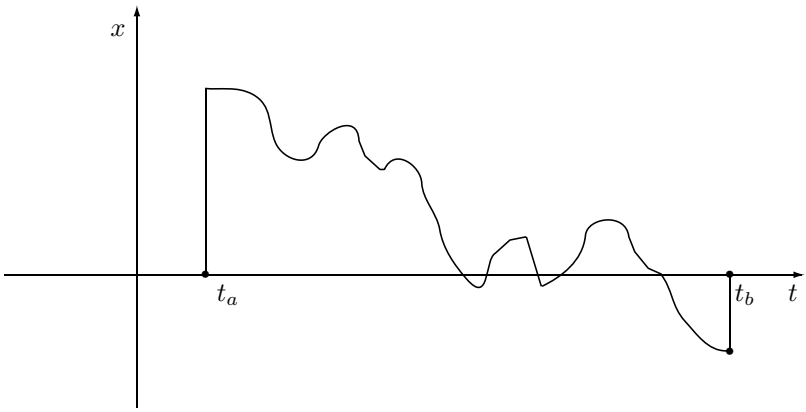

Fig. 1.1

We can approximate the probability law $\mathbb{P}$ on Ω as follows. Given $t_1, \ldots, t_n$ as above,[19] consider the map

$$\Pi_{t_1,\ldots,t_n} : \Omega \to \mathbb{R}^n$$

associating with a path $x = (x(t))_{t_a \le t \le t_b}$ the collection $(x(t_1), \ldots, x(t_n))$. Then there exists a measure $\mu_{t_1,\ldots,t_n}$ on $\mathbb{R}^n$ given by

$$\mu_{t_1,\ldots,t_n}([a_1, b_1] \times \cdots \times [a_n, b_n]) = \mathbb{P}[a_1 \le x(t_1) \le b_1, \ldots, a_n \le x(t_n) \le b_n]. \tag{1.33}$$

In most cases, $\mu_{t_1,\ldots,t_n}$ is given by a probability density $p(t_1, \ldots, t_n; x_1, \ldots, x_n)$ that is

$$\mu_{t_1,\ldots,t_n}(A) = \int_A \mathrm{d}^n x \; p(t_1, \ldots, t_n; x_1, \ldots, x_n) \tag{1.34}$$

for any Borel subset A in $\mathbb{R}^n$. The measures $\mu_{t_1,\ldots,t_n}$ or the corresponding probability densities are called the *marginals* of the process $(x(t))_{t_a \le t \le t_b}$ modeled by the probability law $\mathbb{P}$ on $\Omega = C^0(\mathbb{T}; \mathbb{R})$.

The marginals satisfy certain coherence rules. We state them in terms of probability density:[20]

- *probability density*:

$$p(t_1, \ldots, t_n; x_1, \ldots, x_n) \ge 0, \tag{1.35}$$

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \; p(t_1, \ldots, t_n; x_1, \ldots, x_n) = 1; \tag{1.36}$$

[19] The restriction $t_1 < \cdots < t_n$ is irrelevant, since any system of *distinct* epochs $t_1, \ldots, t_n$ can always be rearranged.

[20] The general case is formally the same if we interpret the probability density in terms of generalized functions, such as Dirac δ-functions.

- *symmetry*: any permutation of the spacetime points $(t_1, x_1), \ldots, (t_n, x_n)$ leaves $p(t_1, \ldots, t_n; x_1, \ldots, x_n)$ invariant; and
- *compatibility*:

$$p(t_1, \ldots, t_n; x_1, \ldots, x_n) = \int_{\mathbb{R}} \mathrm{d}x_{n+1} p(t_1, \ldots, t_n, t_{n+1}; x_1, \ldots, x_n, x_{n+1}). \tag{1.37}$$

The information carried by the marginals is exhaustive, in the sense that the probability law $\mathbb{P}$ on Ω can be uniquely reconstructed from the marginals.

Promeasures

The converse of the previous construction is fundamental. Assume given for each sequence $t_1, \ldots, t_n$ of distinct epochs in the time interval $\mathbb{T}$ a probability measure $\mu_{t_1,\ldots,t_n}$ on $\mathbb{R}^n$ satisfying the previous rules of symmetry and compatibility. Such a system is called (by Bourbaki) a *promeasure*, or (by other authors) a *cylindrical measure*. Such a promeasure enables us to define the mean value $\mathbb{E}[F]$ for a certain class of functionals of the process. Namely, if F is given by

$$F(x) = f(x(t_1), \ldots, x(t_n)) \tag{1.38}$$

for a suitable Borel function f in $\mathbb{R}^n$ (let it be bounded in order to get a finite result), then $\mathbb{E}[F]$ is unambiguously defined by

$$\mathbb{E}[F] = \int_{\mathbb{R}^n} \mathrm{d}^n x \, f(x_1, \ldots, x_n) \, p(t_1, \ldots, t_n; x_1, \ldots, x_n). \tag{1.39}$$

The problem is that of how to construct a probability measure $\mathbb{P}$ on Ω such that $\mathbb{E}[F] = \int_\Omega F \, \mathrm{d}\mathbb{P}$ for F as above. We already know that $\mathbb{P}$ is unique; the question is that of whether it exists.

For the existence, the best result is given by the following criterion, due to Prokhorov [13]:

Criterion. In order that there exists a probability measure $\mathbb{P}$ on Ω with marginals $\mu_{t_1,\ldots,t_n}$, it is necessary and sufficient that, for any given $\epsilon > 0$, there exists a compact set K in the Polish space Ω such that, for all sequences $t_1, \ldots, t_n$ in $\mathbb{T}$, the measure under $\mu_{t_1,\ldots,t_n}$ of the projected set $\Pi_{t_1,\ldots,t_n}(K)$ in $\mathbb{R}^n$ be at least $1 - \epsilon$.

Notice that, by regularity of the measures, given $t_1, \ldots, t_n$ there exists a compact set $K_{t_1,\ldots,t_n}$ in $\mathbb{R}^n$ such that

$$\mu_{t_1,\ldots,t_n}(K_{t_1,\ldots,t_n}) \geq 1 - \epsilon. \tag{1.40}$$

The important point is the *uniformity*, that is the various sets $K_{t_1,\dots,t_n}$ are obtained as the projection of a unique compact set K in Ω.

Prokhorov's criterion was distilled from the many known particular cases. The most conspicuous one is the brownian motion. The main assumptions about this process can be formulated as follows:

- the successive differences $x(t_1)-x(t_a), x(t_2)-x(t_1), \dots, x(t_n)-x(t_{n-1})$ (for $t_a \le t_1 < \cdots < t_n \le t_b$) are stochastically independent; and
- a given difference $x(t_2) - x(t_1)$ obeys a gaussian law with variance $D \cdot (t_2 - t_1)$, where $D > 0$ is a given constant. This gives immediately

$$p(t_1, \dots, t_n; x_1, \dots, x_n) = (2\pi D)^{-n/2} \prod_{i=1}^{n} (t_i - t_{i-1})^{-1/2} \times \exp\left(- \sum_{i=1}^{n} \frac{(x_i - x_{i-1})^2}{2D \cdot (t_i - t_{i-1})} \right) \tag{1.41}$$

(where $t_0 = t_a$ and $x_0 = x_a$, the initial position at time t_a).

The coherence rules are easily checked. To use Prokhorov's criterion, we need Arzela's theorem, which implies that, for given constants $C > 0$ and $\alpha > 0$, the set $K_{C,\alpha}$ of functions satisfying a Lipschitz condition

$$|x(t_2) - x(t_1)| \le C|t_2 - t_1|^\alpha \qquad \text{for } t_1 < t_2 \tag{1.42}$$

is compact in the Banach space $C^0(\mathbb{T}; \mathbb{R})$. Then we need to estimate $\mu_{t_1,\dots,t_n}(\Pi_{t_1,\dots,t_n}(K_{C,\alpha}))$, a problem in the geometry of the euclidean space $\mathbb{R}^n$, but with constants independent of n (and of $t_1, \dots, t_n$). The uniformity is crucial, and typical of such problems of integration in infinite-dimensional spaces!

1.7 Fourier transformation and prodistributions[21]

Characteristic functions

Let Ω be a Polish space, endowed with a probability measure $\mathbb{P}$. If X is a random variable, that is a Borel function $X : \Omega \to \mathbb{R}$, the *probability law of* X is the measure μ_X on $\mathbb{R}$ such that

$$\mu_X([a, b]) = \mathbb{P}[a \le X \le b] \tag{1.43}$$

for real numbers a, b such that $a \le b$, where the right-hand side denotes the probability of the event that $a \le X \le b$. Very often, there exists a

[21] An informal presentation of prodistributions (which were originally introduced as pseudomeasures [8]) accessible to nonspecialists can be found in [14].

probability density p_X such that

$$\mathbb{P}[a \le X \le b] = \int_a^b \mathrm{d}x\, p_X(x). \tag{1.44}$$

Then, for any function $f(x)$ of a real variable, we have

$$\mathbb{E}[f(X)] = \int_{\mathbb{R}} \mathrm{d}x\, p_X(x) f(x). \tag{1.45}$$

In particular, for $f(x) = \mathrm{e}^{\mathrm{i}px}$ we get

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}pX}] = \int_{\mathbb{R}} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}px} p_X(x). \tag{1.46}$$

This is the so-called *characteristic function* of X, or else the Fourier transform of the probability law μ_X (or the probability density p_X).

Given a collection $(X^1, \ldots, X^n)$ of random variables, or, better said, a random vector $\vec{X}$, we define the probability law $\mu_{\vec{X}}$ as a probability measure in $\mathbb{R}^n$, such that

$$\mathbb{P}[\vec{X} \in A] = \mu_{\vec{X}}(A) \tag{1.47}$$

for every Borel subset A of $\mathbb{R}^n$, or equivalently

$$\mathbb{E}[f(\vec{X})] = \int_{\mathbb{R}^n} \mathrm{d}\mu_{\vec{X}}(\vec{x}) f(\vec{x}) \tag{1.48}$$

for every (bounded) Borel function $f(\vec{x})$ on $\mathbb{R}^n$. In particular, the Fourier transform of $\mu_{\vec{X}}$ is given by

$$\chi_{\vec{X}}(\vec{p}) := \int_{\mathbb{R}^n} \mathrm{d}\mu_{\vec{X}}(\vec{x}) \mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}} \tag{1.49}$$

for $\vec{p} = (p_1, \ldots, p_n)$, $\vec{x} = (x^1, \ldots, x^n)$, and $\vec{p} \cdot \vec{x} = \sum_{j=1}^n p_j x^j$. In probabilistic terms, we obtain

$$\chi_{\vec{X}}(\vec{p}) = \mathbb{E}[\exp(\mathrm{i}\vec{p} \cdot \vec{X})], \tag{1.50}$$

where, as expected, we define $\vec{p} \cdot \vec{X}$ as the random variable $\sum_{j=1}^n p_j X^j$.

Hence the conclusion that *knowing the characteristic function of all linear combinations $p_1 X^1 + \cdots + p_n X^n$ with nonrandom coefficients $p_1, \ldots, p_n$ is equivalent to knowledge of the probability law $\mu_{\vec{X}}$ of the random vector $\vec{X}$.*

The characteristic functional

The idea of the characteristic functional was introduced by Bochner [20] in 1955. As in Section 1.6, we consider a stochastic process $(X(t))_{t_a \le t \le t_b}$.

Assuming that the trajectories are continuous (at least with probability unity!), our process is modeled by a probability measure $\mathbb{P}$ on the Banach space[22] $C^0(\mathbb{T};\mathbb{R})$, or equivalently by the marginals $\mu_{t_1,\dots,t_n}$ for $t_a \le t_1 < \cdots < t_n \le t_b$. These marginals are just the probability laws of the random vectors $(X(t_1),\dots,X(t_n))$ obtained by sampling the path at the given times. It is therefore advisable to consider their characteristic functions

$$\chi_{t_1,\dots,t_n}(p_1,\dots,p_n) = \int_{\mathbb{R}^n} \mathrm{d}\mu_{t_1,\dots,t_n}(x^1,\dots,x^n)\mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}}. \tag{1.51}$$

Knowledge of the marginals, that is of the promeasure corresponding to the process, is therefore equivalent to knowledge of the characteristic functions $\chi_{t_1,\dots,t_n}(p_1,\dots,p_n)$. But we have the obvious relations

$$\chi_{t_1,\dots,t_n}(p_1,\dots,p_n) = \mathbb{E}\left[\exp\left(\mathrm{i}\sum_{j=1}^{n} p_j X(t_j)\right)\right], \tag{1.52}$$

$$\sum_{j=1}^{n} p_j X(t_j) = \int_{\mathbb{T}} \mathrm{d}\lambda(t) X(t), \tag{1.53}$$

where λ is the measure $\sum_{j=1}^{n} p_j \delta(t-t_j)$ on $\mathbb{T}$. We are led to introduce the characteristic functional

$$\Phi(\lambda) = \mathbb{E}\left[\exp\left(\mathrm{i}\int_{\mathbb{T}} \mathrm{d}\lambda(t) X(t)\right)\right], \tag{1.54}$$

where λ runs over the (real) measures on the compact space $\mathbb{T}$, that is (by the Riesz–Markoff theorem) over the continuous linear forms over the Banach space $C^0(\mathbb{T};\mathbb{R})$. On going backwards from equation (1.54) to equation (1.51), we see how to *define* the marginals $\mu_{t_1,\dots,t_n}$ starting from the characteristic functional $\Phi(\lambda)$ over the dual space $C^0(\mathbb{T};\mathbb{R})'$ of $C^0(\mathbb{T};\mathbb{R})$. But then the coherence rules (see equations (1.35)–(1.37)) are tautologically satisfied. Hence the probability law $\mathbb{P}$ of the process is completely characterized by the characteristic functional. Notice that we have a kind of infinite-dimensional Fourier transform,[23] namely

$$\Phi(\lambda) = \int_{C^0(\mathbb{T};\mathbb{R})} \mathrm{d}\mathbb{P}(X)\mathrm{e}^{\mathrm{i}\langle\lambda,X\rangle} \tag{1.55}$$

where $\langle\lambda, X\rangle = \int_{\mathbb{T}} \mathrm{d}\lambda\, X$ defines the duality on the Banach space $C^0(\mathbb{T};\mathbb{R})$.

[22] Of course $\mathbb{T}$ is the interval $[t_a, t_b]$.

[23] Later in this book, we shall modify the normalization of the Fourier transform, by putting $2\pi\mathrm{i}$ instead of i in the exponential.

As an example, the characteristic functional of the brownian motion is given by[24]

$$\Phi(\lambda) = \exp\left(-\frac{D}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \mathrm{d}\lambda(t)\mathrm{d}\lambda(t')\inf(t,t')\right). \tag{1.56}$$

Prodistributions

As we shall explain in Part C, going from Kac's formula to Feynman's formula requires analytic continuation. One way to do it is for instance to replace D by iD in the last formula.

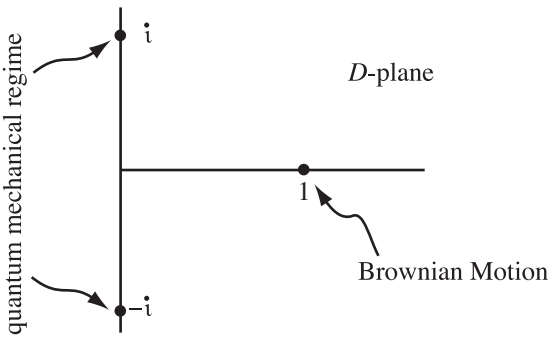

Fig. 1.2

Changing D into iD in the definition (1.41) of the marginals for the brownian motion causes no difficulty, but now $p(t_1,\ldots,t_n;x_1,\ldots,x_n)$ is of constant modulus (for fixed $t_1,\ldots,t_n$) and hence cannot be integrated over the whole space. In order to define the Fourier transformation, we need to resort to the theory of tempered distributions. So we come to the definition of a *prodistribution*[25] (see [8] and [14]) as a collection of marginals $\mu_{t_1,\ldots,t_n}(x_1,\ldots,x_n)$ that are now (tempered) distributions on the spaces $\mathbb{R}^n$. In order to formulate the compatibility condition (1.37), we restrict our distributions to those whose Fourier transform is a continuous function (necessarily of polynomial growth at infinity). As in the case of promeasures, it is convenient to summarize the marginals into the characteristic functional $\Phi(\lambda)$. The correct analytic assumption is an estimate

$$|\Phi(\lambda)| \leq C(\|\lambda\|+1)^N \tag{1.57}$$

with constants $C>0$ and $N\geq 0$, and the continuity of the functional Φ in the dual of the Banach space $C^0(\mathbb{T};\mathbb{R})$. These restrictions are fulfilled

[24] See Section 3.1.
[25] Called "pseudomeasures" in [8], where they were originally introduced.

if we make D purely imaginary in (1.56). This is one way to make sense of the analytic continuation from Wiener–Kac to Feynman. We shall develop it in Part C.

C: Feynman's integral versus Kac's integral

Kac's integral is an application of brownian motion, and is nowadays used in problems of statistical mechanics, especially in euclidean quantum field theories, and in the study of bound states. Feynman's integral is mostly used to study the dynamics of quantum systems. Formally, *one goes from Kac's integral to Feynman's integral by using analytic continuation* with respect to suitable parameters. We would like to argue in favor of the "equation"

$$\text{inverse temperature} \sim \text{imaginary time}.$$

1.8 Planck's blackbody radiation law

The methods of thermodynamics were used in the derivation of the (pre-quantum) laws for the blackbody radiation.

Stefan's law: in a given volume V, the total energy E carried by the blackbody radiation is proportional to the fourth power of the absolute temperature T. (The homogeneity of radiation assumes also that E is proportional to V.)

Wien's displacement law: by reference to the spectral structure of the radiation, it states that the frequency ν_{m} of maximum energy density is proportional to T.

To put these two laws into proper perspective, we need some *dimensional analysis*. Ever since the pioneering work of Gabor in the 1940s, it has been customary to analyze oscillations in a *time–frequency diagram*[26] (figure 1.3).

Here t is the time variable, and $\omega = 2\pi\nu$ is the *pulsation* (ν being the *frequency*). The product ωt is a phase, hence it is a pure number using the radian as the natural phase unit. Hence the area element $\mathrm{d}t \cdot \mathrm{d}\omega$ is without physical dimensions, and it makes sense to speak of the energy in a given cell with $\Delta t \cdot \Delta\omega = 1$. Similarly, for a moving wave with wave-vector $\vec{k}$, the dot product $\vec{k} \cdot \vec{x}$, where $\vec{x}$ is the position vector, is a phase; hence there is a dimensionless volume element $\mathrm{d}^3\vec{x} \cdot \mathrm{d}^3\vec{k}$ in the phase diagram with coordinates $x^1, x^2, x^3, k^1, k^2, k^3$. This is plainly visible in the Fourier

[26] Compare this with the standard representation of musical scores, as well as the modern wavelet analysis.

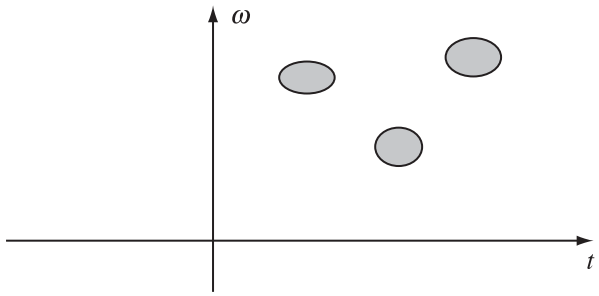

Fig. 1.3

inversion formula, which reads

$$f(0) = (2\pi)^{-3} \int\int \mathrm{d}^3\vec{x} \cdot \mathrm{d}^3\vec{k}\ \mathrm{e}^{-\mathrm{i}\vec{k}\cdot\vec{x}} f(\vec{x}). \tag{1.58}$$

Hence, in the spectral analysis of the blackbody radiation, one can speak of the energy per unit cell $\Delta^3\vec{x} \cdot \Delta^3\vec{k} = 1$ around the point $\vec{x}$, $\vec{k}$ in the phase diagram. Since it depends also on the temperature T, we write it as $E(\vec{x}, \vec{k}, T)$. It obeys the following laws:

homogeneity: the spectral energy density $E(\vec{x}, \vec{k}, T)$ is independent of the space position $\vec{x}$; and

isotropy: the spectral energy density $E(\vec{x}, \vec{k}, T)$ depends only on the length $|\vec{k}|$ of the wave-vector, or rather on the pulsation $\omega = c \cdot |\vec{k}|$ (where c is the speed of light).

Hence the spectral energy density is of the form

$$E(\vec{x}, \vec{k}, T) = E(\omega, T).$$

Stefan's and Wien's law can be reformulated as scaling:

$$E(\vec{x}, \lambda\vec{k}, \lambda T) = \lambda E(\vec{x}, \vec{k}, T) \tag{1.59}$$

for an arbitrary scalar $\lambda > 0$ (or $E(\lambda\omega, \lambda T) = \lambda E(\omega, T)$).

To conclude, the spectral energy density is given by

$$E(\vec{x}, \vec{k}, T) = \omega f(\omega/T), \tag{1.60}$$

where f is a universal function. What one measures in the observations is the energy per unit volume in the small pulsation interval $[\omega, \omega + \Delta\omega]$ of the form

$$\Delta\mathcal{E} = E(\omega, T)\frac{4\pi\omega^2}{c^3}\Delta\omega. \tag{1.61}$$

For a given temperature, we get a unimodal curve (figure 1.4).

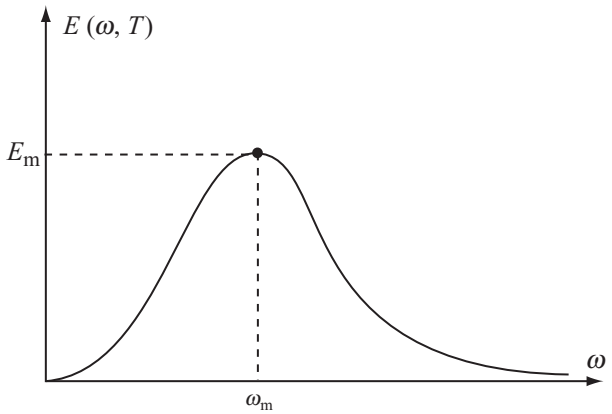

Fig. 1.4

Hence, the coordinates of the maximum define a pulsation $\omega_{\rm m} = \omega_{\rm m}(T)$ and an energy

$$E_{\rm m}(T) = \frac{c^3}{4\pi\omega_{\rm m}^2}\left(\frac{\Delta\mathcal{E}}{\Delta\omega}\right)_{\rm m}.$$

The scaling law is rewritten as[27]

$$\omega_{\rm m}(\lambda T) = \lambda\omega_{\rm m}(T), \qquad E_{\rm m}(\lambda T) = \lambda E_{\rm m}(T)$$

and amounts to an *identification of the three scales* temperature, pulsation, and energy.

This identification is made explicit in *Planck's Ansatz*

$$\Delta\mathcal{E} = \frac{A\nu^3}{{\rm e}^{B\nu/T} - 1} \cdot \Delta\nu \tag{1.62}$$

using the frequency $\nu = \omega/(2\pi)$ and *two* universal constants A and B. With a little algebra, we rewrite this in the standard form (using $A = 8\pi h/c^3$ and $B = h/k_{\rm B}$)

$$\Delta\mathcal{E} = \frac{8\pi h}{c^3}\frac{\nu^3}{{\rm e}^{h\nu/(k_{\rm B}T)} - 1} \cdot \Delta\nu. \tag{1.63}$$

We have now *two* Planck constants,[28] h and $k_{\rm B}$, and two laws

$$E = h\nu, \qquad E = k_{\rm B}T, \tag{1.64}$$

identifying energies E, frequencies ν, and temperatures T.

[27] The first formula is Wien's law, the second Stefan's law.

[28] The constant $k_{\rm B}$ was first considered by Planck and called "Boltzmann's constant" by him in his reformulation of the Boltzmann–Gibbs laws of statistical mechanics.

Here is a better formulation of Planck's law: in the phase diagram, the elementary spectral energy of the blackbody radiation is given by[29]

$$\langle E\rangle \times (2\pi)^{-3}\mathrm{d}^3\vec{x}\cdot\mathrm{d}^3\vec{k}\times 2, \tag{1.65a}$$

where the last factor of 2 corresponds to the two states of polarization of light and

$$\langle E\rangle = \frac{h\nu}{\mathrm{e}^{h\nu/(k_{\mathrm{B}}T)}-1} \tag{1.65b}$$

is the average thermal energy.

1.9 Imaginary time and inverse temperature

As usual, we associate with a temperature the inverse $\beta = 1/(k_{\mathrm{B}}T)$. According to (1.64), βE is dimensionless, and the laws of statistical mechanics are summarized as follows.

In a mechanical system with energy levels $E_0, E_1, \ldots$, the thermal equilibrium distribution at temperature T is given by weights proportional to $\mathrm{e}^{-\beta E_0}$, $\mathrm{e}^{-\beta E_1}, \ldots$ and hence the average thermal energy is given by

$$\langle E\rangle_\beta = \frac{\sum_n E_n \mathrm{e}^{-\beta E_n}}{\sum_n \mathrm{e}^{-\beta E_n}}, \tag{1.66}$$

or by the equivalent form

$$\langle E\rangle_\beta = -\frac{\mathrm{d}}{\mathrm{d}\beta}\ln Z(\beta), \tag{1.67}$$

with the *partition function*

$$Z(\beta) = \sum_n \mathrm{e}^{-\beta E_n}. \tag{1.68}$$

The thermodynamical explanation of Planck's law (1.65) is then that $\langle E\rangle$ is the average thermal energy of an oscillator of frequency ν, the energy being quantized as

$$E_n = nh\nu, \qquad \text{where } n = 0, 1, \ldots \tag{1.69}$$

Let's go back to the two fundamental laws $E = h\nu = k_{\mathrm{B}}T$. We have now a physical picture: in a thermal bath at temperature T, an oscillator of frequency ν corresponds to a Boltzmann weight $\mathrm{e}^{-h\nu/(k_{\mathrm{B}}T)}$ and hence to a phase factor $\mathrm{e}^{-2\pi\mathrm{i}\nu t_{\mathrm{B}}}$ upon introducing an *imaginary Boltzmann time*

$$t_{\mathrm{B}} = -\mathrm{i}\hbar\beta = -\mathrm{i}\hbar/(k_{\mathrm{B}}T) \tag{1.70}$$

[29] It follows from formula (1.58) that the natural volume element in the phase diagram is $(2\pi)^{-3}\mathrm{d}^3\vec{x}\cdot\mathrm{d}^3\vec{k} = \prod_{i=1}^3 \mathrm{d}x^i\mathrm{d}k_i/(2\pi)$, not $\mathrm{d}^3\vec{x}\cdot\mathrm{d}^3\vec{k}$. We refer the reader to the conventions given earlier for normalization constants in the Fourier transformation.