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Automorphic Forms and L-Functions for the Group GL(*n*,R)

DORIAN GOLDFELD

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AUTOMORPHIC FORMS AND L-FUNCTIONS FOR THE GROUP *GL* (n, R)

L-functions associated with automorphic forms encode all classical number theoretic information. They are akin to elementary particles in physics. This book provides an entirely self-contained introduction to the theory of L-functions in a style accessible to graduate students with a basic knowledge of classical analysis, complex variable theory, and algebra. Also within the volume are many new results not yet found in the literature. The exposition provides complete detailed proofs of results in an easy-to-read format using many examples and without the need to know and remember many complex definitions. The main themes of the book are first worked out for GL(2,R) and GL(3,R), and then for the general case of GL(n,R). In an appendix to the book, a set of *Mathematica*® functions is presented, designed to allow the reader to explore the theory from a computational point of view.

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Automorphic Forms and L-Functions for the Group $GL(n, \mathbb{R})$

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Dedicated to Ada, Dahlia, and Iris

Contents

Introduction

The theory of automorphic forms and L-functions for the group of $n \times n$ invertible real matrices (denoted $GL(n, \mathbb{R})$) with $n \geq 3$ is a relatively new subject. The current literature is rife with 150+ page papers requiring knowledge of a large breadth of modern mathematics making it difficult for a novice to begin working in the subject. The main aim of this book is to provide an essentially self-contained introduction to the subject that can be read by someone with a mathematical background consisting only of classical analysis, complex variable theory, and basic algebra – groups, rings, fields. Preparation in selected topics from advanced linear algebra (such as wedge products) and from the theory of differential forms would be helpful, but is not strictly necessary for a successful reading of the text. Any Lie or representation theory required is developed from first principles.

This is a low definition text which means that it is not necessary for the reader to memorize a large number of definitions. While there are many definitions, they are repeated over and over again; in fact, the book is designed so that a reader can open to almost any page and understand the material at hand without having to backtrack and awkwardly hunt for definitions of symbols and terms.

The philosophy of the exposition is to demonstrate the theory by simple, fully worked out examples. Thus, the book is restricted to the action of the discrete group $SL(n, \mathbb{Z})$ (the group of invertible $n \times n$ matrices with integer coefficients) acting on $GL(n, \mathbb{R})$. The main themes are first developed for $SL(2, \mathbb{Z})$ then repeated again for $SL(3, \mathbb{Z})$, and yet again repeated in the more general case of $SL(n, \mathbb{Z})$ with $n \geq 2$ arbitrary. All of the proofs are carefully worked out over the real numbers $\mathbb{R}$, but the knowledgeable reader will see that the proofs will generalize to any local field. In line with the philosophy of understanding by simple example, we have avoided the use of adeles, and as much as possible the theory of representations of Lie groups. This very explicit language appears particularly useful for analytic number theory where precise growth estimates of L-functions and automorphic forms play a major role.

The theory of L-functions and automorphic forms is an old subject with roots going back to Gauss, Dirichlet, and Riemann. An L-function is a Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where the coefficients $a_n,\ n = 1, 2, \ldots,$ are interesting number theoretic functions. A simple example is where a_n is the number of representations of n as a sum of two squares. If we knew a lot about this series as an analytic function of s then we would obtain deep knowledge about the statistical distribution of the values of a_n. An automorphic form is a function that satisfies a certain differential equation and also satisfies a group of periodicity relations. An example is given by the exponential function $e^{2\pi i x}$ which is periodic (i.e., it has the same value if we transform $x \to x + 1$) and it satisfies the differential equation $\frac{d^2}{dx^2} e^{2\pi i x} = -4\pi^2 e^{2\pi i x}$. In this example the group of periodicity relations is just the infinite additive group of integers, denoted $\mathbb{Z}$. Remarkably, a vast theory has been developed exposing the relationship between L-functions and automorphic forms associated to various infinite dimensional Lie groups such as $GL(n, \mathbb{R})$.

The choice of material covered is very much guided by the beautiful paper (Jacquet, 1981), titled *Dirichlet series for the group* $GL(n)$, a presentation of which I heard in person in Bombay, 1979, where a classical outline of the theory of L-functions for the group $GL(n, \mathbb{R})$ is presented, but without any proofs. Our aim has been to fill in the gaps and to give detailed proofs. Another motivating factor has been the grand vision of Langlands' philosophy wherein L-functions are akin to elementary particles which can be combined in the same way as one combines representations of Lie groups. The entire book builds upon this underlying hidden theme which then explodes in the last chapter.

In the appendix a set of Mathematica functions is presented. These have been designed to assist the reader to explore many of the concepts and results contained in the chapters that go before. The software can be downloaded by going to the website given in the appendix.

This book could not have been written without the help I have received from many people. I am particularly grateful to Qiao Zhang for his painstaking reading of the entire manuscript. Hervé Jacquet, Daniel Bump, and Adrian Diaconu have provided invaluable help to me in clarifying many points in the theory. I would also like to express my deep gratitude to Xiaoqing Li, Elon Lindenstrauss, Meera Thillainatesan, and Akshay Venkatesh for allowing me to include their original material as sections in the text. I would like to especially thank

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Dorian Goldfeld

1
Discrete group actions

The genesis of analytic number theory formally began with the epoch making memoir of Riemann (1859) where he introduced the zeta function,

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \quad (\Re(s) > 1),$$

and obtained its meromorphic continuation and functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \qquad \Gamma(s) = \int_0^{\infty} e^{-u}u^s\,\frac{du}{u}.$$

Riemann showed that the Euler product representation

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

together with precise knowledge of the analytic behavior of $\zeta(s)$ could be used to obtain deep information on the distribution of prime numbers.

One of Riemann's original proofs of the functional equation is based on the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(ny) = y^{-1}\sum_{n\in\mathbb{Z}} \hat{f}(ny^{-1}),$$

where f is a function with rapid decay as $y \to \infty$ and

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ity}\,dt,$$

is the Fourier transform of f. This is proved by expanding the periodic function

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$$

in a Fourier series. If f is an even function, the Poisson summation formula may be rewritten as

$$\sum_{n=1}^{\infty} f(ny^{-1}) = y \sum_{n=1}^{\infty} \hat{f}(ny) - \frac{1}{2}(y\hat{f}(0) - f(0)),$$

from which it follows that for $\Re(s) > 1$,

$$\begin{aligned}
\zeta(s) \int_0^\infty f(y) y^s \, \frac{dy}{y} &= \int_0^\infty \sum_{n=1}^{\infty} f(ny) y^s \, \frac{dy}{y} \\
&= \int_1^\infty \sum_{n=1}^{\infty} \left(f(ny) y^s + f(ny^{-1}) y^{-s} \right) \frac{dy}{y} \\
&= \int_1^\infty \sum_{n=1}^{\infty} \left(f(ny) y^s + \hat{f}(ny) y^{1-s} \right) \frac{dy}{y} - \frac{1}{2} \left(\frac{f(0)}{s} + \frac{\hat{f}(0)}{1-s} \right).
\end{aligned}$$

If $f(y)$ and $\hat{f}(y)$ have sufficient decay as $y \to \infty$, then the integral above converges absolutely for all complex s and, therefore, defines an entire function of s. Let

$$\tilde{f}(s) = \int_0^\infty f(y) y^s \, \frac{dy}{y}$$

denote the Mellin transform of f, then we see from the above integral representation and the fact that $\hat{\hat{f}}(y) = f(-y) = f(y)$ (for an even function f) that

$$\zeta(s)\tilde{f}(s) = \zeta(1-s)\tilde{\hat{f}}\,(1-s).$$

Choosing $f(y) = e^{-\pi y^2}$, a function with the property that it is invariant under Fourier transform, we obtain Riemann's original form of the functional equation. This idea of introducing an arbitrary test function f in the proof of the functional equation first appeared in Tate's thesis (Tate, 1950).

A more profound understanding of the above proof did not emerge until much later. If we choose $f(y) = e^{-\pi y^2}$ in the Poisson summation formula, then since $\hat{f}(y) = f(y)$, one observes that for $y > 0$,

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 y} = \frac{1}{\sqrt{y}} \sum_{n=-\infty}^{\infty} e^{-\pi n^2 / y}.$$

This identity is at the heart of the functional equation of the Riemann zeta function, and is a known transformation formula for Jacobi's theta function

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z},$$

where $z = x + iy$ with $x \in \mathbb{R}$ and $y > 0$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with integer coefficients a, b, c, d satisfiying $ad - bc = 1$, $c \equiv 0 \pmod 4$, $c \neq 0$, then the Poisson summation formula can be used to obtain the more general transformation formula (Shimura, 1973)

$$\theta\left(\frac{az+b}{cz+d}\right) = \epsilon_d^{-1}\chi_c(d)(cz+d)^{\frac{1}{2}}\theta(z).$$

Here χ_c is the primitive character of order ≤ 2 corresponding to the field extension $\mathbb{Q}(c^{\frac{1}{2}})/\mathbb{Q}$,

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv -1 \pmod 4, \end{cases}$$

and $(cz+d)^{\frac{1}{2}}$ is the "principal determination" of the square root of $cz + d$, i.e., the one whose real part is > 0.

It is now well understood that underlying the functional equation of the Riemann zeta function are the above transformation formulae for $\theta(z)$. These transformation formulae are induced from the action of a group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on the upper half-plane $\mathfrak{h} = \{x + iy \mid x \in \mathbb{R},\, y > 0\}$ given by

$$z \mapsto \frac{az+b}{cz+d}.$$

The concept of a group acting on a topological space appears to be absolutely fundamental in analytic number theory and should be the starting point for any serious investigations.

1.1 Action of a group on a topological space

Definition 1.1.1 *Given a topological space X and a group G, we say that G* **acts continuously** *on X (on the left) if there exists a map $\circ : G \to \mathrm{Func}(X \to X)$ (functions from X to X), $g \mapsto g\circ$ which satisfies:*

- $x \mapsto g \circ x$ *is a continuous function of x for all $g \in G$;*
- $g \circ (g' \circ x) = (g \cdot g') \circ x$, *for all $g, g' \in G, x \in X$ where $\cdot$ denotes the internal operation in the group G;*
- $e \circ x = x$, *for all $x \in X$ and e = identity element in G.*

Example 1.1.2 Let G denote the additive group of integers $\mathbb{Z}$. Then it is easy to verify that the group $\mathbb{Z}$ acts continuously on the real numbers $\mathbb{R}$ with group action $\circ$ defined by

$$n \circ x := n + x,$$

for all $n \in \mathbb{Z}$, $x \in \mathbb{R}$. In this case $e = 0$.

Example 1.1.3 Let $G = GL(2, \mathbb{R})^+$ denote the group of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$ and determinant $ad - bc > 0$. Let

$$\mathfrak{h} := \left\{ x + iy \,\middle|\, x \in \mathbb{R},\ y > 0 \right\}$$

denote the upper half-plane. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})^+$ and $z \in \mathfrak{h}$ define:

$$g \circ z := \frac{az + b}{cz + d}.$$

Since

$$\frac{az + b}{cz + d} = \frac{ac|z|^2 + (ad + bc)x + bd}{|cz + d|^2} + i \cdot \frac{(ad - bc) \cdot y}{|cz + d|^2}$$

it immediately follows that $g \circ z \in \mathfrak{h}$. We leave as an exercise to the reader, the verification that $\circ$ satisfies the additional axioms of a continuous action. One usually extends this action to the larger space $\mathfrak{h}^* = \mathfrak{h} \cup \{\infty\}$, by defining

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \infty = \begin{cases} a/c & \text{if } c \neq 0, \\ \infty & \text{if } c = 0. \end{cases}$$

Assume that a group G acts continously on a topological space X. Two elements $x_1, x_2 \in X$ are said to be equivalent (mod G) if there exists $g \in G$ such that $x_2 = g \circ x_1$. We define

$$Gx := \left\{ g \circ x \,\middle|\, g \in G \right\}$$

to be the equivalence class or orbit of x, and let $G \backslash X$ denote the set of equivalence classes.

Definition 1.1.4 *Let a group G act continuously on a topological space X. We say a subset $\Gamma \subset G$ is* **discrete** *if for any two compact subsets $A, B \subset X$, there are only finitely many $g \in \Gamma$ such that $(g \circ A) \cap B \neq \phi$, where ϕ denotes the empty set.*

Example 1.1.5 The discrete subgroup $SL(2,\mathbb{Z})$. Let

$$\Gamma = SL(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\},$$

and let

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \;\middle|\; m \in \mathbb{Z} \right\}$$

be the subgroup of Γ which fixes ∞. Note that $\Gamma_\infty\backslash\Gamma$ is just a set of coset representatives of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where for each pair of relatively prime integers $(c,d)=1$ we choose a unique a,b satisfying $ad-bc=1$. This follows immediately from the identity

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+mc & b+md \\ c & d \end{pmatrix}.$$

The fact that $SL(2,\mathbb{Z})$ is discrete will be deduced from the following lemma.

Lemma 1.1.6 *Fix real numbers* $0<r$, $0<\delta<1$. *Let* $R_{r,\delta}$ *denote the rectangle*

$$R_{r,\delta} = \left\{ x+iy \;\middle|\; -r \le x \le r,\ 0<\delta\le y\le \delta^{-1} \right\}.$$

Then for every $\epsilon>0$, *and any fixed set* $\mathcal{S}$ *of coset representatives for* $\Gamma_\infty\backslash SL(2,\mathbb{Z})$, *there are at most* $4+(4(r+1)/\epsilon\delta)$ *elements* $g\in\mathcal{S}$ *such that* $\mathrm{Im}(g\circ z)>\epsilon$ *holds for some* $z\in R_{r,\delta}$.

Proof Let $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then for $z\in R_{r,\delta}$,

$$\mathrm{Im}(g\circ z) = \frac{y}{c^2y^2+(cx+d)^2} < \epsilon$$

if $|c|>(y\epsilon)^{-\frac{1}{2}}$. On the other hand, for $|c|\le (y\epsilon)^{-\frac{1}{2}} \le (\delta\epsilon)^{-\frac{1}{2}}$, we have

$$\frac{y}{(cx+d)^2} < \epsilon$$

if the following inequalities hold:

$$|d| > |c|r + (y\epsilon^{-1})^{\frac{1}{2}} \ge |c|r + (\epsilon\delta)^{-\frac{1}{2}}.$$

Consequently, $\mathrm{Im}(g\circ z)>\epsilon$ only if

$$|c| \le (\delta\epsilon)^{-\frac{1}{2}} \quad \text{and} \quad |d| \le (\epsilon\delta)^{-\frac{1}{2}}(r+1),$$

and the total number of such pairs (not counting $(c,d)=(0,\pm1),(\pm1,0)$) is at most $4(\epsilon\delta)^{-1}(r+1)$. □

It follows from Lemma 1.1.6 that $\Gamma = SL(2,\mathbb{Z})$ is a discrete subgroup of $SL(2,\mathbb{R})$. This is because:

(1) it is enough to show that for any compact subset $A \subset \mathfrak{h}$ there are only finitely many $g \in SL(2,\mathbb{Z})$ such that $(g \circ A) \cap A \neq \phi$;
(2) every compact subset of $A \subset \mathfrak{h}$ is contained in a rectangle $R_{r,\delta}$ for some $r > 0$ and $0 < \delta < \delta^{-1}$;
(3) $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$, except for finitely many $\alpha \in \Gamma_\infty,\ g \in \Gamma_\infty\backslash\Gamma$.

To prove (3), note that Lemma 1.1.6 implies that $(g \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$ except for finitely many $g \in \Gamma_\infty\backslash\Gamma$. Let $S \subset \Gamma_\infty\backslash\Gamma$ denote this finite set of such elements g. If $g \notin S$, then Lemma 1.1.6 tells us that it is because $\mathrm{Im}(gz) < \delta$ for all $z \in R_{r,\delta}$. Since $\mathrm{Im}(\alpha g z) = \mathrm{Im}(gz)$ for $\alpha \in \Gamma_\infty$, it is enough to show that for each $g \in S$, there are only finitely many $\alpha \in \Gamma_\infty$ such that $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} \neq \phi$. This last statement follows from the fact that $g \circ R_{r,\delta}$ itself lies in some other rectangle $R_{r',\delta'}$, and every $\alpha \in \Gamma_\infty$ is of the form $\alpha = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ $(m \in \mathbb{Z})$, so that

$$\alpha \circ R_{r',\delta'} = \left\{x + iy \;\middle|\; -r' + m \le x \le r' + m,\ 0 < \delta' \le \delta'^{-1}\right\},$$

which implies $(\alpha \circ R_{r',\delta'}) \cap R_{r,\delta} = \phi$ for $|m|$ sufficiently large.

Definition 1.1.7 *Suppose the group G acts continuously on a connected topological space X. A* **fundamental domain** *for $G\backslash X$ is a connected region $D \subset X$ such that every $x \in X$ is equivalent* $(\mathrm{mod}\ G)$ *to a point in D and such that no two points in D are equivalent to each other.*

Example 1.1.8 A fundamental domain for the action of $\mathbb{Z}$ on $\mathbb{R}$ of Example 1.1.2 is given by

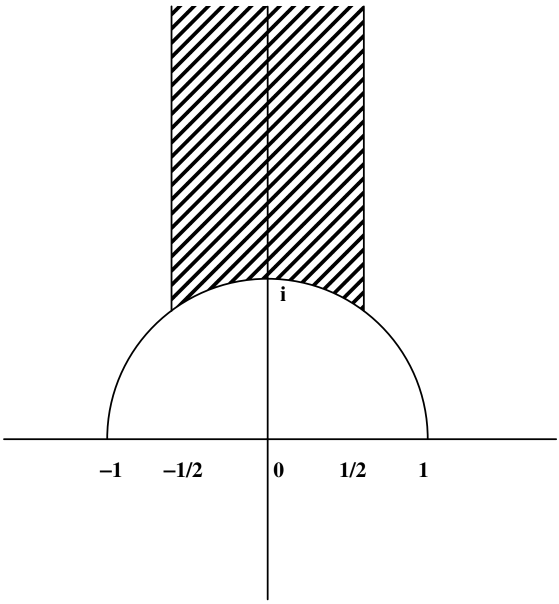
$$\mathbb{Z}\backslash\mathbb{R} = \{0 \le x < 1 \mid x \in \mathbb{R}\}.$$

The proof of this is left as an easy exercise for the reader.

Example 1.1.9 A fundamental domain for $SL(2,\mathbb{Z})\backslash\mathfrak{h}$ can be given as the region $\mathcal{D} \subset \mathfrak{h}$ where

$$\mathcal{D} = \left\{ z \;\middle|\; -\frac{1}{2} \le \mathrm{Re}(z) \le \frac{1}{2},\ \ |z| \ge 1 \right\},$$

with congruent boundary points symmetric with respect to the imaginary axis.



Note that the vertical line $V' := \left\{-\frac{1}{2} + iy \;\middle|\; y \geq \frac{\sqrt{3}}{2}\right\}$ is equivalent to the vertical line $V := \left\{\frac{1}{2} + iy \;\middle|\; y \geq \frac{\sqrt{3}}{2}\right\}$ under the transformation $z \mapsto z + 1$. Furthermore, the arc $A' := \left\{z \;\middle|\; -\frac{1}{2} \leq \mathrm{Re}(z) < 0,\ |z| = 1\right\}$ is equivalent to the reflected arc $A := \left\{z \;\middle|\; 0 < \mathrm{Re}(z) \leq \frac{1}{2},\ |z| = 1\right\}$, under the transformation $z \mapsto -1/z$. To show that $\mathcal{D}$ is a fundamental domain, we must prove:

(1) *For any $z \in \mathfrak{h}$, there exists $g \in SL(2, \mathbb{Z})$ such that $g \circ z \in \mathcal{D}$;*
(2) *If two distinct points $z, z' \in \mathcal{D}$ are congruent* $\pmod{SL(2, \mathbb{Z})}$ *then* $\mathrm{Re}(z) = \pm\frac{1}{2}$ *and* $z' = z \pm 1$, or $|z| = 1$ *and* $z' = -1/z$.

We first prove (1). Fix $z \in \mathfrak{h}$. It follows from Lemma 1.1.6 that for every $\epsilon > 0$, there are at most finitely many $g \in SL(2, \mathbb{Z})$ such that $g \circ z$ lies in the strip

$$D_\epsilon := \left\{ w \;\middle|\; -\frac{1}{2} \leq \mathrm{Re}(w) \leq \frac{1}{2}, \quad \epsilon \leq \mathrm{Im}(w) \right\}.$$

Let B_ϵ denote the finite set of such $g \in SL(2, \mathbb{Z})$. Clearly, for sufficiently small ϵ, the set B_ϵ contains at least one element. We will show that there is at least one $g \in B_\epsilon$ such that $g \circ z \in \mathcal{D}$. Among these finitely many $g \in B_\epsilon$, choose one such that $\mathrm{Im}(g \circ z)$ is maximal in D_ϵ. If $|g \circ z| < 1$, then for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and any $m \in \mathbb{Z}$,

$$\operatorname{Im}(T^m Sg \circ z) = \operatorname{Im}\left(\frac{-1}{g \circ z}\right) = \frac{\operatorname{Im}(g \circ z)}{|g \circ z|^2} > \operatorname{Im}(g \circ z).$$

This is a contradiction because we can always choose m so that $T^m Sg \circ z \in D_\epsilon$. So in fact, $g \circ z$ must be in $\mathcal{D}$.

To complete the verification that $\mathcal{D}$ is a fundamental domain, it only remains to prove the assertion (2). Let $z \in \mathcal{D}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, and assume that $g \circ z \in \mathcal{D}$. Without loss of generality, we may assume that

$$\operatorname{Im}(g \circ z) = \frac{y}{|cz + d|^2} \geq \operatorname{Im}(z),$$

(otherwise just interchange z and $g \circ z$ and use g^{-1}). This implies that $|cz + d| \leq 1$ which implies that $1 \geq |cy| \geq \frac{\sqrt{3}}{2}|c|$. This is clearly impossible if $|c| \geq 2$. So we only have to consider the cases $c = 0, \pm 1$. If $c = 0$ then $d = \pm 1$ and g is a translation by b. Since $-\frac{1}{2} \leq \operatorname{Re}(z), \operatorname{Re}(g \circ z) \leq \frac{1}{2}$, this implies that either $b = 0$ and $z = g \circ z$ or else $b = \pm 1$ and $\operatorname{Re}(z) = \pm\frac{1}{2}$ while $\operatorname{Re}(g \circ z) = \mp\frac{1}{2}$. If $c = 1$, then $|z + d| \leq 1$ implies that $d = 0$ unless $z = e^{2\pi i/3}$ and $d = 0, 1$ or $z = e^{\pi i/3}$ and $d = 0, -1$. The case $d = 0$ implies that $|z| \leq 1$ which implies $|z| = 1$. Also, in this case, $c = 1, d = 0$, we must have $b = -1$ because $ad - bc = 1$. Then $g \circ z = a - \frac{1}{z}$. It follows that $a = 0$. If $z = e^{2\pi i/3}$ and $d = 1$, then we must have $a - b = 1$. It follows that $g \circ e^{2\pi i/3} = a - \frac{1}{1+e^{2\pi i/3}} = a + e^{2\pi i/3}$, which implies that $a = 0$ or 1. A similar argument holds when $z = e^{\pi i/3}$ and $d = -1$. Finally, the case $c = -1$ can be reduced to the previous case $c = 1$ by reversing the signs of a, b, c, d.

1.2 Iwasawa decomposition

This monograph focusses on the general linear group $GL(n, \mathbb{R})$ with $n \geq 2$. This is the multiplicative group of all $n \times n$ matrices with coefficients in $\mathbb{R}$ and non-zero determinant. We will show that every matrix in $GL(n, \mathbb{R})$ can be written as an upper triangular matrix times an orthogonal matrix. This is called the Iwasawa decomposition (Iwasawa, 1949).

The Iwasawa decomposition, in the special case of $GL(2, \mathbb{R})$, states that every $g \in GL(2, \mathbb{R})$ can be written in the form:

$$g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \tag{1.2.1}$$

where $y > 0, x, d \in \mathbb{R}$ with $d \neq 0$ and

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in O(2, \mathbb{R}),$$

where

$$O(n, \mathbb{R}) = \left\{ g \in GL(n, \mathbb{R}) \;\middle|\; g \cdot {}^t g = I \right\}$$

is the orthogonal group. Here I denotes the identity matrix on $GL(n, \mathbb{R})$ and ${}^t g$ denotes the transpose of the matrix g. The matrix $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ in the decomposition (1.2.1) is actually uniquely determined. Furthermore, the matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$ are uniquely determined up to multiplication by $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.

Note that explicitly,

$$O(2, \mathbb{R}) = \left\{ \begin{pmatrix} \pm \cos t & -\sin t \\ \pm \sin t & \cos t \end{pmatrix} \;\middle|\; 0 \leq t \leq 2\pi \right\}.$$

We shall shortly give a detailed proof of (1.2.1) for $GL(n, \mathbb{R})$ with $n \geq 2$.

The decomposition (1.2.1) allows us to realize the upper half-plane

$$\mathfrak{h} = \left\{ x + iy \;\middle|\; x \in \mathbb{R}, \, y > 0 \right\}$$

as the set of two by two matrices of type

$$\left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R}, \; y > 0 \right\},$$

or by the isomorphism

$$\mathfrak{h} \equiv GL(2, \mathbb{R}) \big/ \big\langle O(2, \mathbb{R}), Z_2 \big\rangle, \tag{1.2.2}$$

where

$$Z_n = \left\{ \begin{pmatrix} d & & 0 \\ & \ddots & \\ 0 & & d \end{pmatrix} \;\middle|\; d \in \mathbb{R}, \; d \neq 0 \right\}$$

is the center of $GL(n, \mathbb{R})$, and $\langle O(2, \mathbb{R}), Z_2 \rangle$ denotes the group generated by $O(2, \mathbb{R})$ and Z_2.

The isomorphism (1.2.2) is the starting point for generalizing the classical theory of modular forms on $GL(2, \mathbb{R})$ to $GL(n, \mathbb{R})$ with $n > 2$. Accordingly, we define the generalized upper half-plane $\mathfrak{h}^n$ associated to $GL(n, \mathbb{R})$.

Definition 1.2.3 *Let $n \geq 2$. The* **generalized upper half-plane** $\mathfrak{h}^n$ *associated to $GL(n, \mathbb{R})$ is defined to be the set of all $n \times n$ matrices of the form $z = x \cdot y$ where*

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & & x_{1,n} \\ & 1 & x_{2,3} & \cdots & & x_{2,n} \\ & & \ddots & & & \vdots \\ & & & & 1 & x_{n-1,n} \\ & & & & & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y'_{n-1} & & & & \\ & y'_{n-2} & & & \\ & & \ddots & & \\ & & & y'_1 & \\ & & & & 1 \end{pmatrix},$$

with $x_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq n$ and $y'_i > 0$ for $1 \leq i \leq n-1$.

To simplify later formulae and notation in this book, we will always express y in the form:

$$y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

with $y_i > 0$ for $1 \leq i \leq n-1$. Note that this can always be done since $y'_i \neq 0$ for $1 \leq i \leq n-1$.

Explicitly, x is an upper triangular matrix with 1s on the diagonal and y is a diagonal matrix beginning with a 1 in the lowest right entry. Note that x is parameterized by $n \cdot (n-1)/2$ real variables $x_{i,j}$ and y is parameterized by $n-1$ positive real variables y_i.

Example 1.2.4 The generalized upper half plane $\mathfrak{h}^3$ is the set of all matrices $z = x \cdot y$ with

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} \\ 0 & 1 & x_{2,3} \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $x_{1,2}, x_{1,3}, x_{2,3} \in \mathbb{R},\ y_1, y_2 > 0$. Explicitly, every $z \in \mathfrak{h}^3$ can be written in the form

$$z = \begin{pmatrix} y_1 y_2 & x_{1,2} y_1 & x_{1,3} \\ 0 & y_1 & x_{2,3} \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 1.2.5 The generalized upper half-plane $\mathfrak{h}^3$ does not have a complex structure. Thus $\mathfrak{h}^3$ is quite different from $\mathfrak{h}^2$, which does have a complex structure.

Proposition 1.2.6 *Fix $n \ge 2$. Then we have the Iwasawa decomposition:*

$$GL(n, \mathbb{R}) = \mathfrak{h}^n \cdot O(n, \mathbb{R}) \cdot Z_n,$$

i.e., every $g \in GL(n, \mathbb{R})$ may be expressed in the form

$$g = z \cdot k \cdot d, \qquad (\cdot \text{ denotes matrix multiplication})$$

where $z \in \mathfrak{h}^n$ is uniquely determined, $k \in O(n, \mathbb{R})$, and $d \in Z_n$ is a non-zero diagonal matrix which lies in the center of $GL(n, \mathbb{R})$. Further, k and d are also uniquely determined up to multiplication by $\pm I$ where I is the identity matrix on $GL(n, \mathbb{R})$.

Remark Note that for every $n = 1, 2, 3, \ldots,$ we have $Z_n \cong \mathbb{R}^\times$. We shall, henceforth, write

$$\mathfrak{h}^n \cong GL(n, \mathbb{R})/(O(n, \mathbb{R}) \cdot \mathbb{R}^\times).$$

Proof Let $g \in GL(n, \mathbb{R})$. Then $g \cdot {}^t g$ is a positive definite non–singular matrix. We claim there exists $u, \ell \in GL(n, \mathbb{R})$, where u is upper triangular with 1s on the diagonal and ℓ is lower triangular with 1s on the diagonal, such that

$$u \cdot g \cdot {}^t g = \ell \cdot d \tag{1.2.7}$$

with

$$d = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}, \qquad d_1, \ldots, d_n > 0.$$

For example, consider $n = 2$, and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$g \cdot {}^t g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}.$$

If we set $u = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, then u satisfies (1.2.7) if

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix},$$

so that we may take $t = (-ac - bd)/(c^2 + d^2)$. More generally, the upper triangular matrix u will have $n(n-1)/2$ free variables, and we will have to solve $n(n-1)/2$ equations to satisfy (1.2.7). This system of linear equations has a unique solution because its matrix $g \cdot {}^t g$ is non–singular.

It immediately follows from (1.2.7) that $u^{-1}\ell d = g \cdot {}^t g = d \cdot {}^t\ell({}^t u)^{-1}$, or equivalently

$$\underbrace{\ell \cdot d \cdot {}^t u}_{\text{lower } \triangle} = \underbrace{u \cdot d \cdot {}^t \ell}_{\text{upper } \triangle} = d.$$

The above follows from the fact that a lower triangular matrix can only equal an upper triangular matrix if it is diagonal, and that this diagonal matrix must be d by comparing diagonal entries. The entries $d_i > 0$ because $g \cdot {}^t g$ is positive definite.

Consequently $\ell d = d({}^t u)^{-1}$. Substituting this into (1.2.7) gives

$$u \cdot g \cdot {}^t g \cdot {}^t u = d = a^{-1} \cdot ({}^t a)^{-1}$$

for

$$a = \begin{pmatrix} d_1^{-\frac{1}{2}} & & \\ & \ddots & \\ & & d_n^{-\frac{1}{2}} \end{pmatrix}.$$

Hence $aug \cdot ({}^t g \cdot {}^t u \cdot {}^t a) = I$ so that $aug \in O(n, \mathbb{R})$. Thus, we have expressed g in the form

$$g = (au)^{-1} \cdot (aug),$$

from which the Iwasawa decomposition immediately follows after dividing and multiplying by the scalar $d_n^{-\frac{1}{2}}$ to arrange the bottom right entry of $(au)^{-1}$ to be 1.

It only remains to show the uniqueness of the Iwasawa decomposition. Suppose that $zkd = z'k'd'$ with $z, z' \in \mathfrak{h}^n,\ k, k' \in O(n, \mathbb{R}),\ d, d' \in Z_n$. Then, since the only matrices in $\mathfrak{h}^n$ and $O(n, \mathbb{R})$ which lie in Z_n are $\pm I$ where I is the identity matrix, it follows that $d' = \pm d$. Further, the only matrix in $\mathfrak{h}^n \cap O(n, \mathbb{R})$ is I. Consequently $z = z'$ and $k = \pm k'$. $\square$

We shall now work out some important instances of the Iwasawa decomposition which will be useful later.

Proposition 1.2.8 *Let I denote the identity matrix on $GL(n,\mathbb{R})$, and for every $1 \le j < i \le n$, let $E_{i,j}$ denote the matrix with a 1 at the $\{i, j\}$th position and zeros elsewhere. Then, for an arbitrary real number t, we have*

$$I + tE_{i,j} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & \frac{1}{(t^2+1)^{\frac12}} & \cdots & \frac{t}{(t^2+1)^{\frac12}} & \\ & & & \ddots & \vdots & \\ & & & & (t^2+1)^{\frac12} & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \big(\mathrm{mod}\,(O(n,\mathbb{R})\cdot\mathbb{R}^\times)\big),$$

where, in the above matrix, $\frac{1}{(t^2+1)^{\frac12}}$ occurs at position $\{j, j\}$, $(t^2+1)^{\frac12}$ occurs at position $\{i, i\}$, all other diagonal entries are ones, $\frac{t}{(t^2+1)^{\frac12}}$ occurs at position $\{j, i\}$, and, otherwise, all other entries are zero.

Proof Let $g = I + tE_{i,j}$. Then

$$g \cdot {}^t g = (I + tE_{i,j}) \cdot (I + tE_{j,i}) = I + tE_{i,j} + tE_{j,i} + t^2 E_{i,i}.$$

If we define a matrix $u = I - (t/(t^2+1))E_{j,i}$, then $u \cdot g \cdot {}^t g \cdot {}^t u$ must be a diagonal matrix d. Setting $d = a^{-1} \cdot ({}^t a)^{-1}$, we may directly compute:

$$\begin{aligned} u \cdot g \cdot {}^t g \cdot {}^t u &= I + t^2 E_{i,i} - \frac{t^2}{t^2+1} E_{j,j}, \\ u^{-1} &= I + \frac{t}{t^2+1} E_{j,i}, \\ a^{-1} &= I + \left(\frac{1}{\sqrt{t^2+1}} - 1\right) E_{j,j} + \left(\sqrt{t^2+1} - 1\right) E_{i,i}. \end{aligned}$$

Therefore,

$$u^{-1} a^{-1} = I + \left(\frac{1}{\sqrt{t^2+1}} - 1\right) E_{j,j} + \left(\sqrt{t^2+1} - 1\right) E_{i,i} + \frac{t}{\sqrt{t^2+1}} E_{j,i}.$$

As in the proof of Proposition 1.2.6, we have $g = u^{-1} \cdot a^{-1}$ $(\mathrm{mod}\,(O(n,\mathbb{R}), \mathbb{R}^\times))$. □

Proposition 1.2.9 *Let $n \geq 2$, and let $z = xy \in \mathfrak{h}^n$ have the form given in Definition 1.2.3. For $i = 1, 2, \ldots, n-1$, define*

$$\omega_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix},$$

to be the $n \times n$ identity matrix except for the ith and $(i+1)$th rows where we have $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on the diagonal. Then

$$\omega_i z \equiv \begin{pmatrix} 1 & x'_{1,2} & x'_{1,3} & \cdots & & x'_{1,n} \\ & 1 & x'_{2,3} & \cdots & & x'_{2,n} \\ & & \ddots & & & \vdots \\ & & & & 1 & x'_{n-1,n} \\ & & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} y'_1 y'_2 \cdots y'_{n-1} & & & & \\ & y'_1 y'_2 \cdots y'_{n-2} & & & \\ & & \ddots & & \\ & & & y'_1 & \\ & & & & 1 \end{pmatrix}$$

$\big(\mathrm{mod}\,(O(n,\mathbb{R}) \cdot \mathbb{R}^\times)\big)$*, where $y'_k = y_k$ except for $k = n-i+1, n-i, n-i-1$, in which case*

$$y'_{n-i} = \frac{y_{n-i}}{x_{i,i+1}^2 + y_{n-i}^2}, \qquad y'_{n-i\pm1} = y_{n-i\pm1} \cdot \sqrt{x_{i,i+1}^2 + y_{n-i}^2},$$

and $x_{k,\ell} = x'_{k,\ell}$ except for $\ell = i, i+1$, in which case

$$x'_{i-j,i} = x_{i-j,i+1} - x_{i-j,i} x_{i,i+1}, \qquad x'_{i-j,i+1} = \frac{x_{i-j,i} y_{n-i}^2 + x_{i-j,i+1} x_{i,i+1}}{x_{i,i+1}^2 + y_{n-i}^2},$$

for $j = 1, 2, \ldots, i-2$.

Proof Brute force computation which is omitted. □

Proposition 1.2.10 *The group $GL(n, \mathbb{Z})$ acts on $\mathfrak{h}^n$.*

Proof Recall the definition of a group acting on a topological space given in Definition 1.1.1. The fact that $GL(n, \mathbb{Z})$ acts on $GL(n, \mathbb{R})$ follows immediately from the fact that $GL(n, \mathbb{Z})$ acts on the left on $GL(n, \mathbb{R})$ by matrix multiplication and that we have the realization $\mathfrak{h}^n = GL(n, \mathbb{R})/(O(n, \mathbb{R}) \cdot \mathbb{R}^\times)$, as a set of cosets, by the Iwasawa decomposition given in Proposition 1.2.6. □

1.3 Siegel sets

We would like to show that $\Gamma^n = GL(n, \mathbb{Z})$ acts discretely on the generalized upper half-plane $\mathfrak{h}^n$ defined in Definition 1.2.3. This was already proved for $n = 2$ in Lemma 1.1.6, but the generalization to $n > 2$ requires more subtle arguments. In order to find an approximation to a fundamental domain for $GL(n, \mathbb{Z})\backslash\mathfrak{h}^n$, we shall introduce for every $t, u \geq 0$ the Siegel set $\Sigma_{t,u}$.

Definition 1.3.1 *Let $a, b \geq 0$ be fixed. We define the Siegel set $\Sigma_{a,b} \subset \mathfrak{h}^n$ to be the set of all*

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & & x_{1,n} \\ & 1 & x_{2,3} & \cdots & & x_{2,n} \\ & & \ddots & & & \vdots \\ & & & & 1 & x_{n-1,n} \\ & & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}$$

with $|x_{i,j}| \leq b$ for $1 \leq i < j \leq n$, and $y_i > a$ for $1 \leq i \leq n-1$.

Let $\Gamma^n = GL(n, \mathbb{Z})$ and $\Gamma^n_\infty \subset \Gamma^n$ denote the subgroup of upper triangular matrices with 1s on the diagonal. We have shown in Proposition 1.2.10 that Γ^n acts on $\mathfrak{h}^n$. For $g \in \Gamma^n$ and $z \in \mathfrak{h}^n$, we shall denote this action by $g \circ z$. The following proposition proves that the action is discrete and that $\Sigma_{\frac{\sqrt{3}}{2},\frac{1}{2}}$ is a good approximation to a fundamental domain.

Proposition 1.3.2 *Fix an integer $n \geq 2$. For any $z \in \mathfrak{h}^n$ there are only finitely many $g \in \Gamma^n$ such that $g \circ z \in \Sigma_{\frac{\sqrt{3}}{2},\frac{1}{2}}$. Furthermore,*

$$GL(n, \mathbb{R}) = \bigcup_{g \in \Gamma^n} g \circ \Sigma_{\frac{\sqrt{3}}{2},\frac{1}{2}}. \tag{1.3.3}$$

Remarks The bound $\frac{\sqrt{3}}{2}$ is implicit in the work of Hermite, and a proof can be found in (Korkine and Zolotareff, 1873). The first part of Proposition 1.3.2 is a well known theorem due to Siegel (1939). For the proof, we follow the exposition of Borel and Harish-Chandra (1962).

Proof of Proposition 1.3.2 In order to prove (1.3.3), it is enough to show that

$$SL(n, \mathbb{R}) = \bigcup_{g \in SL(n,\mathbb{Z})} g \circ \Sigma^*_{\frac{\sqrt{3}}{2},\frac{1}{2}}, \tag{1.3.4}$$

where $\Sigma^*_{t,u}$ denotes the subset of matrices $\Sigma_{t,u} \cdot Z_n$ which have determinant 1 and $\circ$ denotes the action of $SL(n, \mathbb{Z})$ on $\Sigma^*_{0,\infty}$. Note that every element in $\Sigma^*_{a,b}$

is of the form

$$\begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & & x_{1,n} \\ & 1 & x_{2,3} & \cdots & & x_{2,n} \\ & & \ddots & & & \vdots \\ & & & & 1 & x_{n-1,n} \\ & & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} dy_1y_2\cdots y_{n-1} & & & & \\ & dy_1y_2\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & dy_1 & \\ & & & & d \end{pmatrix} \tag{1.3.5}$$

where the determinant

$$\mathrm{Det}\begin{pmatrix} dy_1y_2\cdots y_{n-1} & & & & \\ & dy_1y_2\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & dy_1 & \\ & & & & d \end{pmatrix} = 1,$$

so that

$$d = \left(\prod_{i=1}^{n-1} y_i^{n-i}\right)^{-1/n}.$$

In view of the Iwasawa decomposition of Proposition 1.2.6, we may identify $\Sigma^*_{0,\infty}$ as the set of coset representatives $SL(n,\mathbb{R})/SO(n,\mathbb{R})$, where $SO(n,\mathbb{R})$ denotes the subgroup $O(n,\mathbb{R}) \cap SL(n,\mathbb{R})$. □

In order to prove (1.3.4), we first introduce some basic notation. Let

$$e_1 = (1,0,\ldots,0), \quad e_2 = (0,1,\ldots,0), \quad \ldots, \quad e_n = (0,0,\ldots,1),$$

denote the canonical basis for $\mathbb{R}^n$. For $1 \le i \le n$ and any matrix $g \in GL(n,\mathbb{R})$, let $e_i \cdot g$ denote the usual multiplication of a $1 \times n$ matrix with an $n \times n$ matrix. For an arbitrary $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, define the norm: $||v|| := \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$. We now introduce a function

$$\phi : SL(n,\mathbb{R}) \to \mathbb{R}^{>0}$$

from $SL(n,\mathbb{R})$ to the positive real numbers. For all $g = (g_{i,j})_{1\le i,j\le n}$ in $SL(n,\mathbb{R})$ we define

$$\phi(g) := ||e_n \cdot g|| = \sqrt{g_{n,1}^2 + g_{n,2}^2 + \cdots + g_{n,n}^2}.$$

Claim *The function ϕ is well defined on the quotient space $SL(n,\mathbb{R})/SO(n,\mathbb{R})$.*

To verify the claim, note that for $k \in SO(n, \mathbb{R})$, and $v \in \mathbb{R}^n$, we have

$$||v \cdot k|| = \sqrt{(v \cdot k) \cdot {}^t(v \cdot k)} = \sqrt{v \cdot k \cdot {}^tk \cdot {}^tv} = \sqrt{v \cdot {}^tv} = ||v||.$$

This immediately implies that $\phi(gk) = \phi(g)$, i.e., the claim is true.

Note that if $z \in \Sigma^*_{0,\infty}$ is of the form (1.3.5), then

$$\phi(z) = d = \left(\prod_{i=1}^{n-1} y_i^{(n-i)}\right)^{-1/n}. \tag{1.3.6}$$

Now, if $z \in \Sigma^*_{0,\infty}$ is fixed, then

$$e_n \cdot SL(n, \mathbb{Z}) \cdot z \subset \big(\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n - \{(0, 0, \ldots, 0)\}\big) \cdot z, \tag{1.3.7}$$

where $\cdot$ denotes matrix multiplication. The right-hand side of (1.3.7) consists of non–zero points of a lattice in $\mathbb{R}^n$. This implies that ϕ achieves a **positive minimum** on the coset $SL(n, \mathbb{Z}) \cdot z$. The key to the proof of Proposition 1.3.2 will be the following lemma from which Proposition 1.3.2 follows immediately.

Lemma 1.3.8 *Let $z \in \Sigma^*_{0,\infty}$. Then the minimum of ϕ on $SL(n, \mathbb{Z}) \circ z$ is achieved at a point of $\Sigma^*_{\frac{\sqrt{3}}{2},\frac{1}{2}}$.*

Proof It is enough to prove that the minimum of ϕ is achieved at a point of $\Sigma^*_{\frac{\sqrt{3}}{2},\infty}$ because we can always translate by an upper triangular matrix

$$u = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ & 1 & u_{2,3} & \cdots & u_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & u_{n-1,n} \\ & & & & 1 \end{pmatrix} \in SL(n, \mathbb{Z})$$

to arrange that the minimum of ϕ lies in $\Sigma^*_{\frac{\sqrt{3}}{2},\frac{1}{2}}$. This does not change the value of ϕ because of the identity $\phi(u \cdot z) = ||e_n \cdot u \cdot z|| = ||e_n \cdot z||$. We shall use induction on n. We have already proved a stronger statement for $n = 2$ in Example 1.1.9. Fix $\gamma \in SL(n, \mathbb{Z})$ such that $\phi(\gamma \circ z)$ is minimized. We set $\gamma \circ z = x \cdot y$ with

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix}, \quad y = \begin{pmatrix} dy_1y_2\cdots y_{n-1} & & & & \\ & dy_1y_2\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & dy_1 & \\ & & & & d \end{pmatrix},$$

with $d = (\prod_{i=1}^{n-1} y_i^{n-i})^{-1/n}$ as before. We must show $y_i \geq \frac{\sqrt{3}}{2}$ for $i = 1, 2, \ldots, n-1$. The proof proceeds in 3 steps.

Step 1 $y_1 \geq \frac{\sqrt{3}}{2}$.

This follows from the action of $\alpha := \begin{pmatrix} I_{n-2} & & \\ & 0 & -1 \\ & 1 & 0 \end{pmatrix}$ on $\gamma \circ z$. Here I_{n-2} denotes the identity $(n-2) \times (n-2)$–matrix. First of all

$$\begin{aligned}\phi(\alpha \circ \gamma \circ z) &= ||e_n \cdot \alpha \circ \gamma \circ z|| = ||e_{n-1} \cdot x \cdot y|| = ||(e_{n-1} + x_{n-1,n} e_n) \cdot y|| \\ &= d\sqrt{y_1^2 + x_{n-1,n}^2}.\end{aligned}$$

Since $|x_{n-1,n}| \leq \frac{1}{2}$ we see that $\phi(\alpha\gamma z)^2 \leq d^2(y_1^2 + \frac{1}{4})$. On the other hand, the assumption of minimality forces $\phi(\gamma z)^2 = d^2 \leq d^2 \left(y_1^2 + \frac{1}{4}\right)$. This implies that $y_1 \geq \frac{\sqrt{3}}{2}$.

Step 2 Let $g' \in SL(n-1, \mathbb{Z})$, $g = \begin{pmatrix} g' & 1 \\ 0 & 1 \end{pmatrix}$. Then $\phi(g\gamma z) = \phi(\gamma z)$. This follows immediately from the fact that $e_n \cdot g = e_n$.

Step 3 $y_i \geq \frac{\sqrt{3}}{2}$ for $i = 2, 3, \ldots, n-1$.

Let us write $\gamma \circ z = \begin{pmatrix} z' \cdot d' & * \\ & d \end{pmatrix}$ with $z' \in SL(n-1, \mathbb{R})$ and $d' \in Z_{n-1}$ a suitable diagonal matrix. By induction, there exists $g' \in SL(n-1, \mathbb{Z})$ such that $g' \circ z' = x' \cdot y' \in \Sigma^*_{\frac{\sqrt{3}}{2}, \frac{1}{2}} \subset \mathfrak{h}^{n-1}$, the Siegel set for $GL(n-1, \mathbb{R})$. This is equivalent to the fact that

$$y' = \begin{pmatrix} a_{n-1} & & & \\ & a_{n-2} & & \\ & & \ddots & \\ & & & a_1 \end{pmatrix}$$

and

$$\frac{a_{j+1}}{a_j} \geq \frac{\sqrt{3}}{2} \quad \text{for } j = 1, 2, \ldots, n-2. \tag{1.3.9}$$

Define $g := \begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix} \in SL(n, \mathbb{Z})$. Then

$$g \circ \gamma \circ z = \begin{pmatrix} g' & 0 \\ 0 & 1 \end{pmatrix} \circ \begin{pmatrix} z' \cdot d' & * \\ & d \end{pmatrix} = \begin{pmatrix} g' \circ z' \cdot d' & * \\ 0 & d \end{pmatrix} = x'' \cdot y'',$$

where $y'' = \begin{pmatrix} y'd' & 0 \\ 0 & d \end{pmatrix}$, $x'' = \begin{pmatrix} x' & * \\ 0 & 1 \end{pmatrix}$. The inequalities (1.3.9) applied to

$$y'' = \begin{pmatrix} y'd' & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} d & & & \\ & \ddots & & \\ & & y_1 d & \\ & & & d \end{pmatrix},$$

imply that $y_i \geq \frac{\sqrt{3}}{2}$ for $i = 2, 3, \ldots, n-1$. Step 2 insures that multiplying by g on the left does not change the value of $\phi(\gamma z)$. Step 1 gives $y_1 \geq \frac{\sqrt{3}}{2}$. □

1.4 Haar measure

Let $n \geq 2$. The discrete subgroup $SL(n, \mathbb{Z})$ acts on $SL(n, \mathbb{R})$ by left multiplication. The quotient space $SL(n, \mathbb{Z}) \backslash SL(n, \mathbb{R})$ turns out to be of fundamental importance in number theory. Now, we turn our attention to a theory of integration on this quotient space.

We briefly review the theory of Haar measure and integration on locally compact Hausdorff topological groups. Good references for this material are (Halmos, 1974), (Lang, 1969), (Hewitt and Ross, 1979). Excellent introductory books on matrix groups and elementary Lie theory are (Curtis, 1984), (Baker, 2002), (Lang, 2002).

Recall that a topological group G is a topological space G where G is also a group and the map

$$(g, h) \mapsto g \cdot h^{-1}$$

of $G \times G$ onto G is continuous in both variables. Here $\cdot$ again denotes the internal group operation and h^{-1} denotes the inverse of the element h. The assumption that G is locally compact means that every point has a compact neighborhood. Recall that G is termed Hausdorff provided every pair of distinct elements in G lie in disjoint open sets.

Example 1.4.1 The general linear group $GL(n, \mathbb{R})$ is a locally compact Hausdorff topological group.

Let $\mathfrak{gl}(n, \mathbb{R})$ denote the Lie algebra of $GL(n, \mathbb{R})$. Viewed as a set, $\mathfrak{gl}(n, \mathbb{R})$ is just the set of all $n \times n$ matrices with coefficients in $\mathbb{R}$. We assign a topology to $\mathfrak{gl}(n, \mathbb{R})$ by identifying every matrix

$$g = \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,n} \\ \vdots & & \cdots & \vdots \\ g_{n,1} & g_{n,2} & \cdots & g_{n,n} \end{pmatrix}$$

with a point

$$(g_{1,1}, g_{1,2}, \ldots, g_{1,n}, g_{2,1}, g_{2,2}, \ldots, g_{2,n}, \ldots, g_{n,n}) \in \mathbb{R}^{n^2}.$$

This identification is a one–to–one correspondence. One checks that $\mathfrak{gl}(n, \mathbb{R})$ is a locally compact Hausdorff topological space under the usual Euclidean topology on $\mathbb{R}^{n^2}$. The determinant function $\mathrm{Det} : \mathfrak{gl}(n, \mathbb{R}) \to \mathbb{R}$ is clearly continuous. It follows that

$$GL(n, \mathbb{R}) = \mathfrak{gl}(n, \mathbb{R}) - \mathrm{Det}^{-1}(0)$$

must be an open set since $\{0\}$ is closed. Also, the operations of addition and multiplication of matrices in $\mathfrak{gl}(n, \mathbb{R})$ are continuous maps from

$$\mathfrak{gl}(n, \mathbb{R}) \times \mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R}).$$

The inverse map

$$\mathrm{Inv} : GL(n, \mathbb{R}) \to GL(n, \mathbb{R}),$$

given by $\mathrm{Inv}(g) = g^{-1}$ for all $g \in GL(n, \mathbb{R})$, is also continuous since each entry of g^{-1} is a polynomial in the entries of g divided by $\mathrm{Det}(g)$. Thus, $GL(n, \mathbb{R})$ is a topological subspace of $\mathfrak{gl}(n, \mathbb{R})$ and we may view $GL(n, \mathbb{R}) \times GL(n, \mathbb{R})$ as the product space. Since the multiplication and inversion maps: $GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ are continuous, it follows that $GL(n, \mathbb{R})$ is a topological group.

By a **left Haar measure** on a locally compact Hausdorff topological group G, we mean a positive Borel measure (Halmos, 1974)

$$\mu : \{\text{measurable subsets of } G\} \to \mathbb{R}^+,$$

which is left invariant under the action of G on G via left multiplication. This means that for every measurable set $E \subset G$ and every $g \in G$, we have

$$\mu(gE) = \mu(E).$$

In a similar manner, one may define a **right Haar measure**. If every left invariant Haar measure on G is also a right invariant Haar measure, then we say that G is **unimodular**.

Given a left invariant Haar measure μ on G, one may define (in the usual manner) a differential one-form $d\mu(g)$, and for compactly supported functions $f : G \to \mathbb{C}$ an integral

$$\int_G f(g)\, d\mu(g),$$

which is characterized by the fact that

$$\int_E d\mu(g) = \mu(E)$$

for every measurable set E. We shall also refer to $d\mu(g)$ as a Haar measure. The fundamental theorem in the subject is due to Haar.

Theorem 1.4.2 (Haar) *Let G be a locally compact Hausdorff topological group. Then there exists a left Haar measure on G. Further, any two such Haar measures must be positive real multiples of each other.*

We shall not need this general existence theorem, because in the situations we are interested in, we can explicitly construct the Haar measure and Haar integral. For unimodular groups, the uniqueness of Haar measure follows easily from Fubini's theorem. The proof goes as follows. Assume we have two Haar measures μ, ν on G, which are both left and right invariant. Let $h : G \to \mathbb{C}$ be a compactly supported function satisfying

$$\int_G h(g)\, d\mu(g) = 1.$$

Then for an arbitrary compactly supported function $f : G \to \mathbb{C}$,

$$\begin{aligned}
\int_G f(g) d\nu(g) &= \int_G h(g') d\mu(g') \int_G f(g) d\nu(g) \\
&= \int_G \int_G h(g') f(g) d\nu(g) d\mu(g') \\
&= \int_G \int_G h(g') f(g \cdot g')\, d\nu(g)\, d\mu(g') \\
&= \int_G \int_G h(g') f(g \cdot g')\, d\mu(g')\, d\nu(g) \\
&= \int_G \int_G h(g^{-1} \cdot g') f(g')\, d\mu(g')\, d\nu(g) \\
&= \int_G \int_G h(g^{-1} \cdot g') f(g')\, d\nu(g)\, d\mu(g') \\
&= c \cdot \int_G f(g')\, d\mu(g')
\end{aligned}$$

where $c = \int_G h(g^{-1}) d\nu(g)$.

Proposition 1.4.3 *For $n = 1, 2, \ldots,$ let*

$$g = \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,n} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,n} \\ \vdots & & \cdots & \vdots \\ g_{n,1} & g_{n,2} & \cdots & g_{n,n} \end{pmatrix} \in GL(n, \mathbb{R}),$$

where $g_{1,1}, g_{1,2}, \ldots, g_{1,n}, g_{2,1}, \ldots, g_{n,n}$ are n^2 real variables. Define

$$d\mu(g) := \frac{\prod\limits_{1 \le i,j \le n} dg_{i,j}}{\mathrm{Det}(g)^n}, \qquad (\textit{wedge product of differential one-forms})$$

where $dg_{i,j}$ denotes the usual differential one–form on $\mathbb{R}$ and $\mathrm{Det}(g)$ *denotes the determinant of the matrix g. Then $d\mu(g)$ is the unique left–right invariant Haar measure on $GL(n, \mathbb{R})$.*

Proof Every matrix in $GL(n, \mathbb{R})$ may be expressed as a product of a diagonal matrix in Z_n and matrices of the form $\tilde{x}_{r,s}$ (with $1 \le r, s \le n$) where $\tilde{x}_{r,s}$ denotes the matrix with the real number $x_{r,s}$ at position r, s, and, otherwise, has 1s on the diagonal and zeros off the diagonal. It is easy to see that

$$d\mu(g) = d\mu(ag)$$

for $a \in Z_n$. To complete the proof, it is, therefore, enough to check that

$$d\mu(\tilde{x}_{r,s} \cdot g) = d\mu(g \cdot \tilde{x}_{r,s}) = d\mu(g),$$

for all $1 \le r, s \le n$. We check the left invariance and leave the right invariance to the reader.

It follows from the definition that in the case $r \neq s$,

$$d\mu(\tilde{x}_{r,s} \cdot g) = \frac{\left(\prod\limits_{\substack{1 \le i,j \le n \\ i \neq r}} dg_{i,j} \right) \left(\prod\limits_{1 \le j \le n} d(g_{r,j} + g_{s,j} x_{r,s}) \right)}{\mathrm{Det}(\tilde{x}_{r,s} \cdot g)^n}.$$

First of all,

$$\mathrm{Det}(\tilde{x}_{r,s} \cdot g) = \mathrm{Det}(\tilde{x}_{r,s}) \cdot \mathrm{Det}(g) = \mathrm{Det}(g)$$

because $\mathrm{Det}(\tilde{x}_{r,s}) = 1$.

Second, for any $1 \le j \le n$,

$$\left(\prod\limits_{\substack{1 \le i,j \le n \\ i \neq r}} dg_{i,j} \right) \wedge dg_{s,j} = 0$$

because $g_{s,j}$ also occurs in the product $\left(\prod_{\substack{1\le i<j\le n\\ i\ne r}} dg_{i,j}\right)$ and $dg_{s,j}\wedge dg_{s,j}=0$. Consequently, the measure is invariant under left multiplication by $\tilde{x}_{r,s}$. □

On the other hand, if $r=s$, then

$$\begin{aligned} d\mu(\tilde{x}_{r,s}\cdot g) &= \frac{\left(\prod_{\substack{1\le i,j\le n\\ i\ne r}} dg_{i,j}\right)\left(\prod_{1\le j\le n}(x_{r,s}\cdot dg_{r,j})\right)}{\mathrm{Det}(\tilde{x}_{r,s}\cdot g)^n}\\ &= d\mu(g)\cdot\frac{x_{r,s}^n}{\mathrm{Det}(\tilde{x}_{r,s})^n}\\ &= d\mu(g). \end{aligned}$$

1.5 Invariant measure on coset spaces

This monograph focusses on the coset space

$$GL(n,\mathbb{R})/(O(n,\mathbb{R})\cdot\mathbb{R}^\times).$$

We need to establish explicit invariant measures on this space. The basic principle which allows us to define invariant measures on coset spaces, in general, is given in the following theorem.

Theorem 1.5.1 *Let G be a locally compact Hausdorff topological group, and let H be a compact closed subgroup of G. Let μ be a Haar measure on G, and let ν be a Haar measure on H, normalized so that $\int_H d\nu(h)=1$. Then there exists a unique (up to scalar multiple) quotient measure $\tilde{\mu}$ on G/H. Furthermore*

$$\int_G f(g)\,d\mu(g)=\int_{G/H}\left(\int_H f(gh)\,d\nu(h)\right)d\tilde{\mu}(gH),$$

for all integrable functions $f:G\to\mathbb{C}$.

Proof For a proof see (Halmos, 1974). We indicate, however, why the formula in Theorem 1.5.1 holds. First of all note that if $f:G\to\mathbb{C}$, is an integrable function on G, and if we define a new function, $f^H:G\to\mathbb{C}$, by the recipe

$$f^H(g):=\int_H f(gh)\,d\nu(h),$$

then $f^H(gh)=f^H(g)$ for all $h\in H$. Thus, f^H is well defined on the coset space G/H. We write $f^H(g)=f^H(gH)$, to stress that f^H is a function on the coset space. For any measurable subset $E \subset G/H$, we may easily choose a measurable function $\delta_E : G \to \mathbb{C}$ so that

$$\delta_E(g) = \delta_E^H(gH) = \begin{cases} 1 & \text{if } gH \in E, \\ 0 & \text{if } gH \notin E. \end{cases}$$

We may then define an H–invariant quotient measure $\tilde{\mu}$ satisfying:

$$\tilde{\mu}(E) = \int_G \delta_E(g)\, d\mu(g) = \int_{G/H} \delta_E^H(gH)\, d\tilde{\mu}(gH),$$

and

$$\int_G f(g)\, d\mu(g) = \int_{G/H} f^H(gH)\, d\tilde{\mu}(gH),$$

for all integrable functions $f : G \to \mathbb{C}$. □

Remarks There is an analogous version of Theorem 1.5.1 for left coset spaces $H\backslash G$. Note that we are not assuming that H is a normal subgroup of G. Thus G/H (respectively $H\backslash G$) may not be a group.

Example 1.5.2 (Left invariant measure on $GL(n, \mathbb{R})/(O(n, \mathbb{R}) \cdot \mathbb{R}^\times)$)

For $n \geq 2$, we now explicitly construct a left invariant measure on the generalized upper half-plane $\mathfrak{h}^n = GL(n, \mathbb{R})/(O(n, \mathbb{R}) \cdot \mathbb{R}^\times)$. Returning to the Iwasawa decomposition (Proposition 1.2.6), every $z \in \mathfrak{h}^n$ has a representation in the form $z = xy$ with

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & & x_{1,n} \\ & 1 & x_{2,3} & \cdots & & x_{2,n} \\ & & \ddots & & & \vdots \\ & & & & 1 & x_{n-1,n} \\ & & & & & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

with $x_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq n$ and $y_i > 0$ for $1 \leq i \leq n-1$. Let d^*z denote the left invariant measure on $\mathfrak{h}^n$. Then d^*z has the property that

$$d^*(gz) = d^*z$$

for all $g \in GL(n, \mathbb{R})$.

Proposition 1.5.3 *The left invariant $GL(n,\mathbb{R})$–measure d^*z on $\mathfrak{h}^n$ can be given explicitly by the formula*

$$d^*z = d^*x\ d^*y$$

where

$$d^*x = \prod_{1\le i<j\le n} dx_{i,j}, \quad d^*y = \prod_{k=1}^{n-1} y_k^{-k(n-k)-1}\, dy_k. \tag{1.5.4}$$

For example, for $n=2$, with $z = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$, we have $d^*z = \frac{dxdy}{y^2}$, while for $n=3$ with

$$z = \begin{pmatrix} y_1y_2 & x_{1,2}y_1 & x_{1,3} \\ 0 & y_1 & x_{2,3} \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$d^*z = dx_{1,2}dx_{1,3}dx_{2,3}\,\frac{dy_1dy_2}{(y_1y_2)^3}.$$

Proof We sketch the proof. The group $GL(n,\mathbb{R})$ is generated by diagonal matrices, upper triangular matrices with 1s on the diagonal, and the Weyl group W_n which consists of all $n\times n$ matrices with exactly one 1 in each row and column and zeros everywhere else. For example,

$$W_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

$$W_3 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\right.$$
$$\left. \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \right\}.$$

Note that the Weyl group W_n has order $n!$ and is simply the symmetric group on n symbols. It is clear that $d^*(gz) = d^*z$ if g is an upper triangular matrix with 1s on the diagonal. This is because the measures $dx_{i,j}$ (with $1\le i<j\le n$) are all invariant under translation. It is clear that the differential d^*z is Z_n-invariant where $Z_n \cong \mathbb{R}^\times$ denotes the center of $GL(n,\mathbb{R})$. So, without loss of generality,

we may define a diagonal matrix a with its lower-right entry to be one:

$$a = \begin{pmatrix} a_1a_2\cdots a_{n-1} & & & & \\ & a_1a_2\cdots a_{n-2} & & & \\ & & \ddots & & \\ & & & a_1 & \\ & & & & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} az = axy &= (axa^{-1}) \cdot ay \\ &= \begin{pmatrix} 1 & a_{n-1}x_{1,2} & a_{n-1}a_{n-2}x_{1,3} & \cdots & a_{n-1}\cdots a_1\, x_{1,n} \\ & 1 & a_{n-2}x_{2,3} & \cdots & a_{n-2}\cdots a_1\, x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & a_1\, x_{n-1,n} \\ & & & & 1 \end{pmatrix} \\ &\qquad \times \begin{pmatrix} a_1y_1\cdots a_{n-1}y_{n-1} & & & \\ & \ddots & & \\ & & a_1y_1 & \\ & & & 1 \end{pmatrix}. \end{aligned}$$

Thus $d^*(axa^{-1}) = \left(\prod_{k=1}^{n-1} a_k^{k(n-k)}\right) d^*x$. It easily follows that

$$d^*(az) = d^*(axa^{-1} \cdot ay) = d^*z.$$

It remains to check the invariance of d^*z under the Weyl group W_n. Now, if $w \in W_n$ and

$$d = \begin{pmatrix} d_n & & & \\ & d_{n-1} & & \\ & & \ddots & \\ & & & d_1 \end{pmatrix} \in GL(n, \mathbb{R})$$

is a diagonal matrix, then wdw^{-1} is again a diagonal matrix whose diagonal entries are a permutation of $\{d_1, d_2, \ldots, d_n\}$. The Weyl group is generated by the transpositions ω_i $(i = 1, 2, \ldots n-1)$ given in Proposition 1.2.9 which interchange (transpose) d_i and d_{i+1} when d is conjugated by ω_i. After a tedious calculation using Proposition 1.2.9 one checks that $d^*(\omega_i z) = d^*z$. □

1.6 Volume of $SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n,\mathbb{R})$

Following earlier work of Minkowski, Siegel (1936) showed that the volume of

$$\begin{aligned} SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n,\mathbb{R}) &\cong SL(n,\mathbb{Z})\backslash GL(n,\mathbb{R})/(O(n,\mathbb{R})\cdot\mathbb{R}^\times) \\ &\cong SL(n,\mathbb{Z})\backslash\mathfrak{h}^n, \end{aligned}$$

can be given in terms of

$$\zeta(2)\cdot\zeta(3)\cdots\zeta(n)$$

where $\zeta(s)$ is the Riemann zeta function. The fact that the special values (taken at integral points) of the Riemann zeta function appear in the formula for the volume is remarkable. Later, Weil (1946) found another method to prove such results based on a direct application of the Poisson summation formula. A vast generalization of Siegel's computation of fundamental domains for the case of arithmetic subgroups acting on Chevalley groups was obtained by Langlands (1966). See also (Terras, 1988) for interesting discussions on the history of this subject.

The main aim of this section is to explicitly compute the volume

$$\int\limits_{SL(n,\mathbb{Z})\backslash SL(n,\mathbb{R})/SO(n,\mathbb{R})} d^*z,$$

where d^*z is the left–invariant measure given in Proposition 1.5.3. We follow the exposition of Garret (2002).

Theorem 1.6.1 *Let $n\ge 2$. As in Proposition 1.5.3, fix*

$$d^*z = \prod_{1\le i<j\le n} dx_{i,j}\prod_{k=1}^{n-1} y_k^{-k(n-k)-1}\,dy_k$$

to be the left $SL(n,\mathbb{R})$–invariant measure on $\mathfrak{h}^n = SL(n,\mathbb{R})/SO(n,\mathbb{R})$. Then

$$\int_{SL(n,\mathbb{Z})\backslash\mathfrak{h}^n} d^*z = n\,2^{n-1}\cdot\prod_{\ell=2}^{n}\frac{\zeta(\ell)}{\mathrm{Vol}(S^{\ell-1})},$$

where

$$\mathrm{Vol}(S^{\ell-1}) = \frac{2(\sqrt{\pi})^\ell}{\Gamma(\ell/2)}$$

denotes the volume of the $(\ell-1)$–dimensional sphere $S^{\ell-1}$ and $\zeta(\ell)=\sum\limits_{n=1}^{\infty} n^{-\ell}$ denotes the Riemann zeta function.

Proof for the case of $SL(2, \mathbb{R})$ We first prove the theorem for $SL(2, \mathbb{R})$. The more general result will follow by induction. Let $K = SO(2, \mathbb{R})$ denote the maximal compact subgroup of $SL(2, \mathbb{R})$. We use the Iwasawa decomposition which says that

$$SL(2, \mathbb{R})/K \cong \left\{ z = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \;\middle|\; x \in \mathbb{R},\ y > 0 \right\}.$$

Let $f : \mathbb{R}^2/K \to \mathbb{C}$ be an arbitrary smooth compactly supported function. Then, by definition, $f((u, v) \cdot k) = f((u, v))$ for all $(u, v) \in \mathbb{R}^2$ and all $k \in K$. We can define a function $F : SL(2, \mathbb{R})/K \to \mathbb{C}$ by letting

$$F(z) := \sum_{(m,n)\in\mathbb{Z}^2} f((m, n) \cdot z).$$

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL\,2(\mathbb{Z})$, then

$$\begin{aligned} F(\gamma z) &= \sum_{(m,n)\in\mathbb{Z}^2} f\left((m, n) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \right) \\ &= \sum_{(m,n)\in\mathbb{Z}^2} f\big((ma + nc, mb + nd) \cdot z\big) \\ &= F(z). \end{aligned}$$

Thus, $F(z)$ is $SL(2, \mathbb{Z})$–invariant.

Note that we may express

$$\{(m, n) \in \mathbb{Z}^2\} = (0, 0) \;\cup\; \left\{ \ell \cdot (0, 1) \cdot \gamma \;\middle|\; 0 < \ell \in \mathbb{Z},\ \gamma \in \Gamma_\infty \backslash SL(2, \mathbb{Z}) \right\}, \tag{1.6.2}$$

where

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix} \;\middle|\; r \in \mathbb{Z} \right\}.$$

We now integrate F over $\Gamma\backslash\mathfrak{h}^2$, where $\mathfrak{h}^2 = SL(2, \mathbb{R})/K$, $\Gamma = SL(2, \mathbb{Z})$, and $dxdy/y^2$ is the invariant measure on $\mathfrak{h}^2$ given in Proposition 1.5.3. It immediately follows from (1.6.2) that

$$\begin{aligned}
\int\limits_{\Gamma\backslash\mathfrak{h}^2} F(z)\,\frac{dxdy}{y^2} &= f((0,0))\cdot \mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2)\\
&\qquad + \sum_{\ell>0}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\ \int\limits_{\Gamma\backslash\mathfrak{h}^2} f\big(\ell(0,1)\cdot\gamma\cdot z\big)\,\frac{dxdy}{y^2}\\
&= f((0,0))\cdot \mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + 2\sum_{\ell>0}\ \int\limits_{\Gamma_\infty\backslash\mathfrak{h}^2} f\big(\ell(0,1)\cdot z\big)\,\frac{dxdy}{y^2}.
\end{aligned}$$

The factor 2 occurs because $\begin{pmatrix}-1 & \\ & -1\end{pmatrix}$ acts trivially on $\mathfrak{h}^2$. We easily observe that

$$f\big(\ell(0,1)\cdot z\big) = f\left(\ell(0,1)\cdot\begin{pmatrix} y^{\frac12} & 0\\ 0 & y^{-\frac12}\end{pmatrix}\right) = f\left(\left(0,\ell y^{-\frac12}\right)\right).$$

It follows, after making the elementary transformations

$$y\mapsto \ell^2 y,\qquad y\mapsto y^{-2}$$

that

$$\int\limits_{\Gamma\backslash\mathfrak{h}^2} F(z)\,\frac{dxdy}{y^2} = f((0,0))\cdot\mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + 2^2\zeta(2)\int\limits_0^\infty f((0,y))\,ydy. \quad (1.6.3)$$

Now, the function $f((u,v))$ is invariant under multiplication by $k\in K$ on the right. Since $\begin{pmatrix}\sin\theta & -\cos\theta\\ \cos\theta & \sin\theta\end{pmatrix}\in K$, we see that

$$f((0,y)) = f((y\cos\theta,\, y\sin\theta))$$

for any $0\le\theta\le 2\pi$. Consequently

$$\begin{aligned}
\int_0^\infty f((0,y))\,ydy &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^\infty f((y\cos\theta,\ y\sin\theta))\,d\theta\,ydy\\
&= \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} f((u,v))\,dudv\\
&= \frac{1}{2\pi}\hat{f}((0,0)). \qquad (1.6.4)
\end{aligned}$$

Here $\hat{f}$ denotes the Fourier transform of f in $\mathbb{R}^2$. If we now combine (1.6.3) and (1.6.4), we obtain

$$\int\limits_{\Gamma\backslash\mathfrak{h}^2} F(z)\,\frac{dxdy}{y^2} = f((0,0))\cdot \mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + \frac{2\zeta(2)}{\pi}\hat{f}((0,0)). \tag{1.6.5}$$

To complete the proof, we make use of the Poisson summation formula (see appendix) which states that for any $z \in GL(2,\mathbb{R})$

$$\begin{aligned} F(z) = \sum_{(m,n)\in\mathbb{Z}^2} f((m,n)z) &= \frac{1}{|\mathrm{Det}(z)|}\sum_{(m,n)\in\mathbb{Z}^2} \hat{f}\big((m,n)\cdot({}^tz)^{-1}\big) \\ &= \sum_{(m,n)\in\mathbb{Z}^2} \hat{f}\big((m,n)\cdot({}^tz)^{-1}\big), \end{aligned}$$

since $z = \begin{pmatrix} y^{\frac12} & y^{-\frac12}x \\ 0 & y^{-\frac12} \end{pmatrix}$ and $\mathrm{Det}(z) = 1$. We now repeat all our computations with the roles of f and $\hat{f}$ reversed. Since the group Γ is stable under transpose–inverse, one easily sees (from the Poisson summation formula above), by letting $z \mapsto ({}^tz)^{-1}$, that the integral

$$\int\limits_{\Gamma\backslash\mathfrak{h}^2} F(z)\,\frac{dxdy}{y^2}$$

is unchanged if we replace f by $\hat{f}$.

Also, since $\hat{\hat{f}}(x) = f(-x)$, the formula (1.6.5) now becomes

$$\int\limits_{\Gamma\backslash\mathfrak{h}^2} F(z)\,\frac{dxdy}{y^2} = \hat{f}((0,0))\cdot \mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) + \frac{2\zeta(2)}{\pi} f((0,0)). \tag{1.6.6}$$

If we combine (1.6.5) and (1.6.6) and solve for the volume, we obtain

$$\big(f((0,0)) - \hat{f}((0,0))\big)\cdot \mathrm{vol}(\Gamma\backslash\mathfrak{h}^2) = \big(f((0,0)) - \hat{f}((0,0))\big)\cdot\frac{2\zeta(2)}{\pi}.$$

Since f is arbitrary, we can choose f so that $f((0,0)) - \hat{f}((0,0)) \neq 0$. It follows that

$$\mathrm{Vol}(\Gamma\backslash\mathfrak{h}^2) = \frac{2\zeta(2)}{\pi} = \frac{\pi}{3}. \qquad \square$$

Proof for the case of $SL(n,\mathbb{R})$ We shall now complete the proof of Theorem 1.6.1 using induction on n. □

The proof of Theorem 1.6.1 requires two preliminary lemmas which we straightaway state and prove. For $n > 2$, let $U_n(\mathbb{R})$ (respectively $U_n(\mathbb{Z})$) denote the group of all matrices of the form

$$\begin{pmatrix} 1 & & & u_1 \\ & \ddots & & \vdots \\ & & 1 & u_{n-1} \\ & & & 1 \end{pmatrix}$$

with $u_i \in \mathbb{R}$ (respectively, $u_i \in \mathbb{Z}$), for $i = 1, 2, \ldots, n-1$.

Lemma 1.6.7 *Let $n > 2$ and fix an element $\gamma \in SL(n-1,\mathbb{Z})$. Consider the action of $U_n(\mathbb{Z})$ on $\mathbb{R}^{n-1}$ given by left matrix multiplication of $U_n(\mathbb{Z})$ on $\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \cdot U_n(\mathbb{R})$. Then a fundamental domain for this action is given by the set of all matrices*

$$\begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & & & u_1 \\ & \ddots & & \vdots \\ & & 1 & u_{n-1} \\ & & & 1 \end{pmatrix}$$

with $0 \le u_i < 1$ for $1 \le i \le n-1$. In particular,

$$U_n(\mathbb{Z}) \Big\backslash \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \cdot U_n(\mathbb{R}) \cong (\mathbb{Z}\backslash\mathbb{R})^{n-1}.$$

Proof of Lemma 1.6.7 Let m be a column vector with $(m_1, m_2, \ldots, m_{n-1})$ as entries. Then one easily checks that

$$\begin{pmatrix} I_{n-1} & m \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} = \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} I_{n-1} & \gamma^{-1} m \\ & 1 \end{pmatrix},$$

where I_{n-1} denotes the $(n-1) \times (n-1)$ identity matrix. It follows that

$$\begin{aligned} &\bigcup_{m \in \mathbb{Z}^{n-1}} \begin{pmatrix} I_{n-1} & m \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} I_{n-1} & (\mathbb{Z}\backslash\mathbb{R})^{n-1} \\ & 1 \end{pmatrix} \\ &\quad = \bigcup_{m \in \mathbb{Z}^{n-1}} \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} I_{n-1} & \gamma^{-1} m \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} I_{n-1} & (\mathbb{Z}\backslash\mathbb{R})^{n-1} \\ & 1 \end{pmatrix} \\ &\quad = \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot \bigcup_{m \in \mathbb{Z}^{n-1}} \begin{pmatrix} I_{n-1} & (\mathbb{Z}\backslash\mathbb{R})^{n-1} + \gamma^{-1} m \\ & 1 \end{pmatrix} \\ &\quad = \begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} \cdot U_n(\mathbb{R}). \end{aligned}$$

It is also clear that the above union is over non-overlapping sets. This is because $\gamma^{-1}\mathbb{Z}^{n-1} = \mathbb{Z}^{n-1}$ for $\gamma \in SL(n-1,\mathbb{Z})$. □

The second lemma we need is a generalization of the identity (1.6.4).

Lemma 1.6.8 *Let $n > 2$ and let $f : \mathbb{R}^n \to \mathbb{C}$ be a smooth function, with sufficient decay at ∞, which satisfies $f(u_1, \ldots, u_n) = f(v_1, \ldots, v_n)$ whenever $u_1^2 + \cdots + u_n^2 = v_1^2 + \cdots + v_n^2$. Then*

$$\begin{aligned} \int_0^\infty f(0, \ldots, 0, t)\, t^{n-1}\, dt &= \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{\mathbb{R}^n} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n \\ &= \frac{\hat{f}(0)}{\mathrm{Vol}(S^{n-1})}, \end{aligned}$$

where

$$\mathrm{Vol}(S^{n-1}) = \frac{2(\sqrt{\pi})^n}{\Gamma(n/2)}$$

denotes the volume of the $(n-1)$–dimensional sphere S^{n-1}.

Proof of Lemma 1.6.8 For $n \geq 2$ consider the spherical coordinates:

$$\begin{aligned} x_1 &= t \cdot \sin\theta_{n-1} \cdots \sin\theta_2 \sin\theta_1, \\ x_2 &= t \cdot \sin\theta_{n-1} \cdots \sin\theta_2 \cos\theta_1, \\ x_3 &= t \cdot \sin\theta_{n-1} \cdots \sin\theta_3 \cos\theta_2, \qquad (1.6.9) \\ &\vdots \\ x_{n-1} &= t \cdot \sin\theta_{n-1} \cos\theta_{n-2}, \\ x_n &= t \cdot \cos\theta_{n-1}, \end{aligned}$$

with

$$0 < t < \infty, \qquad 0 \leq \theta_1 < 2\pi, \qquad 0 \leq \theta_j < \pi, \ (1 < j < n).$$

Clearly $x_1^2 + \cdots + x_n^2 = t^2$. One may also show that the invariant measure on the sphere S^{n-1} is given by

$$d\mu(\theta) = \prod_{1 \leq j < n} (\sin\theta_j)^{j-1}\, d\theta_j,$$

and that $dx_1 dx_2 \cdots dx_n = t^{n-1} dt\ d\mu(\theta)$. Then the volume of the unit sphere, $\mathrm{Vol}(S^{n-1})$, is given by

$$\mathrm{Vol}(S^{n-1}) = \int_{S^{n-1}} d\mu(\theta) = \frac{2(\sqrt{\pi})^n}{\Gamma(n/2)}.$$

Since f is a rotationally invariant function, it follows that

$$f(0, \ldots, 0, t) = \frac{1}{\mathrm{Vol}(S^{n-1})} \int_{S^{n-1}} f(x_1, \ldots, x_n)\, d\mu(\theta)$$

with $x_1,\ldots,x_n$ given by (1.6.9). Consequently

$$\int_0^\infty f(0,\ldots,0,t)\,t^{n-1}\,dt = \frac{1}{\mathrm{Vol}(S^{n-1})}\int_0^\infty\int_{S^{n-1}} f(x_1,\ldots,x_n)t^{n-1}d\mu(\theta)dt$$
$$= \frac{1}{\mathrm{Vol}(S^{n-1})}\int_{\mathbb{R}^n} f(x_1,\ldots,x_n)\,dx_1\cdots dx_n.$$

□

We now return to the proof of Theorem 1.6.1. Let $K_n = SO(n,\mathbb{R})$ denote the maximal compact subgroup of $SL(n,\mathbb{R})$. In this case, the Iwasawa decomposition (Proposition 1.2.6) says that every $z \in SL(n,\mathbb{R})/K_n$ is of the form $z = xy$ with

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & & x_{1,n} \\ & 1 & x_{2,3} & \cdots & & x_{2,n} \\ & & \ddots & & & \vdots \\ & & & & 1 & x_{n-1,n} \\ & & & & & 1 \end{pmatrix},$$
$$y = \begin{pmatrix} y_1y_2\cdots y_{n-1}t & & & & \\ & y_1y_2\cdots y_{n-2}t & & & \\ & & \ddots & & \\ & & & y_1t & \\ & & & & t \end{pmatrix}, \tag{1.6.10}$$

with $t = \mathrm{Det}(y)^{-1/n} = \left(\prod_{i=1}^{n-1} y_i^{n-i}\right)^{-1/n}$.

In analogy to the previous proof for $SL(2,\mathbb{R})$ we let $f : \mathbb{R}^n/K_n \to \mathbb{C}$ be an arbitrary smooth compactly supported function. We shall also define a function $F : SL(n,\mathbb{R})/K_n \to \mathbb{C}$ by letting

$$F(z) := \sum_{m\in\mathbb{Z}^n} f(m\cdot z).$$

As before, the function $F(z)$ will be invariant under left multiplication by $SL(n,\mathbb{Z})$.

Let

$$P_n = \begin{pmatrix} & & * & & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \in SL(n,\mathbb{Z})$$

denote the set of all $n \times n$ matrices in $SL(n, \mathbb{Z})$ with last row $(0, 0, \dots, 0, 1)$. Let $\Gamma_n = SL(n, \mathbb{Z})$. Then we have as before:

$$F(z) = f(0) + \sum_{0<\ell\in\mathbb{Z}} \sum_{\gamma\in P_n\backslash\Gamma_n} f(\ell e_n \cdot \gamma \cdot z),$$

where $f(0)$ denotes $f((0, 0, \dots, 0))$ and $e_n = (0, 0, \dots, 0, 1)$.

We now integrate $F(z)$ over a fundamental domain for $\Gamma_n\backslash\mathfrak{h}^n$. It follows that

$$\int\limits_{\Gamma_n\backslash\mathfrak{h}^n} F(z)\, d^*z = f(0)\cdot \mathrm{Vol}(\Gamma_n\backslash\mathfrak{h}^n) + 2\sum_{\ell>0} \int\limits_{P_n\backslash\mathfrak{h}^n} f(\ell e_n \cdot z)\, d^*z. \tag{1.6.11}$$

The factor 2 occurs because $-I_n$ ($I_n = n \times n$ identity matrix) acts trivially on $\mathfrak{h}^n$. The computation of the integral above requires some preparations.

We may express $z \in \mathfrak{h}^n$ in the form

$$z = x \cdot \begin{pmatrix} y_1y_2\cdots y_{n-1}t & & & & \\ & y_1y_2\cdots y_{n-2}t & & & \\ & & \ddots & & \\ & & & y_1t & \\ & & & & t \end{pmatrix} \cdot \begin{pmatrix} t^{\frac{1}{n-1}}\cdot I_{n-1} & \\ & t^{-1} \end{pmatrix} \cdot \begin{pmatrix} t^{-\frac{1}{n-1}}\cdot I_{n-1} & \\ & t \end{pmatrix},$$

where x and t are given by (1.6.10). It follows that

$$\begin{aligned} z = {} & \begin{pmatrix} 1 & & & & x_{1,n} \\ & 1 & & & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n-1} & 0 \\ & 1 & x_{2,3} & \cdots & x_{2,n-1} & 0 \\ & & \ddots & & \vdots & \vdots \\ & & & & 1 & 0 \\ & & & & & 1 \end{pmatrix} \\ & \times \begin{pmatrix} y_1y_2\cdots y_{n-1}\cdot t^{\frac{n}{n-1}} & & & & \\ & y_1y_2\cdots y_{n-2}\cdot t^{\frac{n}{n-1}} & & & \\ & & \ddots & & \\ & & & y_1\cdot t^{\frac{n}{n-1}} & \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} t^{-\frac{1}{n-1}}\cdot I_{n-1} & \\ & t \end{pmatrix} \\ = {} & \begin{pmatrix} 1 & & & & x_{1,n} \\ & 1 & & & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} t^{-\frac{1}{n-1}}\cdot I_{n-1} & \\ & t \end{pmatrix}, \end{aligned} \tag{1.6.12}$$

where

$$z' = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n-1} \\ & 1 & x_{2,3} & \cdots & x_{2,n-1} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-2,n-1} \\ & & & & 1 \end{pmatrix}$$
$$\times \begin{pmatrix} y_1 y_2 \cdots y_{n-1} \cdot t^{\frac{n}{n-1}} & & & \\ & y_1 y_2 \cdots y_{n-2} \cdot t^{\frac{n}{n-1}} & & \\ & & \ddots & \\ & & & y_1 \cdot t^{\frac{n}{n-1}} \end{pmatrix}.$$

Now z' represents the Iwasawa coordinate for $SL(n-1,\mathbb{R})/SO(n-1,\mathbb{R}) = \mathfrak{h}^{n-1}$, and the Haar measure d^*z' can be computed using Proposition 1.5.3 and is given by

$$d^*z' = \prod_{1\le i<j\le n-1} dx_{i,j} \prod_{k=1}^{n-2} y_{k+1}^{-k(n-1-k)-1} dy_{k+1}.$$

If we compare this with

$$\begin{aligned} d^*z &= \prod_{1\le i<j\le n} dx_{i,j} \prod_{k=1}^{n-1} y_k^{-k(n-k)-1} dy_k \\ &= \prod_{1\le i<j\le n} dx_{i,j} \prod_{k=0}^{n-2} y_{k+1}^{-(k+1)(n-1-k)-1} dy_k, \end{aligned}$$

we see that

$$d^*z = d^*z' \prod_{j=1}^{n-1} dx_{j,n}\ t^n\ \frac{dy_1}{y_1}. \tag{1.6.13}$$

Here, the product of differentials is understood as a wedge product satisfying the usual rule: $du \wedge du = 0$, given by the theory of differential forms. Since

$$t = y_1^{-(n-1)/n} \prod_{i=2}^{n-1} y_i^{-(n-i)/n},$$

we see that

$$\frac{dt}{t} = -\frac{n-1}{n}\frac{dy_1}{y_1} + \Omega,$$

where Ω is a differential form involving dy_j for each $j = 2, 3, \ldots, n-1$, but