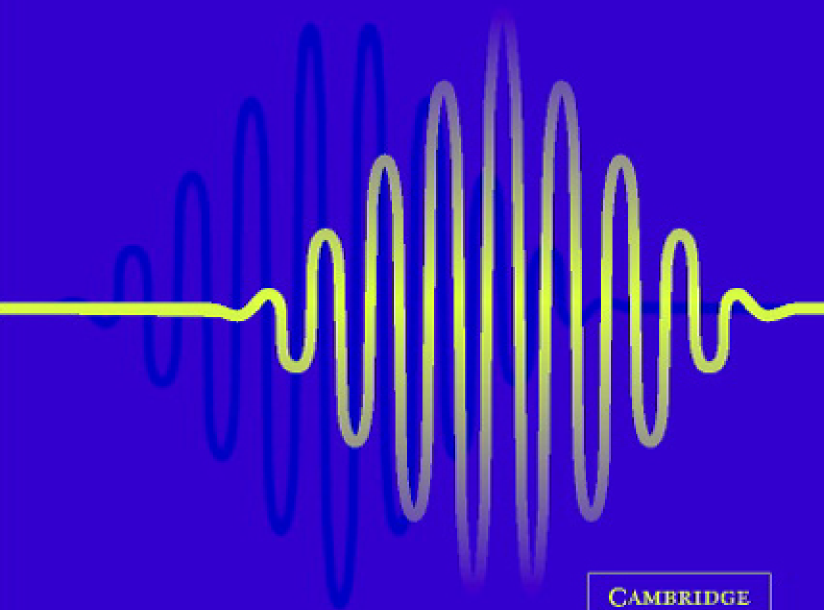
J F James

A Student's Guide to Fourier Transforms

With Applications in Physics and Engineering



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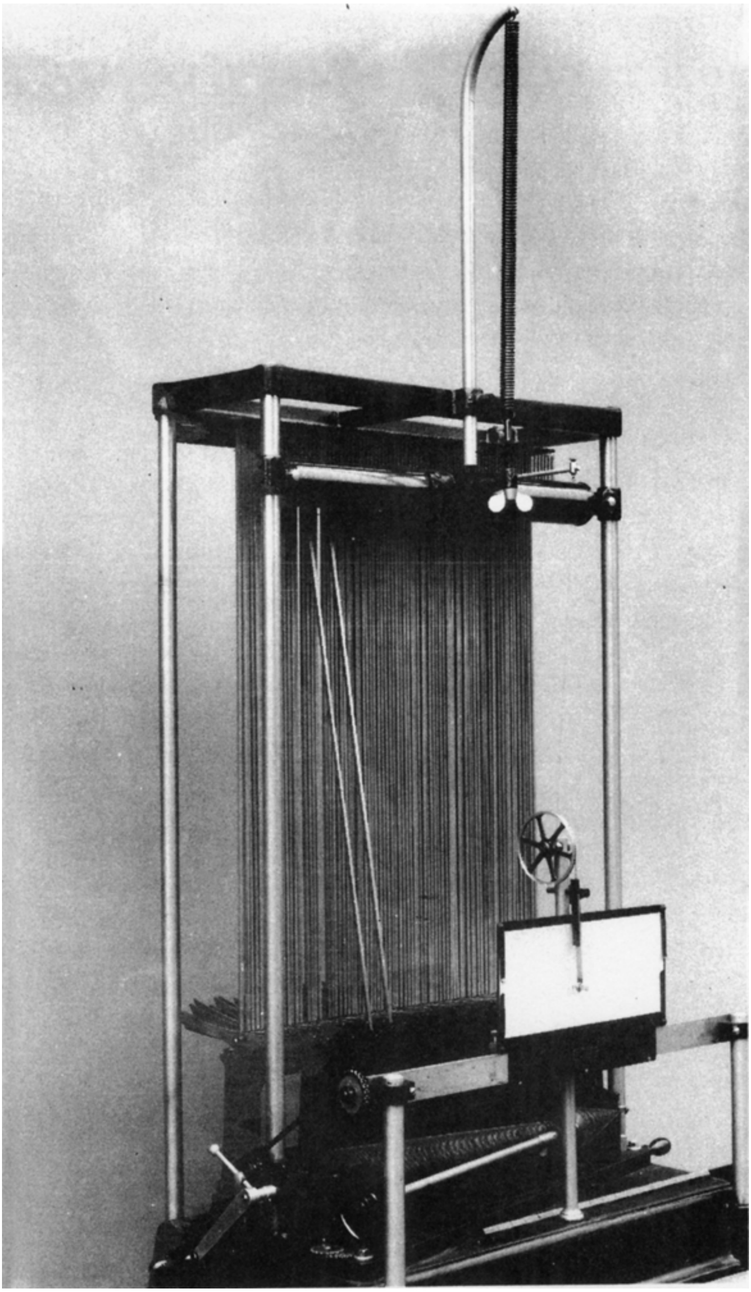
A Student's Guide to Fourier Transforms

Fourier transform theory is of central importance in a vast range of applications in physical science, engineering, and applied mathematics. This new edition of a successful undergraduate text provides a concise introduction to the theory and practice of Fourier transforms, using qualitative arguments wherever possible and avoiding unnecessary mathematics.

After a brief description of the basic ideas and theorems, the power of the technique is then illustrated by referring to particular applications in optics, spectroscopy, electronics and telecommunications. The rarely discussed but important field of multi-dimensional Fourier theory is covered, including a description of computer-aided tomography (CAT-scanning). The final chapter discusses digital methods, with particular attention to the fast Fourier transform. Throughout, discussion of these applications is reinforced by the inclusion of worked examples.

The book assumes no previous knowledge of the subject, and will be invaluable to students of physics, electrical and electronic engineering, and computer science.

JOHN JAMES has held teaching positions at the University of Minnesota, the Queen's University Belfast and the University of Manchester, retiring as Senior Lecturer in 1996. He is currently an Honorary Research Fellow at the University of Glasgow, a Fellow of the Royal Astronomical Society and Member of the Optical Society of America. His research interests include the invention, design and construction of astronomical instruments and their use in astronomy, cosmology and upper-atmosphere. Dr James has led eclipse expeditions to Central America, the Central Sahara, Java and the South Pacific islands. He is the author of about 40 academic papers and co-author with R. S. Sternberg of *The Design of Optical Spectrometers* (Chapman & Hall, 1969).



The Harmonic integrator, designed by Michelson and Stratton (see p. 72). This was the earliest mechanical Fourier transformer, built by Gaertner & Co. of Chicago in 1898. (Reproduced by permission of The Science Museum/Science & Society Picture Library.)

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Second Edition

J. F. JAMES

Honorary Research Fellow,
The University of Glasgow

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Preface to the first edition

Showing a Fourier transform to a physics student generally produces the same reaction as showing a crucifix to Count Dracula. This may be because the subject tends to be taught by theorists who themselves use Fourier methods to solve otherwise intractable differential equations. The result is often a heavy load of mathematical analysis.

This need not be so. Engineers and practical physicists use Fourier theory in quite another way: to treat experimental data, to extract information from noisy signals, to design electrical filters, to 'clean' TV pictures and for many similar practical tasks. The transforms are done digitally and there is a minimum of mathematics involved.

The chief tools of the trade are the theorems in Chapter 2, and an easy familiarity with these is the way to mastery of the subject. In spite of the forest of integration signs throughout the book there is in fact very little integration done and most of that is at high-school level. There are one or two excursions in places to show the breadth of power that the method can give. These are not pursued to any length but are intended to whet the appetite of those who want to follow more theoretical paths.

The book is deliberately incomplete. Many topics are missing and there is no attempt to explain everything: but I have left, here and there, what I hope are tempting clues to stimulate the reader into looking further; and of course, there is a bibliography at the end.

Practical scientists sometimes treat mathematics in general and Fourier theory in particular, in ways quite different from those for which it was invented[1]. The late E. T. Bell, mathematician and writer on mathematics, once described mathematics in a famous book title as 'The Queen and Servant of Science'.

[1] It is a matter of philosophical disputation whether mathematics is invented or discovered. Let us compromise by saying that theorems are discovered; proofs are invented.

The queen appears here in her role as servant and is sometimes treated quite roughly in that role, and furthermore, without apology. We are fairly safe in the knowledge that mathematical functions which describe phenomena in the real world are 'well-behaved' in the mathematical sense. Nature abhors singularities as much as she does a vacuum.

When an equation has several solutions, some are discarded in a most cavalier fashion as 'unphysical'. This is usally quite right[2]. Mathematics is after all only a concise shorthand description of the world and if a position-finding calculation based, say, on trigonometry and stellar observations, gives two results, equally valid, that you are either in Greenland or Barbados, you are entitled to discard one of the solutions if it is snowing outside. So we use Fourier transforms as a guide to what is happening or what to do next, but we remember that for solving practical problems the blackboard-and-chalk diagram, the computer screen and the simple theorems described here are to be preferred to the precise tedious calculations of integrals.

Manchester, January 1994 J. F. James

[2] But Dirac's Equation, with its positive and negative roots, predicted the positron.

Preface to the second edition

This edition follows much advice and constructive criticism which the author has received from all quarters of globe, in consequence of which various typos and misprints have been corrected and some ambiguous statements and anfractuosities have been replaced by more clear and direct derivations. Chapter 7 has been largely rewritten to demonstrate the way in which Fourier transforms are used in CAT-scanning, an application of more than usual ingenuity and importance: but overall this edition represents a renewed effort to rescue Fourier transforms from the clutches of the pure mathematicians and present them as a working tool to the horny-handed toilers who strive in the fields of electronic engineering and experimental physics.

Glasgow, January 2001 J. F. James

Chapter 1

Physics and Fourier transforms

1.1 The qualitative approach

Ninety percent of all physics is concerned with vibrations and waves of one sort or another. The same basic thread runs through most branches of physical science, from accoustics through engineering, fluid mechanics, optics, electromagnetic theory and X-rays to quantum mechanics and information theory. It is closely bound to the idea of a *signal* and its *spectrum*. To take a simple example: imagine an experiment in which a musician plays a steady note on a trumpet or a violin, and a microphone produces a voltage proportional to the the instantaneous air pressure. An oscilloscope will display a graph of pressure against time, $F(t)$, which is periodic. The reciprocal of the period is the frequency of the note, 256 Hz, say, for a well-tempered middle C.

The waveform is not a pure sinusoid, and it would be boring and colourless if it were. It contains 'harmonics' or 'overtones': multiples of the fundamental frequency, with various amplitudes and in various phases[1], depending on the timbre of the note, the type of instrument being played and on the player. The waveform can be *analysed* to find the amplitudes of the overtones, and a list can be made of the amplitudes and phases of the sinusoids which it comprises. Alternatively a graph, $A(\nu)$, can be plotted (the sound-spectrum) of the amplitudes against frequency.

$A(\nu)$ is the Fourier transform of *F(t)*.

Actually it is the *modular* transform, but at this stage that is a detail.

Suppose that the sound is not periodic – a squawk, a drumbeat or a crash instead of a pure note. Then to describe it requires not just a set of overtones

[1] 'phase' here is an angle, used to define the 'retardation' of one wave or vibration with respect to another. One wavelength retardation for example, is equivalent to a phase difference of 2π. Each harmonic will have its own phase, ϕ_m, indicating its position within the period.

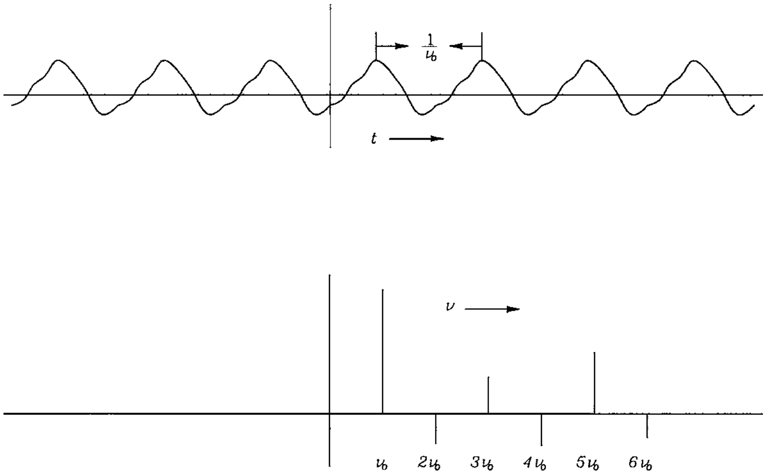

Fig. 1.1. The spectrum of a steady note: fundamental and overtones.

with their amplitudes, but a continuous range of frequencies, each present in an infinitesimal amount. The two curves would then look like Fig. 1.2.

The uses of a Fourier transform can be imagined: the identification of a valuable violin; the analysis of the sound of an aero-engine to detect a faulty gear-wheel; of an electrocardiogram to detect a heart defect; of the light curve of a periodic variable star to determine the underlying physical causes of the variation: all these are current applications of Fourier transforms.

1.2 Fourier series

For a steady note the description requires only the fundamental frequency, its amplitude and the amplitudes of its harmonics. A discrete sum is sufficient. We could write:

$$\begin{aligned}F(t) = {} & a_0 + a_1 \cos 2\pi\nu_0 t + b_1 \sin 2\pi\nu_0 t + a_2 \cos 4\pi\nu_0 t + b_2 \sin 4\pi\nu_0 t \\ & + a_3 \cos 6\pi\nu_0 t + \cdots\end{aligned}$$

where ν_0 is the fundamental frequency of the note. Sines as well as cosines are required because the harmonics are not necessarily ‘in step’ (i.e. ‘in phase’) with the fundamental or with each other.

More formally:

$$F(t) = \sum_{n=-\infty}^{\infty} a_n \cos(2\pi n\nu_0 t) + b_n \sin(2\pi n\nu_0 t) \qquad (1.1)$$

and the sum is taken from $-\infty$ to ∞ for the sake of mathematical symmetry.

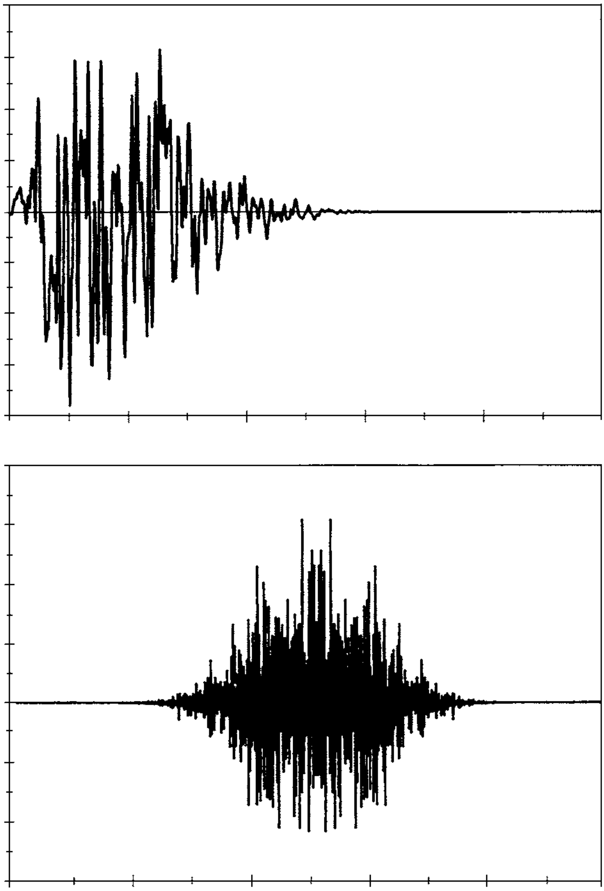

Fig. 1.2. The spectrum of a crash: all frequencies are present.

This process of constructing a waveform by adding together a fundamental frequency and overtones or harmonics of various amplitudes, is called Fourier synthesis.

There are alternative ways of writing this expression: since $\cos x = \cos(-x)$ and $\sin x = -\sin(-x)$ we can write:

$$F(t) = A_0/2 + \sum_{n=1}^{\infty} A_n \cos(2\pi n\nu_0 t) + B_n \sin(2\pi n\nu_0 t) \tag{1.2}$$

and the two expressions are identical provided that we set $A_n = a_{-n} + a_n$ and $B_n = b_n - b_{-n}$. A_0 is divided by two to avoid counting it twice: as it is, A_0 can be found by the same formula that will be used to find all the A_n's.