Path Integrals and Anomalies in Curved Space

FIORENZO BASTIANELLI
AND PETER VAN NIEUWENHUIZEN

CAMBRIDGE MONOGRAPHS
ON MATHEMATICAL PHYSICS

CAMBRIDGE www.cambridge.org/9780521847612

This page intentionally left blank

Path Integrals and Anomalies in Curved Space

Path integrals provide a powerful method for describing quantum phenomena, first introduced in physics by Dirac and Feynman. This book introduces the quantum mechanics of particles that move in curved space by employing the path integral method, and uses this formalism to compute anomalies in quantum field theories.

The authors start by deriving path integrals for particles moving in curved space (one-dimensional nonlinear sigma models), and their supersymmetric generalizations. Coherent states are used for fermionic particles. They then discuss the regularization and renormalization schemes essential to constructing and computing these path integrals.

In the second part of the book, the authors apply these methods to discuss and calculate anomalies in quantum field theories, with external gravitational and/or (non) abelian gauge fields. Anomalies constitute one of the most important aspects of quantum field theory; requiring that there are no anomalies is an enormous constraint in the search for physical theories of elementary particles, quantum gravity and string theories. In particular, the authors include explicit calculations of the gravitational anomalies, reviewing the seminal work of Alvarez-Gaumé and Witten in an original way, and their own work on trace anomalies.

Fiorenzo Bastianelli is Professor of Theoretical Physics at the Department of Physics, University of Bologna. Following his Ph.D., from SUNY at Stony Brook in New York, he has worked at ITP, Stockholm University, and at the Niels Bohr Institute, University of Copenhagen. In 1995 he moved to Italy, initially at the University of Modena, and since 1999 at the University of Bologna. His research interests include quantum field theory and string theory.

Peter van Nieuwenhuizen is a Distinguished Professor at the C. N. Yang Institute of SUNY, Stony Brook. After his Ph.D. from Utrecht University he worked at CERN (Geneva), the Ecole Normale Superieure (Paris) and Brandeis University (Mass.) In 1975 he moved to Stony Brook, where he co-discovered supergravity in 1976, for which he received the Dirac award in 1993, and the Heinemann prize in 2005. His research interests are in supersymmetry, supergravity, quantum field theory, modern general relativity and string theory, about which he has published over 300 articles in scientific journals.

CAMBRIDGE MONOGRAPHS ON MATHEMATICAL PHYSICS

General editors: P. V. Landshoff, D. R. Nelson, S. Weinberg

S. J. Aarseth *Gravitational N-Body Simulations*
J. Ambjørn, B. Durhuus and T. Jonsson *Quantum Geometry: A Statistical Field Theory Approach*
A. M. Anile *Relativistic Fluids and Magneto-Fluids*
J. A. de Azcárrage and J. M. Izquierdo *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics*†
O. Babelon, D. Bernard and M. Talon *Introduction to Classical Integrable Systems*
F. Bastianelli and P. van Nieuwenhuizen *Path Integrals and Anomalies in Curved Space*
V. Belinkski and E. Verdaguer *Gravitational Solitons*
J. Bernstein *Kinetic Theory in the Expanding Universe*
G. F. Bertsch and R. A. Broglia *Oscillations in Finite Quantum Systems*
N. D. Birrell and P.C.W. Davies *Quantum Fields in Curved Space*†
M. Burgess *Classical Covariant Fields*
S. Carlip *Quantum Gravity in* 2+1 *Dimensions*
J. C. Collins *Renormalization*†
M. Creutz *Quarks, Gluons and Lattices*†
P. D. D'Eath *Supersymmetric Quantum Cosmology*
F. de Felice and C. J. S. Clarke *Relativity on Curved Manifolds*†
B. S. DeWitt *Supermanifolds*, 2nd edition†
P. G. O. Freund *Introduction to Supersymmetry*†
J. Fuchs *Affine Lie Algebras and Quantum Groups*†
J. Fuchs and C. Schweigert *Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists*†
Y. Fujii and K. Maeda *The Scalar–Tensor Theory of Gravitation*
A. S. Galperin, E. A. Ivanov, V. I. Orievetsky and E. S. Sokatchev *Harmonic Superspace*
R. Gambini and J. Pullin *Loops, Knots, Gauge Theories and Quantum Gravity*†
M. Göckeler and T. Schücker *Differential Geometry, Gauge Theories and Gravity*†
C. Gómez, M. Ruiz Altaba and G. Sierra *Quantum Groups in Two-dimensional Physics*
M. B. Green, J. H. Schwarz and E. Witten *Superstring Theory, volume 1: Introduction*†
M. B. Green, J. H. Schwarz and E. Witten *Superstring Theory, volume 2: Loop Amplitudes, Anomalies and Phenomenology*†
V. N. Gribov *The Theory of Complex Angular Momenta*
S. W. Hawking and G. F. R. Ellis *The Large-Scale Structure of Space-Time*†
F. Iachello and A. Arima *The Interacting Boson Model*
F. Iachello and P. van Isacker *The Interacting Boson–Fermion Model*
C. Itzykson and J.-M. Drouffe *Statistical Field Theory, volume 1: From Brownian Motion to Renormalization and Lattice Gauge Theory*†
C. Itzykson and J.-M. Drouffe *Statistical Field Theory, volume 2: Strong Coupling, Monte Carlo Methods, Conformal Field Theory, and Random Systems*†
C. Johnson *D-Branes*
J. I. Kapusta *Finite-Temperature Field Theory*†
V. E. Korepin, A. G. Izergin and N. M. Boguliubov *The Quantum Inverse Scattering Method and Correlation Functions*†
M. Le Bellac *Thermal Field Theory*†
Y. Makeenko *Methods of Contemporary Gauge Theory*
N. Manton and P. Sutcliffe *Topological Solitons*
N. H. March *Liquid Metals: Concepts and Theory*
I. M. Montvay and G. Münster *Quantum Fields on a Lattice*†
L. O' Raifeartaigh *Group Structure of Gauge Theories*†
T. Ortín *Gravity and Strings*
A. Ozorio de Almeida *Hamiltonian Systems: Chaos and Quantization*†
R. Penrose and W. Rindler *Spinors and Space-Time, volume 1: Two-Spinor Calculus and Relativistic Fields*†
R. Penrose and W. Rindler *Spinors and Space-Time, volume 2: Spinor and Twistor Methods in Space-Time Geometry*†
S. Pokorski *Gauge Field Theories*, 2nd edition
J. Polchinski *String Theory, volume 1: An Introduction to the Bosonic String*†
J. Polchinski *String Theory, volume 2: Superstring Theory and Beyond*†
V. N. Popov *Functional Integrals and Collective Excitations*†
R. J. Rivers *Path Integral Methods in Quantum Field Theory*†
R. G. Roberts *The Structure of the Proton*†
C. Rovelli *Quantum Gravity*
W. C. Saslaw *Gravitational Physics of Stellar and Galactic Systems*†
H. Stephani, D. Kramer, M. A. H. MacCallum, C. Hoenselaers and E. Herlt *Exact Solutions of Einstein's Field Equations, 2nd edition*
J. M. Stewart *Advanced General Relativity*†
A. Vilenkin and E. P. S. Shellard *Cosmic Strings and Other Topological Defects*†
R. S. Ward and R. O. Wells Jr *Twistor Geometry and Field Theories*†
J. R. Wilson and G. J. Mathews *Relativistic Numerical Hydrodynamics*

†Issued as a paperback

Path Integrals and Anomalies in Curved Space

FIORENZO BASTIANELLI
Università degli Studi,
Bologna, Italy

PETER VAN NIEUWENHUIZEN
State University of New York at Stony Brook

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 2RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521847612

First published in print format 2006

ISBN-13 978-0-511-21772-2 eBook (Adobe Reader)
ISBN-10 0-511-21772-2 eBook (Adobe Reader)

ISBN-13 978-0-521-84761-2 hardback
ISBN-10 0-521-84761-3 hardback

To Bryce S. DeWitt (1923–2004)
who pioneered quantum mechanics in curved space

Contents

Preface

In 1983, L. Alvarez-Gaumé and E. Witten (AGW) wrote a fundamental article in which they calculated the one-loop gravitational anomalies (anomalies in the local Lorentz symmetry of $(4k+2)$-dimensional Minkowskian quantum field theories coupled to external gravity) of complex chiral spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ fields and real self-dual antisymmetric tensor fields[1] [1]. They used two methods: a straightforward Feynman graph calculation in $4k+2$ dimensions with Pauli–Villars regularization, and a quantum mechanical (QM) path integral method in which corresponding nonlinear sigma models appeared. The former has been discussed in detail in an earlier book [3]. The latter method is the subject of this book. AGW applied their formulas to $N=2B$ supergravity in 10 dimensions, which contains precisely one field of each kind, and found that the sum of the gravitational anomalies cancels. Soon afterwards, M. B. Green and J. H. Schwarz [4] calculated the gravitational anomalies in one-loop string amplitudes, and concluded that these anomalies cancel in string theory, and therefore should also cancel in $N=1$ supergravity in 10 dimensions with suitable gauge groups for the $N=1$ matter couplings. Using the formulas of AGW, one can indeed show that the sum of anomalies in $N=1$ supergravity coupled to super Yang–Mills theory with gauge group $SO(32)$ or $E_8 \times E_8$, though nonvanishing, is in the technical sense exact:

[1] Just as one can always shift the axial anomaly from the vector current to the axial current by adding a suitable counterterm to the action or by using a different regularization scheme, one can also shift the gravitational anomaly from the general coordinate symmetry to the local Lorentz symmetry [2]. Conventionally one chooses to preserve general coordinate invariance. AGW chose the symmetric vielbein gauge, so that the symmetry for which they computed the anomalies was a linear combination of a general coordinate transformation and a compensating local Lorentz transformation. However, they used a regulator that manifestly preserved general coordinate invariance, so that their calculation yielded the anomaly in the local Lorentz symmetry.

it can be removed by adding a local counterterm to the action. These two papers led to an explosion of interest in string theory.

We discussed these two papers in a series of internal seminars for advanced graduate students and faculty at Stony Brook (the "Friday seminars"). Whereas the basic philosophy and methods of the paper by AGW were clear, we stumbled on numerous technical problems and details. Some of these became clearer upon closer reading, some became more baffling. In a desire to clarify these issues we decided to embark on a research project: the AGW program for trace anomalies. Since gravitational and chiral anomalies only contribute at the one-worldline-loop level in the QM method, one need not be careful with definitions of the measure for the path integral, choice of regulators, regularization of divergent graphs, etc. This is explicitly discussed in [1]. However, we soon noticed that for the trace anomalies the opposite is true: if the field theory is defined in $n = 2k$ dimensions, one needs $(k + 1)$-loop graphs on the worldline in the QM method. Consequently, every detail in the calculation matters. Our program of calculating trace anomalies turned into a program of studying path integrals for nonlinear sigma models in phase space and configuration space, a notoriously difficult and controversial subject. As already pointed out by AGW, the QM nonlinear sigma models needed for spacetime fermions (or self-dual antisymmetric tensor fields in spacetime) have $N = 1$ (or $N = 2$) worldline supersymmetry (susy), even though the original field theories were not spacetime supersymmetric. Thus, we also had to wrestle with the role of susy in the careful definitions and calculations of these QM path integrals.

Although it only gradually dawned upon us, we have come to recognize the problems with these susy and nonsusy QM path integrals as problems one should expect to encounter in any quantum field theory (QFT), the only difference being that these particular field theories have a one-dimensional (finite) spacetime, as a result of which infinities in the sum of Feynman graphs for a given process cancel. However, individual Feynman graphs are power-counting divergent (because these models contain double-derivative interactions just like quantum gravity). This cancellation of infinities in the sum of graphs is perhaps the psychological reason why there is no systematic discussion of regularization issues in the early literature on the subject (in the 1950s and 1960s). With the advent of the renormalization of gauge theories in the 1970s, issues of regularization of nonlinear sigma models were also studied. It was found that the regularization schemes used at that time (the time slicing method and the mode regularization method) broke general coordinate invariance at intermediate stages, but it was also noted that by adding noncovariant counterterms [5–9], the final physical results were still general coordinate invariant (we shall use the shorter term Einstein invariance for this symmetry in this book). The question thus arose as to how to determine those counterterms, and to understand the relation between the counterterms in one regularization scheme and those in other schemes. Once again, the answer to this question could be found in the general literature on QFT: the imposition of suitable renormalization conditions.

As we tackled more and more difficult problems (four-loop graphs for trace anomalies in six dimensions) it became clear to us that a scheme which needed only covariant counterterms would be very welcome. Dimensional regularization (DR) is such a scheme [10]. It had been used by Kleinert and Chervyakov [11] for the QM of a one-dimensional target space on an infinite worldline time interval (with a mass term added to regulate infrared divergences). For our purposes we have developed instead a version of dimensional regularization on a compact space; because the space is compact we do not need to add by hand a mass term to regulate the infrared divergences due to massless fields. The counterterms needed in such an approach are indeed covariant (both Einstein and locally Lorentz invariant).

The quantum mechanical path integral formalism can be used to compute anomalies in quantum field theories. This application forms the second part of this book. Chiral spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ fields and selfdual antisymmetric tensor (SAT) fields can produce anomalies in loop graphs with external gravitons and/or external gauge (Yang–Mills) fields. The treatment of the spin $\frac{3}{2}$ and SAT fields formed a major obstacle. For example, in the article by AGW the SAT fields are described by a bispinor $\psi_{\alpha\beta}$. However, the vector index of the spin-$\frac{3}{2}$ field and the β index of $\psi_{\alpha\beta}$ are treated differently from the spinor index of the spin-$\frac{1}{2}$ and spin-$\frac{3}{2}$ fields and the α index of $\psi_{\alpha\beta}$. In [1] one finds the following transformation rule for the spin-$\frac{3}{2}$ field (in their notation):

$$-\delta_\eta \psi_A = \eta^i D_i \psi_A + D_a \eta_b (T^{ab})_{AB} \psi_B \tag{1}$$

where $\eta^i(x)$ parametrizes an infinitesimal coordinate transformation $x^i \to x^i + \eta^i(x)$, and $A = 1, 2, \ldots, n$ is the flat vector index of the spin-$\frac{3}{2}$ (gravitino) field, while $(T^{ab})_{AB} = -i(\delta^a_A \delta^b_B - \delta^b_A \delta^a_B)$ are the matrix elements of the Euclidean Lorentz group $SO(n)$ in the vector representation. One would expect that this transformation rule is a linear combination of an Einstein transformation $\delta_E \psi_{A\alpha} = \eta^i \partial_i \psi_{A\alpha}$ (the vector index A of $\psi_{A\alpha}$ is flat and α is the spin index) and a local Lorentz rotation $\delta_{lL} \psi_{A\alpha} = \frac{1}{4} \eta^i \omega_{iBC} (\gamma^B \gamma^C)_\alpha{}^\beta \psi_{A\beta} + \eta^i \omega_{iA}{}^B \psi_{B\alpha}$. However, on top of this Lorentz rotation with parameter $\eta^i \omega_{iAB}$, one finds the second term in (1) which describes a local Lorentz rotation with parameter $(D_a \eta_b - D_b \eta_a)$ and this local Lorentz transformation only acts on the vector index of the gravitino. If one assumes (1), one finds a beautiful simple relation between the gravitational contribution to the axial (γ_5) anomaly in $4k+4$ dimensions and the gravitational (local Lorentz) anomaly in $4k+2$ dimensions. We shall derive (1) from first principles, and show that it is correct, but only if one uses a particular regulator $\mathcal{R}$.

The regulators for the spin-$\frac{1}{2}$ field λ, for the gravitino ψ_A, and for the bispinor $\psi_{\alpha\beta}$ are in all cases the square of the field operators for the nonchiral spinors $\tilde{\lambda}$, $\tilde{\psi}_A$ and $\tilde{\psi}_{\alpha\beta}$, where the "twiddled fields" $\tilde{\lambda}$, $\tilde{\psi}_A$ and $\tilde{\psi}_{\alpha\beta}$ are obtained from λ, ψ_A and $\psi_{\alpha\beta}$ by multiplication by $g^{1/4} = (\det e_\mu{}^m)^{1/2}$. These regulators are covariant regulators, not consistent regulators, and the anomalies we will obtain are covariant anomalies, not consistent anomalies [2]. However, when we come to the cancellation of anomalies, we shall use the descent equations to convert these covariant anomalies to consistent anomalies, and then construct counterterms whose variations cancel these consistent anomalies.

The twiddled fields were used by Fujikawa, who pioneered the path integral approach to anomalies [12]. An ordinary Einstein transformation of $\tilde{\lambda}$ is given by $\delta\tilde{\lambda} = \frac{1}{2}(\xi^\mu\partial_\mu + \partial_\mu\xi^\mu)\tilde{\lambda}$, where the second derivative ∂_μ can also act on $\tilde{\lambda}$, and if one evaluates the corresponding regulated anomaly $An_E = \mathrm{Tr}\frac{1}{2}(\xi^\mu\partial_\mu + \partial_\mu\xi^\mu)\mathrm{e}^{-\beta\mathcal{R}}$ by inserting a complete set of eigenfunctions $\tilde{\varphi}_k$ of $\mathcal{R}$ with non-negative eigenvalues λ_k, one finds

$$An_E = \lim_{\beta\to 0}\sum_k \int d^n x\, \tilde{\varphi}_k^*(x)\frac{1}{2}(\xi^\mu\partial_\mu + \partial_\mu\xi^\mu)\mathrm{e}^{-\beta\lambda_k}\tilde{\varphi}_k(x)\ . \tag{2}$$

Thus, the Einstein anomaly vanishes (partially integrate the second ∂_μ) as long as the regulator is self-adjoint with respect to the inner product $\langle\tilde{\lambda}_1|\tilde{\lambda}_2\rangle = \int dx\, \tilde{\lambda}_1^*(x)\tilde{\lambda}_2(x)$ (so that $\tilde{\varphi}_k$ form a complete set), and as long as both $\tilde{\varphi}_k(x)$ and $\tilde{\varphi}_k^*(x)$ belong to the same complete set of eigenstates, as in the case of plane waves e^{ikx}. One can always make a unitary transformation from $\tilde{\varphi}_k$ to the set e^{ikx}, and using these plane waves, the calculation of anomalies in the framework of quantum field theory is reduced to a set of n-dimensional Gaussian integrals over k. We shall use the regulator $\mathcal{R}$ discussed above, and twiddled fields, but then cast the calculation of anomalies in terms of quantum mechanics and path integrals. Calculating anomalies using quantum mechanics is much simpler than evaluating the Gaussian integrals of quantum field theory. Using path integrals simplifies the calculations even further.

When we first started studying the problems discussed in this book, we used the shortcuts and plausible arguments which are used by researchers and sometimes mentioned in the literature. However, the more we tried to clarify and complete these shortcuts and arguments, the more we were driven to basic questions and theoretical principles. We have been studying these issues now for over 15 years, and have accumulated a wealth of facts and insights. We decided to write a book in which all ideas and calculation were developed from scratch, with all intermediate steps worked out. The result looks detailed, and at places technical. We have made every effort to keep the text readable by providing verbal descriptions next to formulas, and providing introductory sections and historical reviews. In the end, however, we felt there is no substitute for a complete and fundamental treatment.

We end this preface by summarizing the content of this book. In the first part of this book we give a complete derivation of the path integrals for supersymmetric and nonsupersymmetric nonlinear sigma models describing bosonic and fermionic point particles (commuting coordinates $x^i(t)$ and anticommuting variables $\psi^a(t) = e_i^a(x(t))\psi^i(t)$) in a curved target space with metric $g_{ij}(x) = e_i^a(x)e_j^b(x)\delta_{ab}$. All of our calculations are performed in Euclidean target space. We consider a finite time interval because this is what is needed for the applications to anomalies. As these models contain double-derivative interactions, they are divergent according to power-counting, just as in quantum gravity, but ghost loops arising from the path integral measure cancel the divergences. Only the one- and two-loop graphs are power-counting divergent, hence in general the action may contain extra finite local one- and two-loop counterterms, the coefficients of which should be fixed. They are fixed by imposing suitable renormalization conditions. To regularize individual diagrams we use three different regularization schemes:

(i) time slicing (TS), known from the work of Dirac and Feynman;

(ii) mode regularization (MR), known from instanton and soliton physics;[2] and

(iii) dimensional regularization on a finite time interval (DR), discussed in this book.

The renormalization conditions relate a given quantum Hamiltonian $\hat{H}$ to a corresponding quantum action S, by which we mean the action that appears in the exponent of the path integral. The particular finite one- and two-loop counterterms in S thus obtained are different for each regularization scheme. In principle, any $\hat{H}$ with a definite ordering of the operators can be taken as the starting point, and gives a corresponding path integral (with different counterterms for different regularization schemes), but for our physical applications we shall consider quantum Hamiltonians that maintain reparametrization and local Lorentz invariance in target space (i.e. commute with the quantum generators of these

[2] Actually, the mode expansion had already been used by Feynman and Hibbs to compute the path integral for the harmonic oscillator.

symmetries. The chiral anomaly is then due to the chirality matrix in the Jacobians). Then there are no one-loop counterterms in the three schemes, but only two-loop counterterms. Having defined the regulated path integrals, the continuum limit can be taken and reveals the correct "Feynman rules" (the rules of how to evaluate the integrals over products of distributions and equal-time contractions) for each regularization scheme. All three regularization schemes give the same final answer for the transition amplitude, although the Feynman rules are different.

In the second part of this book we apply our methods to the evaluation of anomalies in n-dimensional relativistic quantum field theories with bosons and fermions in the loops (spin $0, \frac{1}{2}, 1, \frac{3}{2}$ and self-dual antisymmetric tensor fields) coupled to external gauge fields and/or gravity. We regulate the field-theoretical Jacobian for the symmetries whose anomalies we want to compute with a factor of $\exp(-\beta\mathcal{R})$, where $\mathcal{R}$ is the covariant regulator which follows from the corresponding quantum field theory, as discussed before, and β tends to zero only at the end of the calculation. Next, we introduce a quantum mechanical representation of the operators which enter in the field-theoretical calculation. The regulator $\mathcal{R}$ yields a corresponding quantum mechanical Hamiltonian $\hat{H}$. We rewrite the quantum mechanical operator expression for the anomalies as a path integral on the finite time interval $-\beta \leq t \leq 0$ for a linear or nonlinear sigma model with action S. For given spacetime dimension n, in the limit $\beta \to 0$ only graphs with a finite number of loops on the worldline contribute. In this way the calculation of the anomalies is transformed from a field-theoretical problem to a problem in quantum mechanics. We give details of the derivation of the chiral and gravitational anomalies as first given by Alvarez-Gaumé and Witten, and discuss our own work on trace anomalies. For the former one only needs to evaluate one-loop graphs on the worldline, but for the trace anomalies in two dimensions we need two-loop graphs, and for the trace anomalies in four dimensions we compute three-loop graphs. Here a technical but important problem was settled: using time-slicing or mode regularization, counterterms proportional to the product of two Christoffel symbols were found, but it is incorrect to invoke normal coordinates and to ignore these counterterms. Their expansion produces products of two Riemann curvatures which do contribute at 3 loops to trace anomalies. We obtain complete agreement with the results for these anomalies obtained from other methods. We conclude with a detailed analysis of the gravitational anomalies in 10-dimensional supergravities, both for classical and for exceptional gauge groups.

Twenty years have passed since AGW wrote their renowned article. We believe we have solved all major and minor problems we initially ran into.[3] The quantum mechanical approach to quantum field theory can be applied to more problems than only anomalies. If future work on such problems will profit from the detailed account given in this book, our scientific and geographical Odyssey has come to a good ending.

No book is everywhere totally clear and without misprints. We collect and correct them at the following webpage: http://insti.physics.sunysb.edu/itp/books/anomalies/ and we ask readers who have questions or spot misprints or errors to contact us by email at bastianelli@bo.infn.it and vannieu@insti.physics.sunysb.edu.

We would like to thank our respective universities of Bologna and Stony Brook, and the Istituto Nazionale di Fisica Nucleare (INFN) of Italy, the Teyler Foundation of the Netherlands, and the National Science Foundation (NSF) of the USA for financial support. Over the years part of our work was presented at lecture series, summer schools and conferences in Berlin, Brussels, Leiden, Leuven, Los Angeles, Marseille, Rio de Janeiro and Vienna, and we thank the organizers for these opportunities.

Bologna and Stony Brook, January 2005

[3]Except one problem: a rigorous derivation, based only on quantum mechanical path integrals, of the overall normalization of the gravitational anomaly of self-dual antisymmetric tensor fields, see Chapter 8. We fix this normalization by requiring agreement with bosonization formulas of two-dimensional quantum field theories.

Part I

Path integrals for quantum mechanics in curved space

1

Introduction to path integrals

Path integrals play an important role in modern quantum field theory. One usually first encounters them as useful formal devices to derive Feynman rules. For gauge theories they yield straightforwardly the Ward identities. Namely, if BRST symmetry (the "quantum gauge invariance" discovered by Becchi, Rouet, Stora and Tyutin [14]) holds at the quantum level, certain relations between Green functions can be derived from path integrals, but details of the path integral (for example, the precise form of the measure) are not needed for this purpose.[1] Once the BRST Ward identities for gauge theories have been derived, unitarity and renormalizability can be proven, and at this point one may forget about path integrals if one is only interested in perturbative aspects of quantum field theories. One can compute higher-loop Feynman graphs without ever using path integrals.

However, for nonperturbative aspects, path integrals are essential. The first place where one encounters path integrals in nonperturbative quantum field theory is in the study of instantons and solitons. Here advanced methods based on path integrals have been developed. For example, in the case of instantons the correct measure for integration over their collective coordinates (corresponding to the zero modes) is needed. In particular, for supersymmetric nonabelian gauge theories, there are only contributions from these zero modes, while the contributions from the nonzero modes cancel between bosons and fermions. Another area where the path integral

[1] To prove that the BRST symmetry is free from anomalies, one may either use regularization-free cohomological methods, or one may perform explicit loop graph calculations using a particular regularization scheme. When there are no anomalies, but the regularization scheme does not preserve the BRST symmetry, one can always add local counterterms to the action at each loop level to restore the BRST symmetry. In these manipulations the path integral measure is usually not taken into account.

measure is important is quantum gravity. In particular, in modern studies of quantum gravity based on string theory, the measure is crucial in obtaining the correct correlation functions.

One can compute path integrals at the nonperturbative level by going to Euclidean space, discretizing the path integrals on lattices and using powerful computers. In this book we use a continuum approach. We study a class of simple models which lead to path integrals in which no infinite renormalization is needed, but some individual diagrams are divergent and need be regulated, and subtle issues of regularization and measures can be studied explicitly. These models are the quantum mechanical (one-dimensional) nonlinear sigma models. The one- and two-loop diagrams in these models are power-counting divergent, but the infinities cancel in the sum of diagrams for a given process at a given loop level.

Quantum mechanical (QM) nonlinear sigma models can be described by path integrals and are toy models for realistic path integrals in four dimensions. They describe curved target spaces and contain double-derivative interactions (quantum gravity has also double-derivative interactions). The formalism for path integrals in curved space has been discussed in great generality in several books and reviews [15–26]. In the first half of this book we define the path integrals for these models and discuss various subtleties. However, quantum mechanical nonlinear sigma models can also be used to compute anomalies of realistic four- and higher-dimensional quantum field theories, and this application is thoroughly discussed in the second half of this book. Furthermore, quantum mechanical path integrals can be used to compute correlation functions and effective actions. For references in flat space see [27], and for some work in curved space see [28–30].

The study of path integrals in curved space was pioneered by DeWitt [15]. He first extended to curved space a result of Pauli [16] for the transition element for infinitesimal times which was the product of the exponent of the classical action evaluated for a classical trajectory, times the Van Vleck–Morette determinant [17]. He verified that this transition element satisfied a Schrödinger equation with Hamiltonian $\hat{H} + \frac{1}{12}\hbar^2 R$ ($-\frac{1}{12}\hbar^2 R$ in our conventions for R), where $\hat{H} = \frac{1}{2}\hat{g}^{-1/4}\hat{p}_i\hat{g}^{ij}\hat{g}^{1/2}\hat{p}_j\hat{g}^{-1/4}$. He also claimed that this transition element could be written as a path integral with a modified action, which was the sum of the classical action and a term $+\frac{\hbar^2}{12}R$. The latter term comes from the Van Vleck determinant.[2] His work has led to an enormous literature on this subject, with many authors proposing various ingenuous definitions or approximations of the

[2]There exists some confusion in the literature about the coefficient of R in the action in the path integral for the transition element related to the minimal hamiltonian operator $\hat{H}$ ("the counter term with R"). Initially DeWitt obtained $\frac{1}{6}$ [15]. However, recently in [26] he rectified this to $\frac{1}{8}$, a result with which we agree, at least if one uses the regularization schemes discussed in this book, see eqs. (2.81), (3.73), (4.28) and Appendix B. (Note: some of these schemes have additional noncovariant $\Gamma\Gamma$ terms.)

infinitesimal transition element, and various proposals for iterations which should produce the finite transition amplitude, see for example [31–34].

In Part I of this book we show how to define and compute the transition element for finite times using path integrals. This yields, in particular, the transition element for infinitesimal times in a series expansion. Path integrals are of course just one of many ways of computing the transition element, but for the calculation of anomalies the path integral method is far superior as we hope to demonstrate in this book.

1.1 The simplest case: a particle in flat space

Before considering path integrals in curved space, we first review the simple case of a nonrelativistic particle moving in an n-dimensional flat space and subject to a scalar potential $V(x)$. We are going to derive the path integral from the canonical (operatorial) formulation of quantum mechanics. We will also compute the transition amplitude in the free case (i.e. with vanishing potential), a useful result to compare with when we deal with the more complicated case of curved space.

Thus, let us consider a particle with coordinates x^i, conjugate momenta p_i and mass m. As the quantum Hamiltonian we take

$$H(\hat{x},\hat{p}) = \frac{1}{2m}\hat{p}_i\hat{p}^i + V(\hat{x}) \tag{1.1}$$

where, as usual, hats denote quantum mechanical operators. We are interested in deriving a path integral representation of the transition amplitude

$$T(z,y;\beta) \equiv \langle z|\mathrm{e}^{-\frac{\beta}{\hbar}\hat{H}}|y\rangle \tag{1.2}$$

for the particle to propagate from the point y^i to the point z^i in a Euclidean time β. We use a language appropriate to quantum mechanics ("transition amplitude", etc.) even though we consider a Euclidean approach. The usual quantum mechanics in Minkowskian time is obtained by the substitution $\beta \to it$, which corresponds to the so-called Wick rotation, an analytical continuation in the time coordinate that relates statistical mechanics to quantum mechanics, and vice versa.

We use eigenstates $|x\rangle$ and $|p\rangle$ of the position operator $\hat{x}^i$ and momentum operator $\hat{p}_i$, respectively,

$$\hat{x}^i|x\rangle = x^i|x\rangle, \quad \hat{p}_i|p\rangle = p_i|p\rangle, \tag{1.3}$$

together with the completeness relations

$$I = \int d^n x\, |x\rangle\langle x| = \int d^n p\, |p\rangle\langle p| \tag{1.4}$$

and the scalar products

$$\langle x_1|x_2\rangle = \delta^n(x_1 - x_2), \quad \langle p_1|p_2\rangle = \delta^n(p_1 - p_2), \quad \langle x|p\rangle = \frac{1}{(2\pi\hbar)^{n/2}}\mathrm{e}^{\frac{i}{\hbar}p_i x^i}. \tag{1.5}$$

It is easy to show that the transition amplitude should satisfy the Schrödinger equation (see (2.229) and (2.230))

$$-\hbar \frac{\partial}{\partial \beta} T(z, y; \beta) = H(z) T(z, y; \beta) \tag{1.6}$$

with the boundary condition

$$T(z, y; 0) = \delta^n(z - y) \tag{1.7}$$

where the Hamiltonian in the coordinate representation is, of course, given by

$$H(z) = -\frac{\hbar^2}{2m} \frac{\partial}{\partial z^i} \frac{\partial}{\partial z_i} + V(z). \tag{1.8}$$

A similar equation holds at the point y^i.

The derivation of a path integral representation for the transition amplitude is rather standard. The transition amplitude can be split into N factors

$$\begin{aligned} T(z, y; \beta) &= \langle z | \left(\mathrm{e}^{-\frac{\beta}{\hbar N} \hat{H}} \right)^N | y \rangle = \langle z | \underbrace{\mathrm{e}^{-\frac{\epsilon}{\hbar} \hat{H}} \mathrm{e}^{-\frac{\epsilon}{\hbar} \hat{H}} \cdots \mathrm{e}^{-\frac{\epsilon}{\hbar} \hat{H}}}_{N \text{ times}} | y \rangle \\ &= \int \left(\prod_{k=1}^{N-1} d^n x_k \right) \prod_{k=1}^{N} \langle x_k | \mathrm{e}^{-\frac{\epsilon}{\hbar} \hat{H}} | x_{k-1} \rangle \end{aligned} \tag{1.9}$$

where we have denoted $x_0^i = y^i$, $x_N^i = z^i$, $\epsilon = \beta/N$, and used $N-1$ times the completeness relations with position eigenstates. Then one can use N times the completeness relations with momentum eigenstates and obtain

$$T(z, y; \beta) = \int \left(\prod_{k=1}^{N-1} d^n x_k \right) \left(\prod_{k=1}^{N} d^n p_k \right) \prod_{k=1}^{N} \langle x_k | p_k \rangle \langle p_k | \mathrm{e}^{-\frac{\epsilon}{\hbar} \hat{H}} | x_{k-1} \rangle. \tag{1.10}$$

This is still an exact formula, but we are now going to evaluate it using approximations which are correct in the limit $N \to \infty$ ($\epsilon \to 0$). The key point for deriving the path integral is to evaluate the following matrix element

$$\begin{aligned} \langle p | \mathrm{e}^{-\frac{\epsilon}{\hbar} \hat{H}(\hat{x}, \hat{p})} | x \rangle &= \langle p | \left[1 - \frac{\epsilon}{\hbar} \hat{H}(\hat{x}, \hat{p}) + \cdots \right] | x \rangle \\ &= \langle p | x \rangle - \frac{\epsilon}{\hbar} \langle p | \hat{H}(\hat{x}, \hat{p}) | x \rangle + \cdots \\ &= \langle p | x \rangle \left[1 - \frac{\epsilon}{\hbar} H(x, p) + \cdots \right] \\ &= \langle p | x \rangle \, \mathrm{e}^{-\frac{\epsilon}{\hbar} H(x, p) + \cdots}. \end{aligned} \tag{1.11}$$

The replacement $\langle p | \hat{H}(\hat{x}, \hat{p}) | x \rangle = \langle p | x \rangle \, H(x, p)$ follows from the simple structure of the Hamiltonian in (1.1), which allows to act with the position and momentum operators on the corresponding eigenstates, so that these operators are simply replaced by the corresponding eigenvalues. In this way the Hamiltonian operator $\hat{H}(\hat{x},\hat{p})$ is replaced by the Hamiltonian function $H(x,p)=p^2/2m+V(x)$. These approximations are justified in the limit $N\to\infty$ for many physically interesting potentials (i.e. the "dots" in (1.11) can be neglected in this limit), in which cases a rigorous mathematical proof is also available, and goes under the name of the "Trotter formula" [21]. Finally, using the expression for $\langle x|p\rangle$ given in (1.5), and recalling that $\langle p|x\rangle = \langle x|p\rangle^*$, one obtains

$$\langle x_k|p_k\rangle\langle p_k|\mathrm{e}^{-\frac{\epsilon}{\hbar}\hat{H}}|x_{k-1}\rangle = \frac{1}{(2\pi\hbar)^n}\mathrm{e}^{\frac{i}{\hbar}p_k\cdot(x_k-x_{k-1})-\frac{\epsilon}{\hbar}H(x_{k-1},p_k)} \tag{1.12}$$

which can now be inserted into (1.10). At this point the expression of the transition amplitude does not contain any more operators, and reads as

$$\begin{aligned} T(z,y;\beta) &= \lim_{N\to\infty}\int\left(\prod_{k=1}^{N-1}d^nx_k\right)\left(\prod_{k=1}^{N}\frac{d^np_k}{(2\pi\hbar)^n}\right) \\ &\quad\times\exp\left\{-\frac{\epsilon}{\hbar}\sum_{k=1}^{N}\left[-ip_k\cdot\frac{(x_k-x_{k-1})}{\epsilon}+H(x_{k-1},p_k)\right]\right\} \\ &= \int Dx\,Dp\,\mathrm{e}^{-\frac{1}{\hbar}S[x,p]}. \end{aligned} \tag{1.13}$$

This is the path integral in phase space. We recognize in the exponent a discretization of the classical Euclidean phase space action

$$\begin{aligned} S[x,p] &= \int_0^\beta dt\,[-ip\cdot\dot{x}+H(x,p)] \\ &\to \epsilon\sum_{k=1}^{N}\left[-ip_k\cdot\frac{(x_k-x_{k-1})}{\epsilon}+H(x_{k-1},p_k)\right] \end{aligned} \tag{1.14}$$

where again $\beta=N\epsilon$. The last line in (1.13) is symbolic and indicates a formal sum over paths in phase space weighted by the exponential of minus their classical action.

The configuration space path integral is easily derived by integrating out the momenta in (1.13). Completing squares and using Gaussian integration one obtains

$$\begin{aligned} T(z,y;\beta) &= \lim_{N\to\infty}\int\left(\prod_{k=1}^{N-1}d^nx_k\right)\left(\frac{m}{2\pi\hbar\epsilon}\right)^{nN/2} \\ &\quad\times\exp\left\{-\frac{\epsilon}{\hbar}\sum_{k=1}^{N}\left[\frac{m}{2}\left(\frac{x_k-x_{k-1}}{\epsilon}\right)^2+V(x_{k-1})\right]\right\} \\ &= \int Dx\,\mathrm{e}^{-\frac{1}{\hbar}S[x]}. \end{aligned} \tag{1.15}$$

This is the path integral in configuration space. In the exponent one finds a discretization of the classical Euclidean configuration space action

$$\begin{aligned} S[x] &= \int_0^\beta dt \left[\frac{m}{2}\dot{x}^2 + V(x)\right] \\ &\to \epsilon \sum_{k=1}^{N} \left[\frac{m}{2}\Big(\frac{x_k - x_{k-1}}{\epsilon}\Big)^2 + V(x_{k-1})\right]. \end{aligned} \tag{1.16}$$

Again the last line in (1.15) is symbolic, and indicates a sum over paths in configuration space.

For the case of a vanishing potential, the path integral can be evaluated exactly [45, 46, 21]. Performing successive Gaussian integrations one obtains

$$T(z, y; \beta) = \Big(\frac{m}{2\pi\hbar\beta}\Big)^{n/2} \mathrm{e}^{-m(z-y)^2/2\beta\hbar}. \tag{1.17}$$

This final result is very suggestive. Up to a prefactor, it consists of the exponential of the classical action evaluated on the classical trajectory. This is typical for the cases where the semiclassical approximation is exact. The prefactor can be considered as containing the "one-loop" corrections which make up the full result (thus "semiclassical" = "classical + one-loop").

The preceding approach is called time slicing, and will be applied to nonlinear sigma models (models in curved target space) in Chapter 2. In Chapters 3 and 4 we shall use two other equivalent methods of computing path integrals: mode regularization and dimensional regularization.

We shall actually use a somewhat different way to evaluate path integrals, by decomposing $x^i(t)$ as follows. We expand the continuous paths $x^i(t)$ into a fixed classical "background" part $x^i_{bg}(t)$ plus "quantum fluctuations" $q^i(t)$

$$x^i(t) = x^i_{bg}(t) + q^i(t). \tag{1.18}$$

Here $x^i_{bg}(t)$ is a fixed function: it solves the classical equations of motion and takes into account the boundary conditions ($x^i(0) = y^i$ and $x^i(\beta) = z^i$)

$$x^i_{bg}(t) = y^i + (z^i - y^i)\frac{t}{\beta}, \tag{1.19}$$

while the arbitrary fluctuations $q^i(t)$ vanish at the boundaries. One may interpret $x^i_{bg}(t)$ as the origin and $q^i(t)$ as the coordinates of the "space of paths".

Now one can compute the path integral (1.15) for a vanishing potential

$$T(z, y; \beta) = \int Dx\, \mathrm{e}^{-\frac{1}{\hbar}S[x]} = \int D(x_{bg} + q)\, \mathrm{e}^{-\frac{1}{\hbar}S[x_{bg}+q]}$$

$$= \int Dq\, e^{-\frac{1}{\hbar}(S[x_{bg}]+S[q])} = e^{-\frac{1}{\hbar}S[x_{bg}]} \int Dq\, e^{-\frac{1}{\hbar}S[q]}$$
$$= A e^{-\frac{1}{\hbar}S[x_{bg}]} = A e^{-\frac{m(z-y)^2}{2\beta\hbar}} \tag{1.20}$$

where we have used the translational invariance of the path integral measure $Dx = D(x_{bg} + q) = Dq$ (at the discretized level this is evident from writing $d^n x_k = d^n(x_{k,bg} + q_k) = d^n q_k$) and the fact that in the action there is no term linear in q^i (the action is quadratic in q^i, but the term linear in q^i must also be linear in x_{bg}, but then this term must vanish by the equations of motion). Finally, the constant $A = \int Dq \exp(-\frac{1}{\hbar}S[q])$ is not determined by this method, but it can be fixed by requiring that (1.20) solves the Schrödinger equation (1.6) with the boundary condition in (1.7). The value $A = (m/2\pi\hbar\beta)^{n/2}$ is sometimes called the Feynman measure.

1.2 Quantum mechanical path integrals in curved space require regularization

The path integrals for the quantum mechanical systems we shall discuss have a Hamiltonian $\hat{H}(\hat{x}, \hat{p})$ which is more general than $\hat{T}(\hat{p}) + \hat{V}(\hat{x})$. We shall typically be considering models with a Euclidean Lagrangian of the form $L = \frac{1}{2} g_{ij}(x) \frac{dx^i}{dt} \frac{dx^j}{dt} + i A_i(x) \frac{dx^i}{dt} + V(x)$, where $i, j = 1, \ldots, n$. These systems are one-dimensional quantum field theories with double-derivative interactions, and hence they are not ultraviolet finite by power counting; rather, the one- and two-loop diagrams are divergent as we shall discuss in detail in the next section. The ultraviolet infinities cancel in the sum of diagrams, but one needs to regularize individual diagrams which are divergent. The results of individual diagrams are then regularization-scheme dependent, and also the results for the sum of diagrams are finite but scheme dependent. One must then add finite counterterms which are also scheme dependent, and which must be chosen such that certain physical requirements are satisfied (renormalization conditions). Of course, the final physical answers should be the same, no matter which scheme one uses. Since we shall be working with actions defined on a compact time-interval, there are no infrared divergences. We shall also discuss nonlinear sigma models with fermionic point particles $\psi^a(t)$ with again $a = 1, \ldots, n$. Also one- and two-loop diagrams containing fermions can be power-counting divergent. For applications to chiral and gravitational anomalies the most important cases are the rigidly supersymmetric models, in particular the quantum mechanical models with $N = 1$ and $N = 2$ supersymmetry, but nonsupersymmetric models with or without fermions will also be used as they are needed for applications to trace anomalies.

Quantum mechanical path integrals can be used to compute anomalies of n-dimensional quantum field theories. This was first shown by Alvarez-Gaumé and Witten (AGW) [1, 35, 36], who studied various chiral and gravitational anomalies (see also [37, 38]). Subsequently, Bastianelli and van Nieuwenhuizen [39, 40] extended their approach to trace anomalies. With the formalism developed below one can now, in principle, compute any anomaly, and not only chiral anomalies. In the work of Alvarez-Gaumé and Witten, the chiral anomalies themselves were written directly as a path integral in which the fermions have periodic boundary conditions. Similarly, the trace anomalies lead to path integrals with antiperiodic boundary conditions for the fermions. These are, however, only special cases, and in our approach any Jacobian will lead to a corresponding set of boundary conditions.

Because chiral anomalies have a topological character, one would expect details of the path integral to be unimportant and only one-loop graphs on the worldline to contribute. In fact, in the approach of AGW this is indeed the case.[3] On the other hand, for trace anomalies, which have no topological interpretation, the details of the path integral do matter and higher loops on the worldline contribute. In fact, it was precisely because three-loop calculations of the trace anomaly based on quantum mechanical path integrals initially did not agree with results known from other methods, that we started a detailed study of path integrals for nonlinear sigma models. These discrepancies have been resolved in the meantime, and the resulting formalism is presented in this book.

The reason that we do not encounter infinities in loop calculations for QM nonlinear sigma models is different from a corresponding statement for QM linear sigma models. For a linear sigma model with a kinetic term $\frac{1}{2}\dot{x}^i\dot{x}^i$ on an infinite t-interval, the propagator behaves as $1/k^2$ for large momenta, and vertices from $V(x)$ do not contain derivatives, hence loops $\int dk[\cdots]$ will always be finite. For nonlinear sigma models with $L = \frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j$, propagators still behave like k^{-2} but vertices now behave like k^2 (as in ordinary quantum gravity), hence single loops are linearly divergent by power counting and double loops are logarithmically

[3] Their approach uses a particular linear combination of general coordinate and local Lorentz transformations, and for this symmetry one only needs to evaluate single loops on the worldline. However if one directly computes the anomaly of the Lorentz operator $\gamma^{\mu\nu}\gamma_5$, using the same steps as in the case of the chiral operator γ_5 for gauge fields in flat space, one needs higher loops on the worldline. We discuss this at the end of Section 6.3.

divergent. It is clear by inspection of

$$\langle z|\mathrm{e}^{-\frac{\beta}{\hbar}\hat{H}}|y\rangle = \int_{-\infty}^{\infty} \langle z|\mathrm{e}^{-\frac{\beta}{\hbar}\hat{H}}|p\rangle\langle p|y\rangle\, d^n p \tag{1.21}$$

that no infinities should be present: the matrix element $\langle z|\exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle$ is **finite** and **unambiguous**. Indeed, we could in principle insert a complete set of momentum eigenstates as indicated, and then expand the exponent and move all $\hat{p}$ operators to the right and all $\hat{x}$ operators to the left, taking commutators into account. The integral over $d^n p$ is Gaussian and converges. To any given order in β we would then find a finite and well-defined expression.[4] Hence, also **the path integrals should be finite**.

The mechanism by which loops based on path integrals are finite is different in phase space and configuration space path integrals. In the phase space path integrals the momenta are independent variables and the vertices contained in $H(x,p)$ are without derivatives. (The only derivatives are due to the term $p\dot{x}$, whereas the term $\frac{1}{2}p^2$ is free from derivatives.) The propagators and vertices are nonsingular functions (containing at most step functions but no delta functions) which are integrated over the finite domain $[-\beta, 0]$, hence no infinities arise. (We use the interval $[-\beta, 0]$ instead of $[0, \beta]$, but it is easy to change notation to go from one to the other.) In the configuration space path integrals, on the other hand, there are divergences in individual loops, as we mentioned. The reason for this is that although one still integrates over the finite domain $[-\beta, 0]$, single derivatives of the propagators are discontinuous and double derivatives are divergent (they contain delta functions).

However, since the results of configuration-space path integrals should be the same as those of phase-space path integrals, these infinities should not be there in the first place. The resolution of this paradox is that **configuration-space path integrals contain a new kind of ghost.** These ghosts are needed to exponentiate the factors $(\det g_{ij})^{1/2}$ which are produced when one integrates out the momenta. Historically, the cancellation of divergences at the one-loop level was first found by Lee and Yang [41], who studied nonlinear deformations of harmonic oscillators, and who wrote these determinants as new terms in the action of the form

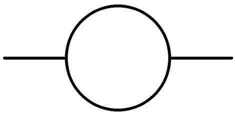
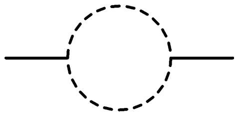
$$\frac{1}{2}\sum_t \ln\det g_{ij}(x(t)) = \frac{1}{2}\delta(0)\int \mathrm{tr}\,\ln\, g_{ij}(x(t))\, dt. \tag{1.22}$$

To obtain the right-hand side one may multiply the left-hand side by $\Delta t/\Delta t$ and replace $1/\Delta t$ by $\delta(0)$ in the continuum limit. For higher loops,

[4] This program is executed in Section 2.5 to order β. For reasons explained there, we count the difference $(z-y)$ as being of order $\beta^{1/2}$.

it is inconvenient to work with $\delta(0)$; rather, we shall use the new ghosts in precisely the same manner as one uses the Faddeev–Popov ghosts in gauge theories: they occur in all possible manners in Feynman diagrams and have their own Feynman rules. These ghosts for quantum mechanical path integrals were first introduced by Bastianelli [39].

In configuration space, loops with ghost particles cancel divergences in corresponding loop graphs without ghost particles. Generically one has

 +  = finite.

However, the fact that the infinities cancel does not mean that the remaining finite parts are unambiguous. One must regularize the divergent graphs, and different regularization schemes can lead to different finite parts, as is well known from field theory. Since our actions are of the form $\int_{-\beta}^{0} L\,dt$, we are dealing with one-dimensional quantum field theories in a finite "spacetime". If one is not dealing with a circle, translational invariance is broken, and propagators depend on t and s, not only on $t-s$. In configuration space the propagators contain singularities. For example, the propagator for a free quantum particle $q(t)$ corresponding to $L=\frac{1}{2}\dot{q}^2$ with boundary conditions $q(-\beta)=q(0) = 0$ is proportional to $\Delta(\sigma,\tau)$, where $\sigma = s/\beta$ and $\tau = t/\beta$, with $-\beta \leq s,t \leq 0$ and $-1 \leq \sigma,\tau \leq 0$

$$\langle q(\sigma)q(\tau)\rangle \approx \Delta(\sigma,\tau) = \sigma(\tau+1)\theta(\sigma-\tau) + \tau(\sigma+1)\theta(\tau-\sigma). \quad (1.23)$$

It is easy to check that $\partial_\sigma^2\Delta(\sigma,\tau) = \delta(\sigma-\tau)$ and $\Delta(\sigma,\tau) = 0$ at $\sigma = -1, 0$ and $\tau = -1, 0$ (use $\partial_\sigma\Delta(\sigma,\tau) = \tau + \theta(\sigma-\tau)$).

It is clear that Wick contractions of $\dot{q}(\sigma)$ with $q(\tau)$ will contain a factor of $\theta(\sigma-\tau)$, and $\dot{q}(\sigma)$ with $\dot{q}(\tau)$ a factor $\delta(\sigma-\tau)$. Also the propagators for the ghosts contain factors of $\delta(\sigma-\tau)$. Thus one needs a consistent, unambiguous and workable regularization scheme for products of the distributions $\delta(\sigma-\tau)$ and $\theta(\sigma-\tau)$. In mathematics the products of distributions are ill-defined [42]. Thus, it comes as no surprise that in physics different regularization schemes give different answers for such integrals. For example, consider the following two familiar ways of evaluating the product of distributions: smoothing of distributions and using Fourier transforms. Suppose one is required to evaluate

$$I = \int_{-1}^{0}\int_{-1}^{0} \delta(\sigma-\tau)\theta(\sigma-\tau)\theta(\sigma-\tau)\,d\sigma\,d\tau. \quad (1.24)$$

Smoothing of a distribution can be achieved by approximating $\delta(\sigma-\tau)$ and $\theta(\sigma-\tau)$ by some smooth functions and requiring that at the regulated level one still has the relation $\delta(\sigma-\tau) = \frac{\partial}{\partial\sigma}\theta(\sigma-\tau)$. One then obtains

$I = \frac{1}{3}\int_{-1}^{0}\int_{-1}^{0}\frac{\partial}{\partial\sigma}[\theta(\sigma-\tau)]^3\,d\sigma\,d\tau = \frac{1}{3}$. On the other hand, if one were to interpret the delta function $\delta(\sigma-\tau)$ to mean that one should evaluate the function $\theta(\sigma-\tau)^2$ at $\sigma=\tau$ one obtains $\frac{1}{4}$. One could also decide to use the representations

$$\begin{aligned}\delta(\sigma-\tau) &= \int_{-\infty}^{\infty}\frac{d\lambda}{2\pi}\,\mathrm{e}^{i\lambda(\sigma-\tau)}\\ \theta(\sigma-\tau) &= \int_{-\infty}^{\infty}\frac{d\lambda}{2\pi i}\frac{\mathrm{e}^{i\lambda(\sigma-\tau)}}{\lambda-i\epsilon} \qquad \text{with } \;\epsilon>0.\end{aligned} \tag{1.25}$$

Formally, $\partial_\sigma\theta(\sigma-\tau) = \delta(\sigma-\tau) - \epsilon\,\theta(\sigma-\tau)$, and upon taking the limit ϵ tending to zero one would again expect to obtain the value $\frac{1}{3}$ for I. However, if one first integrates over σ and τ, one finds

$$I = \left[\int_{-\infty}^{\infty}\frac{dy}{2\pi}\frac{(2-2\cos y)}{y^2}\right]\left(\int_{-\infty}^{\infty}\frac{d\lambda}{2\pi i}\frac{1}{\lambda-i\epsilon}\right)^2. \tag{1.26}$$

Depending on the prescription used to evaluate the last integral, one could obtain different results. Clearly, using different methods to evaluate I leads to different answers. Without further specifications, integrals such as I are indeed ambiguous and make no sense.

In the applications we are going to discuss, we sometimes choose a regularization scheme that reduces the path integral to a finite-dimensional integral. For example, for time slicing one chooses a finite set of intermediate points, and for mode regularization one begins with a finite number of modes for each one-dimensional field. Another scheme we use is dimensional regularization: here one regulates the various Feynman diagrams by moving away from $d=1$ dimensions, and performing partial integrations which make the integral manifestly finite at $d=1$. Afterwards one returns to $d=1$ and computes the values of these finite integrals. One omits boundary terms in the extra dimensions; this can be justified by noting that there are factors of $\mathrm{e}^{i\mathbf{k}(\mathbf{t}-\mathbf{s})}$ in the propagators due to translation invariance in the extra D dimensions. They yield the Dirac delta functions $\delta^D(\mathbf{k}_1+\mathbf{k}_2+\cdots+\mathbf{k}_n)$ upon integration over the extra space coordinates. A derivative with respect to the extra space coordinate which yields, for example, a factor $\mathbf{k}_1$ can be replaced by $-\mathbf{k}_2-\mathbf{k}_3-\cdots-\mathbf{k}_n$ due to the presence of the delta function, and this replacement is equivalent to a partial integration without boundary terms. These are formal manipulations which should be viewed as specifying the regularization scheme.

In time slicing we find the value $I=\frac{1}{4}$ for (1.24): in fact, as we shall see, in this case the delta function is a Kronecker delta which gives the product of the θ functions at the point $\sigma=\tau$. In mode regularization, one finds $I=\frac{1}{3}$ because now $\delta(\sigma-\tau)$ is indeed $\partial_\sigma\theta(\sigma-\tau)$ at the regulated level. In dimensional regularization one must first decide which derivatives are contracted with which derivatives in $D+1$ dimensions (for example, $({}_{\mu}\Delta_{\nu})({}_{\mu}\Delta)(\Delta_{\nu})$). This follows from the form of the action in $D+1$ dimensions. Then one applies the usual manipulations of dimensional regularization in $D+1$ dimensions until one reaches a convergent integral which can directly be evaluated in one dimension.[5]

As we have seen, different regularization schemes lead to finite well-defined results for a given diagram which are in general different, but there are also ambiguities in the vertices: the finite one- and two-loop counterterms have not been fixed. The physical requirement that the theory be based on a given quantum Hamiltonian removes the ambiguities in the counterterms: for time slicing Weyl ordering of $\hat{H}$ directly produces the counterterms, while for the other schemes the requirement that the transition element satisfies the Schrödinger equation with a given Hamiltonian $\hat{H}$ fixes the counterterms. Thus in all of these schemes the renormalization condition is that the transition element should be derived from the same particular Hamiltonian $\hat{H}$.

The first scheme, time slicing (TS), has the advantage that one can deduce it directly from the operatorial formalism of quantum mechanics. This regularization can be considered to be equivalent to lattice regularization of standard quantum field theories. It is the approach followed by Dirac and Feynman. One must specify the Hamiltonian $\hat{H}$ with an a priori fixed operator ordering; this ordering corresponds to the renormalization conditions in this approach. All further steps are finite and unambiguous. This approach breaks general coordinate invariance in target space which is then recovered by a specific finite counterterm ΔV_{TS} in the action of the path integral. (To simplify the notation, we denote these counterterms in later sections by V_{TS} instead of ΔV_{TS}.) This counterterm also follows unambiguously from the initial Hamiltonian and is itself not coordinate invariant either. However, if the initial Hamiltonian is general coordinate invariant (as an operator, see Section 2.5) then the final result (the transition element) will also be general coordinate invariant.

The second scheme, mode regularization (MR), will be constructed directly without referring to the operatorial formalism. It can be thought

[5]For an example of an integral where dimensional regularization is applied, consider

$$\begin{aligned} J &= \int_{-1}^{0} d\sigma \int_{-1}^{0} d\tau \, ({}^{\bullet}\!\Delta^{\bullet}) \, ({}^{\bullet}\!\Delta) \, (\Delta^{\bullet}) \\ &= \int_{-1}^{0} d\sigma \int_{-1}^{0} d\tau \, [1-\delta(\sigma-\tau)] \, [\tau+\theta(\sigma-\tau)] \, [\sigma+\theta(\tau-\sigma)] \end{aligned} \tag{1.27}$$

where dots on the left and right denote derivatives with respect to the first and second variable. One finds $J=-\frac{1}{6}$ for time slicing, see (2.270). Furthermore, $J=-\frac{1}{12}$ for mode regularization, see (3.82). In dimensional regularization one rewrites the integrand as $({}_{\mu}\Delta_{\nu})({}_{\mu}\Delta)(\Delta_{\nu})$ and one finds $J=-\frac{1}{24}$, see (4.24).

of as the equivalent of momentum cut-off in QFT.[6] It is close to the intuitive notion of path integrals, that are meant to give a global picture of the quantum phenomena by summing over entire paths (while one may view the time discretization method as being closer to the local picture of the differential Schrödinger equation, since one imagines the particle propagating by small time steps). Mode regularization gives, in principle, a nonperturbative definition of path integrals in that one does not have to expand the exponential of the interaction part of the action. However, this regularization also breaks general coordinate invariance, and one needs a different finite noncovariant counterterm ΔV_{MR} to recover it.

Finally, the third regularization scheme, dimensional regularization (DR), is based on the dimensional continuation of the ambiguous integrals appearing in the loop expansion. It is inherently a perturbative regularization, but it is the optimal one for perturbative computations in the following sense. It does not break general coordinate invariance at intermediate stages and the counterterm ΔV_{DR} is Einstein and local Lorentz invariant.

All of these different regularization schemes will be presented in separate chapters. Since our derivation of the path integrals contains several steps, each requiring a detailed discussion, we have decided to put all of these special discussions in separate sections after the main derivation. This has the advantage that one can read each section independently. The structure of our discussions is summarized by the flow chart in Fig. 1.1.

We shall first discuss time slicing, the lower part of the flow chart. This discussion is first given for bosonic systems with $x^i(t)$ and afterwards for systems with fermions. In the bosonic case, we first construct discretized phase-space path integrals, then discretized configuration-space path integrals, to be followed by the continuous configuration-space path integrals, and finally the continuous phase-space path integrals. We show that after Weyl ordering of the Hamiltonian operator $\hat{H}(\hat{x}, \hat{p})$ one obtains a path integral with a midpoint rule (Berezin's theorem). Then we repeat the analysis for fermions.

Next, we consider mode number regularization (the upper part of the flow chart). Here we define the path integrals *ab initio* in configuration space with the naive classical action and a counterterm ΔV_{MR} which is at first left unspecified. We then proceed to fix ΔV_{MR} by imposing the

[6] In more complicated cases, such as path integrals in spaces with a topological vacuum (for example, the kink background in Euclidean quantum mechanics), the mode regularization scheme and the momentum regularization scheme with a sharp cut-off are not equivalent (for example, they give different answers for the quantum mass of the kink). However, if one replaces the sharp energy cut-off by a smooth cut-off, those schemes become equivalent [43]. We do not consider such topologically nontrivial backgrounds.

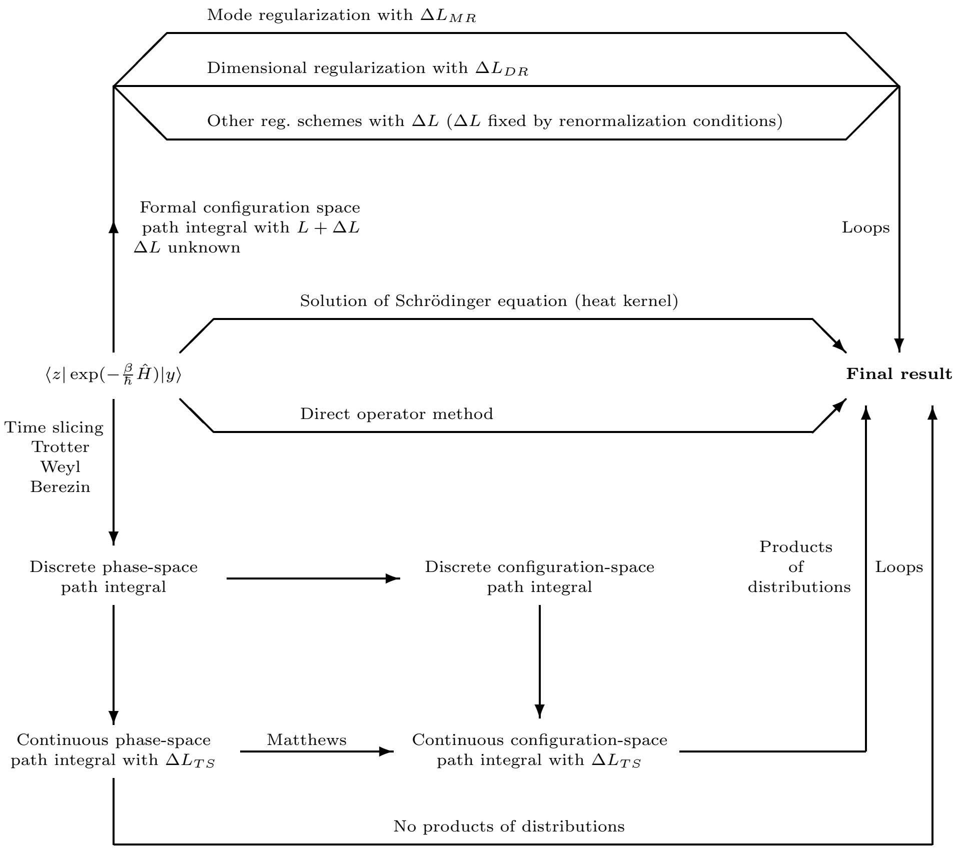


Fig. 1.1 Flow chart of Part I of the book.

requirement that the Schrödinger equation be satisfied with a specific Hamiltonian $\hat{H}$. Having fixed ΔV_{MR}, one can proceed to compute loops at any desired order.

Finally, we present dimensional regularization along similar lines. The counterterm 5 is now denoted by ΔV_{MR}. Each section can be read independently of the previous ones.

In all three cases we define the theory by the Hamiltonian $\hat{H}$ and then construct the path integrals and Feynman rules which correspond to $\hat{H}$. The choice of $\hat{H}$ defines the physical theory. One may be prejudiced about which $\hat{H}$ makes physical sense (for example, many physicists require that $\hat{H}$ preserves general coordinate invariance), but in our work one does not have to restrict oneself to these particular $\hat{H}$. Any $\hat{H}$, no matter how unphysical, leads to a corresponding path integral and corresponding Feynman rules. We repeat that the path integral and Feynman rules depend on the regularization scheme chosen, but the final result for the transition element and correlation functions are the same in each scheme.

In the time-slicing approach we shall solve some of the following basic problems: **given** a Hamiltonian operator $\hat{H}(\hat{x},\hat{p})$ with arbitrary but a-priori fixed operator ordering, find a path integral expression for the matrix element[7] $\langle z|\exp(-\frac{\beta}{\hbar}\hat{H})|y\rangle$. (The bra $\langle z|$ and ket $|y\rangle$ are eigenstates of the position operator $\hat{x}^i$ with eigenvalues z^i and y^i, respectively. For fermions we shall use coherent states as bra and ket.) One way to obtain such a path integral representation is, as we have discussed, to insert complete sets of x- and p-eigenstates (namely N sets of p-eigenstates and $N-1$ sets of x-eigenstates), in the manner first studied by Dirac [44] and Feynman [45, 46], and leads to the following result:

$$\langle z|\mathrm{e}^{-\frac{\beta}{\hbar}\hat{H}}|y\rangle \approx \int Dx\,Dp\;\mathrm{e}^{-\frac{1}{\hbar}\int_{-\beta}^{0}L\,dt} \tag{1.28}$$

where $L=-ip_i(t)\frac{dx^i}{dt}+H(x,p)$ in our Euclidean phase-space approach. However, several questions arise if one studies (1.28).

(i) What is the precise relation between the operator $\hat{H}(\hat{x},\hat{p})$ and the function $H(x,p)$? Different operator orderings of $\hat{H}$ are expected to lead to different functions $H(x,p)$. Are there special orderings of $\hat{H}$ for which $H(x,p)$ is particularly simple? And if so, are these special orderings consistent with general coordinate invariance?

(ii) What is the precise meaning of the measures $Dx\,Dp$ in phase space and Dx in configuration space in theories with external gravitational fields? Is there a normalization constant in front of the path integral? Does the measure depend on the metric? The measure $Dx\,Dp = \prod_{i=1}^{N-1}dx^i\prod_{i=1}^{N}dp_i$ is not a canonically invariant measure (not equal to the Liouville measure) because there is one more dp than dx. Does this have implications?

(iii) What are the boundary conditions one must impose on the paths over which one sums? One expects that all paths must satisfy the Dirichlet boundary conditions $x^i(-\beta)=y^i$ and $x^i(0)=z^i$, but are there also boundary conditions on $p_i(t)$? Is it possible to consider classical paths in phase space which satisfy boundary conditions both at $t=-\beta$ and at $t=0$?

(iv) How does one compute such path integrals in practice? Performing the integrations over dx^i and dp_i for finite N and then taking

[7] The results in this book are for Euclidean path integrals with $L=-ip\dot{x}+H(x,p)$. However, they hold equally well in Minkowskian time, at least at the level of perturbation theory, with operators $\exp(-\frac{i}{\hbar}\hat{H}t)$ and path integrals with $\exp(\frac{i}{\hbar}\int L_M\,dt)$, where L_M is the Lagrangian in Minkowskian time, related to the positive-definite Euclidean Lagrangian L by an inverse a Wick rotation ($t\to+it$) and an extra overall minus sign.

the limit $N \to \infty$ is in practice hardly possible. Is there a simpler scheme by which one can compute the path integral loop-by-loop, and what are the precise Feynman rules for such an approach? Does the measure contribute to the Feynman rules?

(v) It is often advantageous to use a background formalism and to decompose bosonic fields $x(t)$ into background fields $x_{bg}(t)$ and quantum fluctuations $q(t)$. One can then require that $x_{bg}(t)$ satisfies the boundary conditions so that $q(t)$ vanishes at the endpoints. However, inspired by string theory, one can also compactify the interval $[-\beta, 0]$ to a circle, and then decompose $x(t)$ into a center of mass coordinate x_c and quantum fluctuations about it. What is the relation between both approaches?

(vi) When one is dealing with $N = 1$ supersymmetric systems, one has real (Majorana) particles $\psi^a(t)$. How does one define the Hilbert space in which $\hat{H}$ is supposed to act? Must one also impose an initial and a final condition on $\psi^a(t)$, even though the Dirac equation is only linear in (time) derivatives? We shall introduce operators $\hat{\psi}^a$ and $\hat{\psi}^\dagger_a$ and construct coherent states by contracting them with Grassmann variables $\bar{\eta}_a$ and η^a. If $\hat{\psi}^\dagger_a$ is the hermitian conjugate of $\hat{\psi}^a$, then is $\bar{\eta}_a$ the complex conjugate of η^a?

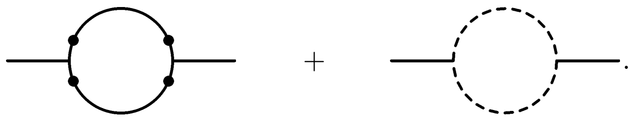
(vii) In certain applications, for example the calculation of trace anomalies, one must evaluate path integrals over fermions with antiperiodic boundary conditions. In the work of AGW the chiral anomalies were expressed in terms of integrals over the zero modes of the fermions. For antiperiodic boundary conditions there are no zero modes. How then should one compute trace anomalies from quantum mechanics?

These are some of the questions which come to mind if one contemplates (1.28) for some time. In the literature one can find discussions of some of these questions [47, 48], but we have made an effort to give a consistent discussion of all of them. Answers to these questions can be found in Chapter 8. New material in this book is an exact evaluation of all discretized expressions in the TS scheme as well as the derivation of the MR and DR schemes in curved space.

1.3 Power counting and divergences

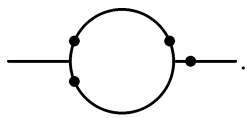
Let us now give some examples of divergent graphs. The precise form of the vertices is given later, in (2.85), but for the discussion in this section we only need the qualitative features of the action. The propagators we are going to use later in this book are not of the simple form $1/k^2$ for a scalar, rather they have the form $\sum_{n=1}^{\infty}(2/\pi^2 n^2)\sin(\pi n\tau)\sin(\pi n\sigma)$ due to boundary conditions. (Even the propagator for time slicing can be cast into this form by Fourier transformation.) However, for ultraviolet divergences the sum of $1/n^2$ is equivalent to an integral over $1/k^2$, and in this section we analyze Feynman graphs with $1/k^2$ propagators. The physical justification is that ultraviolet divergences should not feel the boundaries.

Consider first the self-energy. At the one-loop level the self-energy without external derivatives receives contributions from the following two graphs



We used the vertices from $\frac{1}{2}[g_{ij}(x) - g_{ij}(z)](\dot{q}^i\dot{q}^j + a^i a^j + b^i c^j)$, where $x^i = x^i(\tau) = z^i + q^i(\tau)$ and $z^i = x^i(0)$. Dots indicate derivatives and dashed lines denote the ghost particles a^i, b^i, c^i. The two divergences are proportional to $\delta^2(\sigma - \tau)$ and cancel, but there are ambiguities in the finite part which must be fixed using suitable conditions. (In quantum field theories with divergences one calls these conditions "renormalization conditions".) In momentum space both graphs are linearly divergent, but the linear divergence $\int dk$ cancels in the sums of the graphs and the two remaining logarithmic divergences $\int dk\, k/k^2$ cancel by symmetric integration leaving in general a finite but ambiguous result.

Another example is the self-energy with one external derivative



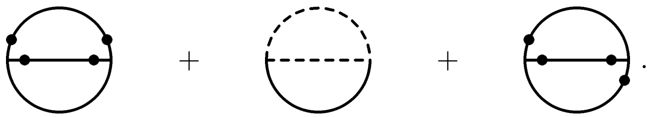
This graph is logarithmically divergent, $\int dk\, k^3/(k^2)^2$, but using symmetric integration it again leaves a finite but ambiguous part.

All three regularization schemes give the same answer for all one-loop graphs, so the one-loop counterterms are the same; in fact, there are no one-loop counterterms at all in any of the schemes if one starts with an Einstein-invariant Hamiltonian.[8]

At the two-loop level, there are similar cancellations and ambiguities. Consider the following vacuum graphs (vacuum graphs will play an

[8]If one were to use the Einstein-noninvariant Hamiltonian $g^{1/4-\alpha}\hat{p}_i\sqrt{g}g^{ij}\hat{p}_j g^{1/4+\alpha}$, one would obtain in the TS scheme a one-loop counterterm proportional to $\hbar p_i g^{ij}\partial_j \ln g$ in phase space or $\hbar\dot{x}^i\partial_i \ln g$ in configuration space (see Appendix B).

important role in the applications to anomalies)



Again the infinities in the upper loop of the first two graphs cancel, but the finite part is ambiguous. The last graph is logarithmically divergent by power counting, and also the two subdivergences are logarithmically divergent by power counting, but actual calculation shows that it is finite but ambiguous (the leading singularities are of the form $\int \frac{dk\,k}{k^2}$ and cancel due to symmetric integration). The sum of the first two graphs yields $(\frac{1}{4}, \frac{1}{4}, \frac{1}{8})$ in TS, MR and DR, respectively, while the last graph yields $(-\frac{1}{6}, -\frac{1}{12}, -\frac{1}{24})$. This explicitly proves that the results for power counting logarithmically divergent graphs are ambiguous, even though the divergences cancel.

It is possible to use standard power-counting methods as used in ordinary quantum field theory to determine all possibly ultraviolet-divergent graphs. Let us interpret our quantum mechanical nonlinear sigma model as a particular QFT in one Euclidean time dimension. We consider a toy model of the type

$$S = \int dt \left[\frac{1}{2} g(\phi)\dot{\phi}\dot{\phi} + A(\phi)\dot{\phi} + V(\phi) \right] \tag{1.29}$$

where the functions $g(\phi), A(\phi)$ and $V(\phi)$ describe the various couplings. For simplicity we omit the indices i and j.

The choice $g(\phi) = 1$, $A(\phi) = 0$ and $V(\phi) = \frac{1}{2}m^2\phi^2$ reproduces a free massive theory, namely a harmonic oscillator of "mass" (frequency) m. The action is dimensionless and the Lagrangian then has the dimension of a mass. From this one deduces that the field ϕ has mass dimension $M^{-1/2}$. Next, let us consider general interactions and expand them in Taylor series

$$V(\phi) = \sum_{n=0}^{\infty} V_n \phi^n, \quad A(\phi) = \sum_{n=0}^{\infty} A_{n+1}\phi^n, \quad g(\phi) = \sum_{n=0}^{\infty} g_{n+2}\phi^n. \tag{1.30}$$

These expansions define the coupling constants V_n, A_n and g_n. We easily deduce the following mass dimensions for such couplings:

$$[V_n] = M^{n/2+1}; \quad [A_n] = M^{n/2}; \quad [g_n] = M^{n/2-1}. \tag{1.31}$$

The interactions correspond to the terms with $n \geq 3$ in (1.31), so all coupling constants have positive mass dimensions. This implies that the theory is super-renormalizable. Namely, from a certain loop level onwards,