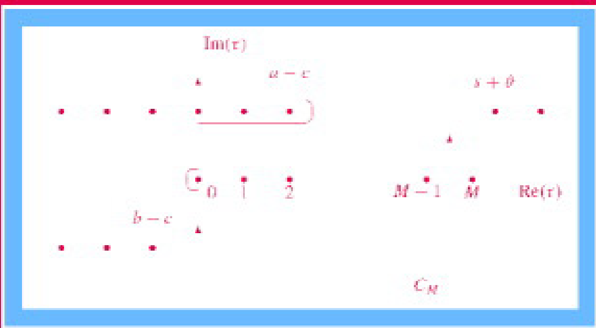
ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS 85

ASYMPTOTICS AND MELLIN-BARNES INTEGRALS



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Asymptotics and Mellin-Barnes Integrals is a comprehensive account of the properties of Mellin-Barnes integrals and their application to problems involving special functions, primarily the determination of asymptotic expansions. An account of the basic analytical properties of Mellin-Barnes integrals and Mellin transforms and their use in applications ranging from number theory to differential and difference equations is followed by a systematic analysis of the asymptotics of Mellin-Barnes representations of many important special functions, including hypergeometric, Bessel and parabolic cylinder functions. An account of the recent developments in the understanding of the Stokes phenomenon and of hyperasymptotics in the setting of Mellin-Barnes integrals ensues. The book concludes with the application of ideas set forth in the earlier parts of the book to higher-dimensional Laplace-type integrals and sophisticated treatments of Euler-Jacobi series, the Riemann zeta function and the Pearcey integral. Detailed numerical illustrations accompany many of the results developed in the text.

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Asymptotics and Mellin-Barnes Integrals

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Preface

Mellin-Barnes integrals are characterised by integrands involving one or more gamma functions (and possibly simple trigonometric or other functions) with integration contours that thread their way around sequences of poles of the integrands. They are a powerful tool in the development of asymptotic expansions of functions defined by integrals, sums or differential equations and, combined with the closely related Mellin transform, form an important part of the toolkit of any practising analyst. The great utility of these integrals resides in the facts that the asymptotic behaviour near the origin and at infinity of the function being represented is related to the singularity structure in the complex plane of the resulting integrand and to the inherent flexibility associated with deformation of the contour of integration over subsets of these singularities.

It is a principal aim of this book to describe the theory of these integrals and to illustrate their power and usefulness in asymptotic analysis. Mellin-Barnes integrals have their early history bound up in the study of hypergeometric functions of the late nineteenth and early twentieth centuries. This association has lent a classical feel to their use and in the domain of asymptotic analysis, the account of their utility in other works has largely been restricted to the analysis of special sums or their role in inversion of Mellin transforms. For their part, Mellin transforms have appeared in several settings within mathematics, as far back as Riemann's memoir on the distribution of primes, and continue to see application through to the present day.

This work gathers a detailed account of the asymptotic analysis of Mellin-Barnes integrals and, conversely, the use of Mellin-Barnes integral representations to problems in asymptotics, from basic results involving their early application to hypergeometric functions, to work that is still appearing at the beginning of this new century. Our account differs from earlier work in the latter half of the twentieth century. For example, texts such as those by Sneddon, *The Use of Integral*

Transforms (1972), and by Davies, *Integral Transforms and Their Application* (1978), are primarily concerned with Mellin transforms and their use in the construction of solutions to differential equations. The well-known monograph by Copson, *Asymptotic Expansions* (1965), barely mentions Mellin-Barnes integrals, and that by Olver, *Asymptotics and Special Functions* (1974), makes little use of them outside of the problem of determining the asymptotics of sums of special type. Mellin-Barnes integrals are more prominently employed in the accounts of Bleistein and Handelsman, *Asymptotic Expansion of Integrals* (1975), of Wong, *Asymptotic Approximation of Integrals* (1989) and of Marichev, *Handbook of Integral Transforms and Higher Transcendental Functions: Theory and Algorithmic Tables* (1982), but there the roles are primarily confined to consequences of the Parseval formula for Mellin transforms. The classic texts on analysis by Whittaker and Watson, *Modern Analysis* (1965), and Copson, *Theory of Functions of a Complex Variable* (1935), and *Generalised Hypergeometric Functions* by Slater (1960), include important sections describing the development of asymptotic expansions of functions represented by Mellin-Barnes integrals. The monograph by Paris and Wood, *Asymptotics of High Order Differential Equations* (1986), contains much of the foundations of the analysis found here, but in more limited scope, and restricted to solutions of differential equations of a particular type. We believe that this present volume, then, is to date the most comprehensive account of Mellin-Barnes integrals and their interactions with asymptotics.

Additionally, this work liberally employs numerical studies to better display the calibre of the asymptotic approximations obtained, a strategy we feel gives the non-expert practitioner a good sense of the concept or method being showcased. A wide-ranging collection of special functions is used to illustrate the ideas under discussion in the fine tradition of the texts mentioned earlier, from Bessel and parabolic cylinder functions, to more exotic functions such as the Mittag-Leffler function and a Riemann-Siegel type of expansion of the zeta function. This book should be accessible to anyone with a solid undergraduate background in functions of a single complex variable.

The book begins with a brief foray into general notions common in asymptotic analysis, and illustrated with the asymptotic behaviour of some classical (and more recent) special functions. The main tools employed in the asymptotics of integrals are found here, including Watson's lemma and the method of steepest descent. Also present is a description of the notion of optimal truncation, which plays a significant role later in the discussion of hyperasymptotics. Brief historical sketches of the namesakes of the type of integrals under examination round out the introductory chapter.

Basic results pertaining to Mellin-Barnes integrals and Mellin transforms are detailed in the next three chapters. Since rational functions of the gamma function are to be found in almost every Mellin-Barnes integral, a thorough account of the behaviour of these rational functions is provided, along with convergence rules for Mellin-Barnes integrals and error estimates for expansions of ratios of gamma functions that occur throughout the remainder of the monograph. Mellin transforms and their properties follow, with applications to the evaluation of slowly convergent sums, number-theoretic sums, and also to differential, integral and difference equations. While these latter applications are not strictly speaking necessarily concerned with asymptotics, they add to the value of the volume and, hopefully, render it more useful as a reference.

The theme of asymptotics comes to the fore in the remaining chapters. In the fifth chapter, a careful and systematic analysis is undertaken which extracts both algebraic and exponential behaviours of Mellin-Barnes-type integrals, with attention paid to the errors committed in the approximation process. These methods are illustrated in the settings of several classical special functions, and the calibre of the approximations illuminated with numerical comparisons. An account of the Stokes phenomenon ensues in the setting of Bessel functions, and the reader is drawn into a detailed account of the recent theory of hyperasymptotics applied to the confluent hypergeometric functions (which incorporate many of the commonly used special functions). An illustration of this theory is made to the exponentially-improved asymptotics of the gamma function and amplified by a study of the numerics of this new expansion.

The penultimate chapter illustrates the manner in which Mellin-Barnes-type integrals may be successfully deployed to extract the algebraic asymptotic behaviour of multidimensional Laplace-type integrals in a systematic manner, and further, interpret the results geometrically. The monograph closes with sophisticated applications of the ideas developed in the text to three particular problems: the determination of the asymptotics of the generalised Euler-Jacobi series, expansions for the zeta function on the critical line and the Pearcey integral, a two-variable generalisation of the classical Airy function.

There is much in this book that is encyclopaedic, but much also is of recent vintage – a good deal of the mathematics present is less than a decade old, and continues to develop apace. We feel we have captured the most important tools and techniques surrounding the analysis and asymptotics of Mellin-Barnes integrals, and by gathering them in a single source, have made the task of their continued application to both mathematics and physical science a more tractable and, we hope, interesting affair.

The authors gratefully acknowledge the long-suffering forbearance of their respective wives, Jocelyne and Laurie, during the lengthy duration of this project. The authors also acknowledge the support of their institutions, the Universities of Abertay Dundee and of Lethbridge, and in the case of the second author, the research funding made available by the Natural Sciences and Engineering Research Council of Canada, which underwrote some of the investigations reported on in this volume. We also owe a considerable debt of gratitude to J. Boersma of Eindhoven University of Technology for the meticulous care with which he studied the text and

for his many critical comments on all but two of the chapters, which have improved the calibre of the volume before you. In spite of our best efforts, however, it is certain that some errors and misprints are bound to have crept into the text, and we ask for the reader's forgiveness for those that prove to be vexing.

R. B. Paris and D. Kaminski

1

Introduction

1.1 Introduction to Asymptotics

Before venturing into our examination of Mellin-Barnes integrals, we present an overview of some of the basic definitions and ideas found in asymptotic analysis. The treatment provided here is not intended to be comprehensive, and several high quality references exist which can provide a more complete treatment than is given here: in particular, we recommend the tracts by Olver (1974), Bleistein & Handelsman (1975) and Wong (1989) as particularly good treatments of asymptotic analysis, each with their own strengths.†

1.1.1 Order Relations

Let us begin our survey by defining the *Landau symbols* O and o and the notion of asymptotic equality.

Let f and g be two functions defined in a neighbourhood of x_0. We say that $f(x) = O(g(x))$ as $x \to x_0$ if there is a constant M for which

$$|f(x)| \leq M\,|g(x)|$$

for x sufficiently close to x_0. The constant M depends only on how close to x_0 we wish the bound to hold. The notation $O(g)$ is read as 'big-oh of g', and the constant M, which is often not explicitly calculated, is termed the *implied constant.*

In a similar fashion, we define $f(x) = o(g(x))$ as $x \to x_0$ to mean that

$$|f(x)/g(x)| \to 0$$

† Olver provides a good balance between techniques used in both integrals and differential equations; Bleistein & Handelsman present a relatively unified treatment of integrals through the use of Mellin convolutions; and Wong develops the theory and application of (Schwartz) distributions in the setting of developing expansions of integrals.

as $x \to x_0$, subject to the proviso that $g(x)$ be nonzero in a neighbourhood of x_0. The expression $o(g)$ is read as 'little-oh of g', and from the preceding definition, it is immediate that $f = o(g)$ implies that $f = O(g)$ (merely take the implied constant to be any (arbitrarily small) positive number).

The last primitive asymptotic notion required is that of asymptotic equality. We write

$$f(x) \sim g(x)$$

as $x \to x_0$ to mean that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1,$$

provided, of course, that g is nonzero sufficiently close to x_0. The tilde here is read 'is asymptotically equal to'. An equivalent formulation of asymptotic equality is readily available: for $x \to x_0$,

$$f(x) \sim g(x) \qquad \text{iff} \qquad f(x) = g(x)\{1 + o(1)\}.$$

Example 1. The function $\log x$ satisfies the order relation $\log x = O(x - 1)$ as $x \to \infty$, since the ratio $(\log x)/(x - 1)$ is bounded for all large x. In fact, it is also true that $\log x = o(x - 1)$ for large x, and for $x \to 1$, $\log x \sim x - 1$.

Example 2. Stirling's formula is a well-known asymptotic equality. For large n, we have

$$n! \sim (2\pi)^{\frac{1}{2}} e^{-n} n^{n+\frac{1}{2}}.$$

This result follows from the asymptotic expansion of the gamma function, a result carefully developed in §2.1.

Example 3. The celebrated Prime Number Theorem is an asymptotic equality. If we denote by $\pi(x)$ the number of primes less than or equal to x, then for large positive x we have the well-known result

$$\pi(x) \sim \frac{x}{\log x}.$$

With the aid of Gauss' logarithmic integral,†

$$\mathrm{li}(x) = \int_2^x \frac{dt}{\log t}$$

we also have the somewhat more accurate form

$$\pi(x) \sim \mathrm{li}(x) \qquad (x \to \infty).$$

† We note here that $\mathrm{li}(x)$ is also used to denote the same integral, but taken over the interval $(0, x)$, with $x > 1$. With this larger interval, the integral is a Cauchy principal value integral. The notation in this example appears to be in use by some number theorists, and is also sometimes written $\mathrm{Li}(x)$.

That both forms hold can be seen from a simple integration by parts:

$$\mathrm{li}(x) = \frac{x}{\log x} - \frac{2}{\log 2} + \int_2^x \frac{dt}{(\log t)^2}.$$

An application of l'Hôpital's rule reveals that the resulting integral on the right-hand side is $o(x/\log x)$, from which the $x/\log x$ form of the Prime Number Theorem follows.

A number of useful relationships exist for manipulating the Landau symbols. The following selections are all easily obtained from the above definitions, and are not established here:

$$\begin{array}{ll} \text{(a) } O(O(f)) = O(f) & \text{(e) } O(f) + O(f) = O(f) \\ \text{(b) } o(o(f)) = o(f) & \text{(f) } o(f) + o(f) = o(f) \\ \text{(c) } O(fg) = O(f) \cdot O(g) & \text{(g) } o(f) + O(f) = O(f) \\ \text{(d) } O(f) \cdot o(g) = o(fg) & \text{(h) } O(o(f)) = o(O(f)) = o(f). \end{array} \tag{1.1.1}$$

It is easy to deduce linearity of Landau symbols using these properties, and it is a simple matter to establish asymptotic equality as an equivalence relation. In the transition to calculus, however, some difficulties surface.

A moment's consideration reveals that differentiation is, in general, often badly behaved in the sense that if $f = O(g)$, then it does not necessarily follow that $f' = O(g')$, as the example $f(x) = x + \sin e^x$ aptly illustrates: for large, real x, we have $f = O(x)$, but the derivative of f is not bounded (i.e., not $O(1)$).

The situation for integration is a good deal better. It is possible to formulate many results concerning integrals of order estimates, but we content ourselves with just two.

Example 4. For functions f and g of a real variable x satisfying $f = O(g)$ as $x \to x_0$ on the real line, we have

$$\int_{x_0}^{x} f(t)\,dt = O\left(\int_{x_0}^{x} |g(t)|\,dt\right) \qquad (x \to x_0).$$

A proof can be fashioned along the following lines: for $f(t) = O(g(t))$, let M be the implied constant so that $|f(t)| \leq M\,|g(t)|$ for t sufficiently close to x_0, say $|t - x_0| \leq \eta$. (For $x_0 = \infty$, a suitable interval would be $t \geq N$ for some large positive N.) Then

$$-M\,|g(t)| \leq f(t) \leq M\,|g(t)| \qquad (|t - x_0| \leq \eta),$$

whence the result follows upon integration.

Example 5. If f is an integrable function of a real variable x, and $f(x) \sim x^\nu$, $\mathrm{Re}(\nu) < -1$ as $x \to \infty$, then

$$\int_x^\infty f(t)\,dt \sim -\frac{x^{\nu+1}}{\nu+1} \qquad (x \to \infty).$$

A proof of this claim follows from $f(x) = x^\nu\{1 + \psi(x)\}$ where $\psi(x) = o(1)$ as $x \to \infty$, for then

$$\int_x^\infty f(t)\,dt = -\frac{x^{\nu+1}}{\nu+1} + \int_x^\infty t^\nu \psi(t)\,dt.$$

But $\psi(t) = o(1)$ implies that for $\epsilon > 0$ arbitrarily small, there is an $x_0 > 0$ for which $|\psi(t)| < \epsilon$ whenever $t > x_0$. Thus, the remaining integral may be bounded as

$$\left|\int_x^\infty t^\nu \psi(t)\,dt\right| < \epsilon \int_x^\infty |t^\nu|\,dt \qquad (x > x_0).$$

Accordingly, we find

$$\int_x^\infty f(t)\,dt = -\frac{x^{\nu+1}}{\nu+1} + o\left(\frac{x^{\nu+1}}{\nu+1}\right) = -\frac{x^{\nu+1}}{\nu+1}\{1 + o(1)\},$$

from which the asymptotic equality is immediate. □

It is in the complex plane that we find differentiation of order estimates becomes better behaved. This is due, in part, to the fact that the Cauchy integral theorem allows us to represent holomorphic functions as integrals which, as we have noted, are better behaved in the setting of Landau symbols. A standard result in this direction is the following:

Lemma 1.1. *Let f be holomorphic in a region containing the closed annular sector $\mathcal{S} = \{z : \alpha \le \arg(z - z_0) \le \beta, |z - z_0| \ge R \ge 0\}$, and suppose $f(z) = O(z^\nu)$ (resp. $f(z) = o(z^\nu)$) as $z \to \infty$ in the sector, for fixed real ν. Then $f^{(n)}(z) = O(z^{\nu-n})$ (resp. $f^{(n)} = o(z^{\nu-n})$) as $z \to \infty$ in any closed annular sector properly interior to $\mathcal{S}$ with common vertex z_0.*

The proof of this result follows from the Cauchy integral formula for $f^{(n)}$, and is available in Olver (1974, p. 9).

1.1.2 Asymptotic Expansions

Let a sequence of continuous functions $\{\phi_n\}$, $n = 0, 1, 2, \ldots$, be defined on some domain, and let x_0 be a (possibly infinite) limit point of this domain. The sequence $\{\phi_n\}$ is termed an *asymptotic scale* if it happens that $\phi_{n+1}(x) = o(\phi_n(x))$ as $x \to x_0$, for every n. If f is some continuous function on the common domain of the asymptotic scale, then by an (infinite) *asymptotic expansion* of f with respect to the asymptotic scale $\{\phi_n\}$ is meant the formal series $\sum_{n=0}^\infty a_n\phi_n(x)$, provided the coefficients a_n, independent of x, are chosen so that for any nonnegative integer N,

$$f(x) = \sum_{n=0}^{N} a_n\phi_n(x) + O(\phi_{N+1}(x)) \qquad (x \to x_0). \tag{1.1.2}$$

In this case we write

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \qquad (x \to x_0).$$

Such a formal series is uniquely determined in view of the fact that the coefficients a_n can be computed from

$$a_N = \lim_{x \to x_0} \frac{1}{\phi_N(x)} \left\{ f(x) - \sum_{n=0}^{N-1} a_n \phi_n(x) \right\} \qquad (N = 0, 1, 2, \ldots).$$

The formal series so obtained is also referred to as an asymptotic expansion of *Poincaré type*, or an asymptotic expansion in the sense of Poincaré or, more simply, a Poincaré expansion. Examples of asymptotic scales and asymptotic expansions built with them are easy to come by. The most commonplace is the asymptotic power series: an *asymptotic power series* is a formal series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^{\nu_n},$$

where the appropriate asymptotic scale is the sequence $\{(x - x_0)^{\nu_n}\}$, $n = 0, 1, 2, \ldots$, and the ν_n are constants for which $(x - x_0)^{\nu_{n+1}} = o\big((x - x_0)^{\nu_n}\big)$ as $x \to x_0$. Any convergent Taylor series expansion of an analytic function f serves as an example of an asymptotic power series, with x_0 a point in the domain of analyticity of f, $\nu_n = n$ for any nonnegative integer n, and the coefficients in the expansion are the familiar Taylor coefficients $a_n = f^{(n)}(x_0)/n!$.

Asymptotic expansions, however, need not be convergent, as the next two examples illustrate.

Example 1. WATSON'S LEMMA. A well-known result of Laplace transform theory is that the Laplace transform of a piecewise continuous function on the interval $[0, +\infty)$ is $o(1)$ as the transform variable grows without bound. By imposing more structure on the small parameter behaviour of the function being transformed, a good deal more can be said about the growth at infinity of the transform.

Lemma 1.2. *Let $g(t)$ be an integrable function of the variable $t > 0$ with asymptotic expansion*

$$g(t) \sim \sum_{n=0}^{\infty} a_n t^{(n+\lambda-\mu)/\mu} \quad (t \to 0+)$$

for some constants $\lambda > 0$, $\mu > 0$. Then, provided the integral converges for all sufficiently large x, the Laplace transform of g, $\mathcal{L}[g; x]$, has the asymptotic behaviour

$$\mathcal{L}[g; x] \equiv \int_0^{\infty} e^{-xt} g(t)\, dt \sim \sum_{n=0}^{\infty} \Gamma\left(\frac{n+\lambda}{\mu}\right) \frac{a_n}{x^{(n+\lambda)/\mu}} \quad (x \to \infty).$$

Proof. To see this, let us put, for positive integer N and $t > 0$,

$$g_N(t) = g(t) - \sum_{n=0}^{N-1} a_n t^{(n+\lambda-\mu)/\mu}$$

so that the Laplace transform has a finite expansion with remainder given by

$$\mathcal{L}[g; x] = \sum_{n=0}^{N-1} \Gamma\left(\frac{n+\lambda}{\mu}\right) \frac{a_n}{x^{(n+\lambda)/\mu}} + \int_0^\infty e^{-xt} g_N(t)\, dt. \tag{1.1.3}$$

Since $g_N(t) = O(t^{(N+\lambda-\mu)/\mu})$, there are constants K_N and t_N for which

$$|g_N(t)| \le K_N\, t^{(N+\lambda-\mu)/\mu} \quad (0 < t \le t_N).$$

Use of this in the remainder term in our finite expansion (1.1.3) allows us to write

$$\begin{aligned} \left| \int_0^{t_N} e^{-xt} g_N(t)\, dt \right| &\le K_N \int_0^{t_N} e^{-xt} t^{(N+\lambda-\mu)/\mu} dt \\ &< \Gamma\left(\frac{N+\lambda}{\mu}\right) \frac{K_N}{x^{(N+\lambda)/\mu}}. \end{aligned} \tag{1.1.4}$$

By hypothesis, $\mathcal{L}[g; x]$ exists for all sufficiently large x, so the Laplace transform of g_N must also exist for all sufficiently large x, by virtue of (1.1.3). Let X be such that $\mathcal{L}[g_N; x]$ exists for all $x \ge X$, and put

$$G_N(t) = \int_{t_N}^{t} e^{-Xv} g_N(v)\, dv.$$

The function G_N so defined is a bounded continuous function on $[t_N, \infty)$, whence the bound

$$L_N = \sup_{[t_N, \infty)} |G_N(t)|$$

exists. Then for $x > X$, we have

$$\begin{aligned} \int_{t_N}^\infty e^{-xt} g_N(t)\, dt &= \int_{t_N}^\infty e^{-(x-X)t} e^{-Xt} g_N(t)\, dt \\ &= (x - X) \int_{t_N}^\infty e^{-(x-X)t} G_N(t)\, dt \end{aligned}$$

after one integration by parts. After applying the uniform bound L_N to the integral that remains, we arrive at

$$\left| \int_{t_N}^\infty e^{-xt} g_N(t)\, dt \right| \le (x - X) L_N \int_{t_N}^\infty e^{-(x-X)t}\, dt = L_N e^{-(x-X)t_N} \tag{1.1.5}$$

for $x > X$.

Together, (1.1.4) and (1.1.5) yield

$$\left|\int_0^\infty e^{-xt} g_N(t)\,dt\right| < \Gamma\left(\frac{N+\lambda}{\mu}\right)\frac{K_N}{x^{(N+\lambda)/\mu}} + L_N e^{-(x-X)t_N}$$

which, since $L_N e^{-(x-X)t_N}$ is $o(x^{-\nu})$ for any positive ν, establishes the asymptotic expansion for $\mathcal{L}[g; x]$. □

As a simple illustration of the use of Watson's lemma, consider the Laplace transform of $(1+t)^{\frac{1}{2}}$. From the binomial theorem, we have the convergent expansion as $t \to 0$

$$(1+t)^{\frac{1}{2}} = 1 + \tfrac{1}{2}t - \tfrac{1}{8}t^2 + \sum_{n=3}^{\infty}(-)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n n!}t^n.$$

Since $(1+t)^{\frac{1}{2}}$ is of algebraic growth, its Laplace transform clearly exists for $x > 0$, and Watson's lemma produces the asymptotic expansion

$$\mathcal{L}[(1+t)^{\frac{1}{2}}; x] \sim \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{4x^3} + \sum_{n=3}^{\infty}(-)^{n-1}\frac{1\cdot 3\cdot 5\cdots(2n-3)}{2^n x^{n+1}}$$

as $x \to \infty$. The resulting asymptotic series is divergent, since the ratio of the $(n+1)$th to nth terms in absolute value is $(2n-1)/(2x)$ which, for fixed x, tends to ∞ with n. The reason for this divergence is a simple consequence of our applying the binomial expansion for $(1+t)^{\frac{1}{2}}$ (valid in $0 \le t \le 1$) in the Laplace integral beyond its interval of convergence.

Example 2. The confluent hypergeometric function† $U(1; 1; z)$ (which equals the exponential integral $e^z E_1(z)$) has the integral representation

$$U(1; 1; z) = \int_0^\infty \frac{e^{-t}dt}{t+z} \tag{1.1.6}$$

for z not a negative number or zero. In fact, it is relatively easy to show that this integral representation converges uniformly in the closed annular sector $\mathcal{S}_{\epsilon,\delta} = \{z : |z| \ge \epsilon, |\arg z| \le \pi - \delta\}$ for every positive ϵ and every positive $\delta < \pi$. Such a demonstration can proceed along the following lines.

Put $\theta = \arg z$ for $z \in \mathcal{S}_{\epsilon,\delta}$ and observe that for any nonnegative t, $|t+z|^2 = t^2 + |z|^2 + 2|z|t\cos\theta \ge t^2 + |z|^2 - 2|z|t\cos\delta \ge |z|^2\sin^2\delta$. Thus, the integrand of (1.1.6) admits the simple bound

$$e^{-t}|t+z|^{-1} \le e^{-t}|z|^{-1}\operatorname{cosec}\delta$$

whence we have, upon integrating the bound,

$$|U(1; 1; z)| \le |z|^{-1}\operatorname{cosec}\delta$$

† An alternative notation for this function is $\Psi(1; 1; z)$.

for $z \in \mathcal{S}_{\epsilon,\delta}$. The uniform convergence of the integral follows, from which we see that $U(1;1;z)$ is holomorphic in the z plane cut along the negative real axis.

Through repeated integration by parts, differentiating in each case the factor $(t+z)^{-k}$ appearing at each step, we arrive at

$$U(1;1;z) = \sum_{k=1}^{n} (-)^{k-1}(k-1)!z^{-k} + R_n(z), \tag{1.1.7}$$

where the remainder term $R_n(z)$ is

$$R_n(z) = (-)^n n! \int_0^\infty \frac{e^{-t}}{(t+z)^{n+1}}\, dt. \tag{1.1.8}$$

Evidently, each term produced in the series in (1.1.7) is a term from the asymptotic scale $\{z^{-j}\}$, $j = 1, 2, \ldots$, so that if we can show that for any n, $R_n(z) = O(z^{-n-1})$, we will have established the asymptotic expansion

$$U(1;1;z) \sim \sum_{k=1}^{\infty} (-)^{k-1}(k-1)!z^{-k}, \tag{1.1.9}$$

for $z \to \infty$ in the sector $|\arg z| \le \pi - \delta < \pi$.

To this end, we observe that the bound used in establishing the uniform convergence of the integral (1.1.6), namely $1/|t+z| \le 1/|z| \sin\delta$, can be brought to bear on (1.1.8) to yield

$$|R_n(z)| \le \frac{n!}{(|z|\sin\delta)^{n+1}}.$$

The expansion (1.1.9) is therefore an asymptotic expansion in the sense of Poincaré. It is, however, quite clearly a divergent series, as ratios of consecutive terms in the asymptotic series diverge to ∞ as $(n!/|z|^{n+1})/((n-1)!/|z|^n) = n/|z|$, as $n \to \infty$, irrespective of the value of z. Nevertheless, the divergent character of this asymptotic series does not detract from its computational utility. □

In Tables† 1.1 and 1.2, we have gathered together computed and approximate values of $U(1;1;z)$, with approximate values derived from the finite series approximation

$$S_n(z) = \sum_{k=1}^{n} (-)^{k-1}(k-1)!z^{-k},$$

obtained by truncating the asymptotic expansion (1.1.9) after n terms. It is apparent from the tables that the calibre of even modest approximations to $U(1;1;z)$ becomes quite good once $|z|$ is of the order of 100, and is good to two or more significant digits for values of $|z|$ as small as 10. This naturally leads one to

† In Tables 1.1 and 1.2 we have adopted the convention of writing $x(y)$ in lieu of the more cumbersome $x \times 10^y$.

Table 1.1. *Computed and approximate values of* $U(1; 1; z)$ *for real values of* z

z	$U(1; 1; z)$	$S_5(z)$	$S_{10}(z)$
10	0.915633(−1)	0.916400(−1)	0.915456(−1)
50	0.196151(−1)	0.196151(−1)	0.196151(−1)
100	0.990194(−2)	0.990194(−2)	0.990194(−2)

Table 1.2. *Computed and approximate values of* $U(1; 1; z)$ *for imaginary values of* z

z	$U(1; 1; z)$
$10i$	$0.948854(-2) - 0.981910(-1)i$
$50i$	$0.399048(-3) - 0.199841(-1)i$
$100i$	$0.999401(-4) - 0.999800(-2)i$
z	$S_5(z)$
$10i$	$0.940000(-2) - 0.982400(-1)i$
$50i$	$0.399040(-3) - 0.199841(-1)i$
$100i$	$0.999400(-4) - 0.999800(-2)i$
z	$S_{10}(z)$
$10i$	$0.950589(-2) - 0.982083(-1)i$
$50i$	$0.399048(-3) - 0.199841(-1)i$
$100i$	$0.999401(-4) - 0.999800(-2)i$

wonder how the best approximation can be obtained, in view of the utility of these finite approximations and the divergence of the full asymptotic expansion: how can we select n so that the approximation furnished by $S_n(z)$ is the best possible?

The strategy we detail here, called *optimal truncation*, is easily stated: for a fixed z, the successive terms in the asymptotic expansion will reach a minimum in absolute value, after which the terms must necessarily increase without bound given the divergent character of the full expansion; see Fig. 1.1. It is readily shown that the terms in $S_n(z)$ attain their smallest absolute value when $k \sim |z|$ (except when $|z|$ is an integer, in which case there are two equally small terms corresponding to $k = |z| - 1$ and $k = |z|$). If the full series is truncated just before this minimum modulus term is reached, then the finite series that results is the optimally truncated series, and will yield the best approximation to the original function, in the present case, $U(1; 1; z)$.

To see that this is so, observe for $U(1; 1; z)$ that for $z > 0$ the remainder in the approximation after n terms of the asymptotic series,

$$R_n(z) = U(1; 1; z) - S_n(z) = (-)^n n! \int_0^\infty \frac{e^{-t} dt}{(t + z)^{n+1}},$$

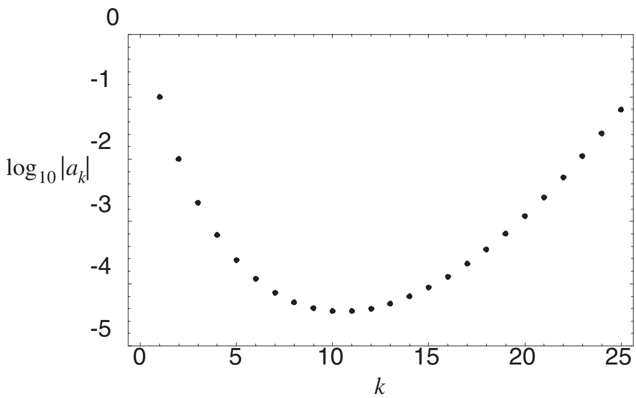


Fig. 1.1. Magnitude of the terms $a_k = (-)^{k-1}\Gamma(k)z^{-k}$ in the expansion $S_n(z)$ against ordinal number k when $z = 10$.

is of the sign opposite to that in the last term in $S_n(z)$ and further, is of the same sign as the first term left in the full asymptotic series after excising $S_n(z)$. In absolute value, we also have

$$|R_n(z)| = \frac{n!}{z^{n+1}} \int_0^\infty \frac{e^{-t}dt}{(1+t/z)^{n+1}} < \frac{n!}{z^{n+1}},$$

so the remainder term is numerically smaller in absolute value than the modulus of the first neglected term. Since the series $S_n(z)$ is an alternating series, it follows that $S_n(z)$ is alternately bigger than $U(1;1;z)$ and less than $U(1;1;z)$ as n increases. The sum $S_n(z)$ will therefore be closest in value to $U(1;1;z)$ precisely when we truncate the full expansion just before the numerically smallest term (in absolute value) in the full expansion. From the preceding inequality, it is easy to note that the remainder term will then be bounded by this minimal term.

To see the order of the remainder term at optimal truncation, we substitute $n \sim z$ ($\gg 1$) in the above bound for $R_n(z)$, and employ Stirling's formula to approximate the factorial, to find

$$|R_n(z)| < \frac{n!}{z^{n+1}} \simeq (2\pi)^{\frac{1}{2}} \frac{e^{-n}n^{n+\frac{1}{2}}}{z^{n+1}} \simeq \left(\frac{2\pi}{z}\right)^{\frac{1}{2}} e^{-z}.$$

This shows that at optimal truncation the remainder term for $U(1;1;z)$ is of order $z^{-\frac{1}{2}}e^{-z}$ as $z \to +\infty$ and consequently that evaluation of the function by this scheme will result in an error that is *exponentially small* in z; these results can be extended to deal with complex values of z – see Olver (1974, p. 523) for a more detailed treatment. We remark that this principle is found to apply to a wide range of asymptotic series yielding in each case an error term at optimal truncation that is typically exponentially small in the asymptotic variable.

We observe that not all asymptotic series present the regular behaviour of the coefficients depicted in Fig. 1.1. In certain compound expansions, with coefficients containing gamma functions in the numerator, it is possible to find situations where some of the arguments of the gamma functions approach a nonpositive integer value. This gives rise to a series of 'peaks' superimposed on the basic structure of Fig. 1.1. A specific example is provided by the compound expansion

$$z^{-2/\mu}(I_1 + I_2), \tag{1.1.10}$$

where $I_r = \sum_{k=0}^{\infty} a_k^{(r)}$ $(r = 1, 2)$ and, for positive parameters m_1, m_2 and μ,

$$a_k^{(1)} = \frac{(-)^k}{k!}\Gamma\left(\frac{1+\mu k}{m_1}\right)\Gamma\left(\frac{m_1 - m_2(1+\mu k)}{m_1\mu}\right)z^{-(1+\mu k)/m_1}$$

with a similar expression for $a_k^{(2)}$ with m_1 and m_2 interchanged. Expansions of this type arise in the treatment of certain Laplace-type integrals discussed in Chapter 7. If the parameters m_1, m_2 and μ are chosen such that the arguments of the second gamma function in $a_k^{(1)}$ and $a_k^{(2)}$ are not close to zero or a negative integer, then the variation of the modulus of the coefficients with ordinal number k will be similar to that shown in Fig. 1.1. If, however, the parameter values are chosen so that these arguments become close to a nonpositive integer† for subsets of k values, then we find that the variation of the coefficients becomes irregular with a sequence of peaks of variable height. Such a situation for the coefficients $a_k^{(1)}$ is shown in Fig. 1.2 for two sets of parameter values. The truncation of such series has been investigated in Liakhovetski & Paris (1998), where it is found that even if the series I_1 is truncated at a peak (provided that the corresponding peak associated with the coefficients $a_k^{(2)}$ is included) increasingly accurate asymptotic approximations are obtained by steadily increasing the truncation indices in the series I_1 and I_2 until they correspond roughly to the global minimum of each curve. An inspection of Fig. 1.2, however, would indicate that these optimal points are not as easily distinguished as in the case of Fig. 1.1.

The notion of optimal truncation will surface in a significant way in the subject matter of the Stokes phenomenon and hyperasymptotics, and so we defer further discussion of it until Chapter 6, where a detailed analysis of remainder terms is undertaken. We do mention, however, that apart from optimally truncating an asymptotic series, one can sometimes obtain dramatic improvements in the numerical utility of an asymptotic expansion if one is able to extract exponentially small (measured against the scale being used) terms prior to developing an asymptotic expansion. This particular situation can be seen in the following example.

† If the parameter values are such that the second gamma-function argument equals a nonpositive integer for a subset of k values, then the expansion (1.1.10) becomes nugatory. In the derivation of (1.1.10) by a Mellin-Barnes approach this would result in a sequence of double poles and the formation of logarithmic terms.

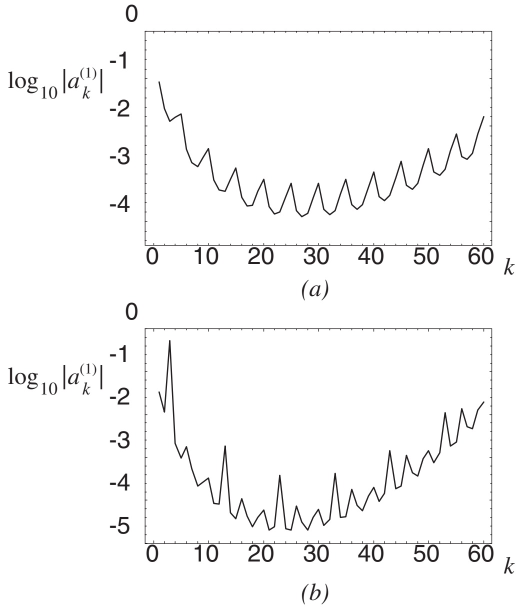


Fig. 1.2. Magnitude of the coefficients $a_k^{(1)}$ against ordinal number k for $\mu = 3$, $m_1 = 1.5$ when (a) $m_2 = 1.2$, $z = 3.0$ and (b) $m_2 = 1.049$, $z = 3.6$. For clarity the points have been joined.

Example 3. Let us consider the finite Fourier integral

$$J(\lambda) = \int_{-1}^{1} e^{i\lambda(x^3/3+x)}dx$$

with λ large and positive. Introduce the change of variable $u = \frac{1}{3}x^3 + x$ and observe that over the interval of integration, the change of variable is one-to-one, fixes the origin and maps ± 1 to $\pm\frac{4}{3}$ respectively, resulting in

$$J(\lambda) = \int_{-4/3}^{4/3} e^{i\lambda u} x'(u)du,$$

where $x(u)$ is the function inverse to the $x \mapsto u$ change of variable. An explicit formula for $x(u)$ is available to us from the classical theory of equations, resulting from the trigonometric solution to the cubic equation, and takes the form

$$x = 2\sinh\theta, \qquad \text{where} \quad 3\theta = \operatorname{arcsinh}\left(\tfrac{3}{2}u\right),$$

or

$$x = \left(\tfrac{3}{2}u + \sqrt{\tfrac{9}{4}u^2 + 1}\right)^{1/3} - \left(\tfrac{3}{2}u + \sqrt{\tfrac{9}{4}u^2 + 1}\right)^{-1/3}.$$

It is a straightforward matter to deduce that $x^{(k)}(-u) = (-)^{k-1}x^{(k)}(u)$, where $x^{(n)}(u)$ as usual indicates the nth derivative of the inverse function.

By repeatedly applying integration by parts, the latter representation for $J(\lambda)$ can be seen to yield a finite asymptotic expansion with remainder,

$$J(\lambda) = \sum_{n=1}^{N} \left\{ e^{4i\lambda/3} x^{(n)}\left(\tfrac{4}{3}\right) - e^{-4i\lambda/3} x^{(n)}\left(-\tfrac{4}{3}\right) \right\} \frac{(-)^{n-1}}{(i\lambda)^n} + \frac{(-)^N}{(i\lambda)^N} \int_{-4/3}^{4/3} e^{i\lambda u} x^{(N+1)}(u)\, du. \tag{1.1.11}$$

In view of the Riemann-Lebesgue lemma, the remainder term is seen to be $o(\lambda^{-N})$, so the finite expansion (1.1.11) leads, after exploiting $x^{(k)}(-\frac{4}{3}) = (-)^{k-1} x^{(k)}(\frac{4}{3})$, to the large-$\lambda$ expansion†

$$J(\lambda) \sim 2\sin\left(\tfrac{4}{3}\lambda\right) \sum_{n=0}^{\infty} \frac{(-)^n}{\lambda^{2n+1}} x^{(2n+1)}\left(\tfrac{4}{3}\right) - 2\cos\left(\tfrac{4}{3}\lambda\right) \sum_{n=1}^{\infty} \frac{(-)^n}{\lambda^{2n}} x^{(2n)}\left(\tfrac{4}{3}\right).$$

If we evaluate the first few derivatives $x^{(n)}(\frac{4}{3})$ and employ optimal truncation for modest values of λ, say $\lambda = 4, 5, 6, 7$, we obtain the approximate values shown in the fourth column of Table 1.3. The columns labelled N_s and N_c show respectively, for each value of λ, the number of terms of the sine and cosine series in the expansion of $J(\lambda)$ retained after optimally truncating each series. As comparison with the last column of Table 1.3 reveals, the asymptotic approximations obtained for these modest values of λ are of poor calibre.

However, an improvement in the numerical utility of the expansion can be obtained by rewriting the integral representation of $J(\lambda)$ in the following manner. Because of the exponential decay in the integrand, we can, by Cauchy's theorem, write

$$J(\lambda) = \left\{ -\int_{1}^{\infty e^{\pi i/6}} + \int_{-1}^{\infty e^{5\pi i/6}} + \int_{\infty e^{5\pi i/6}}^{\infty e^{\pi i/6}} \right\} e^{i\lambda(x^3/3+x)} dx. \tag{1.1.12}$$

The third integral in this sum can be expressed in terms of the Airy function

$$\mathrm{Ai}(z) = \frac{1}{2\pi i} \int_{\infty e^{-\pi i/3}}^{\infty e^{\pi i/3}} \exp\left(\tfrac{1}{3}t^3 - zt\right) dt,$$

namely,

$$2\pi\lambda^{-1/3}\,\mathrm{Ai}(\lambda^{2/3}) = \int_{\infty e^{5\pi i/6}}^{\infty e^{\pi i/6}} e^{i\lambda(x^3/3+x)} dx$$

upon making the substitution $x = it\lambda^{-1/3}$. From this, and integration by parts applied to each of the remaining integrals in (1.1.12), we arrive at the same expansion and approximation for $J(\lambda)$ that we found earlier, only now the expansion

† This expansion does not fit the form of a Poincaré-type expansion as we have defined it previously, but rather is an example (after separating sine and cosine terms) of a compound asymptotic expansion, discussed in the next section.

Table 1.3. *Comparison of optimally truncated asymptotic approximation, asymptotic approximation and exponentially decaying correction and computed values of the Fourier integral $J(\lambda)$*

λ	N_s	N_c	Optimally truncated series	Optimally truncated series with Airy term	$J(\lambda)$
4	2	2	−0.213739	−0.153525	−0.154260
5	3	2	0.055788	0.083551	0.083545
6	6	5	0.164661	0.177709	0.177703
7	6	5	0.022816	0.029031	0.029034

includes the term involving the Airy function:

$$J(\lambda) \sim \frac{2\pi}{\lambda^{1/3}}\mathrm{Ai}(\lambda^{2/3}) + 2\sin\left(\tfrac{4}{3}\lambda\right)\sum_{n=0}^{\infty}\frac{(-)^n}{\lambda^{2n+1}}x^{(2n+1)}\left(\tfrac{4}{3}\right)$$
$$- 2\cos\left(\tfrac{4}{3}\lambda\right)\sum_{n=1}^{\infty}\frac{(-)^n}{\lambda^{2n}}x^{(2n)}\left(\tfrac{4}{3}\right).$$

The Airy function of positive argument can be shown to exhibit exponential decay as the argument increases, so the additional Airy function term in the above expression is $o(\lambda^{-k})$ for any nonnegative integer k and can be eliminated entirely from the asymptotic expansion in view of the definition of asymptotic expansions of Poincaré type. If it is instead retained, the resulting approximations for the same modest values of λ used in Table 1.3 show dramatic improvement, giving several significant figures of the computed values of $J(\lambda)$ as a comparison of the last two columns of Table 1.3 reveals. □

Another interesting fact concerning asymptotic power series stems from the observation that given an arbitrary sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$, there is a function $f(z)$ holomorphic in a region containing a closed annular sector which has the formal series $\sum_{n=0}^{\infty} a_n z^{-n}$ as its asymptotic expansion.

One such construction† proceeds by taking the closed annular sector to be $\mathcal{S} = \{z : |\arg z| \leq \theta,\ |z| \geq R > 0\}$ – other sectors can be used by translating and rotating this initial choice. Then set

$$f(z) = \sum_{n=0}^{\infty}\frac{a_n e_n(z)}{z^n}$$

where for nonzero a_n,

$$e_n(z) = 1 - \exp\bigl(-z^{\phi} r^n/|a_n|\bigr),$$

† This account is drawn from Olver (1974, § I.9). Other examples along this line are also to be found there.

for numbers ϕ and r chosen to satisfy $0 < \phi < \pi/(2\theta)$ and $0 < r < R$. Should an a_n vanish, the corresponding e_n is taken to be the zero function, so that the corresponding term in the sum defining f is effectively excised.

With these terms so defined, in the sector of interest we have $|\arg(z^\phi)| < \frac{1}{2}\pi$ and

$$\left|\frac{a_n e_n(z)}{z^n}\right| \le r^n |z|^{\phi-n} \le |z|^\phi \left(\frac{r}{R}\right)^n, \tag{1.1.13}$$

since $|1-e^{-\zeta}| \le |\zeta|$ when $|\arg\zeta| \le \frac{1}{2}\pi$. The series defining f therefore converges uniformly on compact subsets of our sector, and so defines a holomorphic function there.

That f has the desired asymptotic expansion can be seen from

$$f(z) - \sum_{n=0}^{N-1} \frac{a_n}{z^n} = -\sum_{n=0}^{N-1} \frac{a_n}{z^n} \exp\left(-\frac{z^\phi r^n}{|a_n|}\right) + \sum_{n=N}^{\infty} \frac{a_n e_n(z)}{z^n}, \tag{1.1.14}$$

where it bears noting that the infinite series here is uniformly convergent. Because of the exponential decay of each term in the finite sum on the right, the entire sum is $o(z^{-n})$ for any n as $z \to \infty$ in our sector. The remaining series on the right-hand side is easily bounded using (1.1.13) to give

$$\left|\sum_{n=N}^{\infty} \frac{a_n e_n(z)}{z^n}\right| \le |z|^\phi \sum_{n=N}^{\infty} \left(\frac{r}{|z|}\right)^n = |z|^\phi \left(\frac{r}{|z|}\right)^N \frac{|z|}{|z|-r} = O(z^{\phi-N}).$$

Upon replacing N by $N+\lfloor\phi\rfloor+1$, we obtain a similar expression to that in (1.1.14), for which the right-hand side is $O(z^{-N})$ but for which there are "extra" terms on the left-hand side. These additional terms, $a_n z^{-n}$ for $n \ge N$, are also $O(z^{-N})$ and so can be absorbed into the order estimate that results on the right-hand side.

1.1.3 Other Expansions

Expansions other than Poincaré-type also have currency in asymptotic analysis. Here, we mention but three types.

To begin, let $\{\phi_n\}$ be an asymptotic scale as $x \to x_0$. A formal series $\sum f_n(x)$ is a *generalised asymptotic expansion* of a function $f(x)$ with respect to the asymptotic scale $\{\phi_n\}$ if

$$f(x) = \sum_{n=0}^{N} f_n(x) + o(\phi_N(x)) \qquad (x \to x_0,\ N = 0, 1, 2, \ldots).$$

In this event, we write, as we have for Poincaré-type expansions,

$$f(x) \sim \sum_{n=0}^{\infty} f_n(x) \qquad (x \to x_0, \{\phi_n\}),$$

indicating with the formal series the asymptotic scale used to define the expansion.

The important difference between Poincaré and generalised asymptotic expansions is that the functions f_n appearing in the formal series expansion for f need not, themselves, form an asymptotic scale.

Example 1. Define the sequence of functions $\{f_n\}$, for nonnegative integer n and nonzero x, by

$$f_n(x) = \frac{\cos nx}{x^n}.$$

For $x \to \infty$, it is apparent that each $f_n(x) = O(x^{-n})$, and that $\{\phi_n(x)\} = \{x^{-n}\}$ is an asymptotic scale. However, the sequence $\{f_n(x)\}$ fails to be an asymptotic scale, as a ratio of consecutive elements in the sequence gives

$$\frac{f_{n+1}(x)}{f_n(x)} = \frac{\cos(n+1)x}{x \cos nx},$$

which fails to be $o(1)$ for all x sufficiently large.

Generalised asymptotic expansions are less commonplace than expansions of Poincaré type, and are not used in our development of asymptotic expansions of Mellin-Barnes integrals.

A different mechanism for extending Poincaré-type expansions presents itself naturally in the setting of the method of stationary phase or steepest descent, and in the domain of expansions of solutions of differential equations. The idea here is to replace the series expansion of a function, as in (1.1.2), by several different series, each with different scales.

Put more precisely, by a *compound asymptotic expansion* of a function f, we mean a finite sum of Poincaré-type series expansions

$$\begin{aligned} f(x) \sim A_1(x) \sum_{n=0}^{\infty} a_{1n}\phi_{1n}(x) + A_2(x) \sum_{n=0}^{\infty} a_{2n}\phi_{2n}(x) \\ + \cdots + A_k(x) \sum_{n=0}^{\infty} a_{kn}\phi_{kn}(x) \qquad (x \to x_0), \end{aligned}$$

where, for $1 \le m \le k$, the sequences $\{\phi_{mn}\}$ are asymptotic scales, the coefficient functions $A_m(x)$ are continuous, and for $N_1, N_2, \ldots, N_k \ge 0$, we have

$$\begin{aligned} f(x) = A_1(x) \Bigg\{ \sum_{n=0}^{N_1} a_{1n}\phi_{1n}(x) + O(\phi_{1,N_1+1}(x)) \Bigg\} \\ + A_2(x) \Bigg\{ \sum_{n=0}^{N_2} a_{2n}\phi_{2n}(x) + O(\phi_{2,N_2+1}(x)) \Bigg\} \\ + \cdots + A_k(x) \Bigg\{ \sum_{n=0}^{N_k} a_{kn}\phi_{kn}(x) + O(\phi_{k,N_k+1}(x)) \Bigg\} \qquad (x \to x_0). \end{aligned}$$

It is entirely possible that some of the series $A_j(x)\sum a_{jn}\phi_{jn}(x)$ could, by virtue of the coefficient function $A_j(x)$, or choice of scale $\{\phi_{jn}\}$, be $o(\phi_{mn})$ for some $m \neq j$, and so be absorbed into the error terms implied in other series in the compound expansion. However, in some numerical work, the retention of such negligible terms, when measured against the other scales in the expansion, can add to the numerical accuracy of asymptotic approximations of f, especially for values of x that are at some distance from x_0. This, in turn, extends the utility of such expansions.

In some circumstances, it may be possible to embed the scales $\{\phi_{mn}\}$ in a larger scale, say $\{\psi_\nu\}$, and so collapse the sum of Poincaré expansions into a single-series expansion involving this larger scale $\{\psi_\nu\}$. Success in this direction depends in part on the coefficient functions $A_j(x)$.

Example 2. Steepest descent method. An integral of the form

$$I(\lambda) = \int_C g(z)e^{\lambda f(z)}dz,$$

is said to be of *Laplace type* if the functions f and g are holomorphic in a region containing the contour C, and the integral converges for some λ. In the most common setting, C is an infinite contour, and the parameter λ is large in modulus. Thus, we require that the integral $I(\lambda)$ exist for all λ sufficiently large in some sector.

The idea behind the steepest descent method is deceptively simple: deform the integration contour C into a sum of contours, $C_1, C_2, \ldots, C_k$, so that along each of the contours C_n, the *phase* function $f(z)$ has a single point z_n – a *saddle* or *saddle point*†– at which $f'(z_n)$ vanishes, and as z varies along the contour C_n, $\lambda[f(z) - f(z_n)] \leq 0$, with this difference tending to $-\infty$ as $|z| \to \infty$ along the contour. If this deformation is possible, the contours $C_1, C_2, \ldots, C_k$ are termed *steepest descent contours*, and the integral can be recast as

$$I(\lambda) = \sum_{n=1}^{k} e^{\lambda f(z_n)} \int_{C_n} g(z)e^{\lambda[f(z)-f(z_n)]}dz.$$

In the case where $f''(z_n) \neq 0$ for all saddle points z_n, each integral in the sum can be represented as a Gaussian integral, namely

$$e^{\lambda f(z_n)} \int_{C_n} g(z)e^{\lambda[f(z)-f(z_n)]}dz = e^{\lambda f(z_n)} \int_{-\infty}^{\infty} g(z(t))e^{-|\lambda|t^2} z'(t)\, dt.$$

The transformation $z \mapsto t$ will map one branch of the steepest descent curve from z_n to ∞ into the positive real t axis, and the remainder of the steepest descent curve will be mapped into the negative real t axis. By splitting the integral into integrals

† Saddle points of Fourier-type integrals are often referred to as stationary points.

taken over negative and positive real t axes separately, a further reduction to a sum of two Laplace transforms can be achieved, to each of which Watson's lemma can then be applied.

For a concrete example, we consider the Pearcey integral

$$P(x, y) = \int_{-\infty}^{\infty} \exp\{i(t^4 + xt^2 + yt)\}\, dt,$$

where, for the purpose of illustration, we will assume $|x|$ and $|y|$ are both large, with $x < 0$ and $y > 0$. We will also replace x by $-x$ and take $x > 0$. Thus, we consider

$$P(-x, y) = x^{\frac{1}{2}} \int_{-\infty}^{\infty} \exp\{ix^2(u^4 - u^2 + yx^{-3/2}u)\}\, du, \tag{1.1.15}$$

where we have applied the simple change of variable $t = x^{\frac{1}{2}}u$. Denoting the phase function of this integral by

$$\psi(u) = u^4 - u^2 + yx^{-3/2}u,$$

we have

$$\begin{aligned}\psi'(u) &= 4(u^3 - \tfrac{1}{2}u + \tfrac{1}{4}yx^{-3/2}) \\ &= 4(u^3 - (u_1 + u_2 + u_3)u^2 + (u_1u_2 + u_1u_3 + u_2u_3)u - u_1u_2u_3),\end{aligned}$$

where the roots of $\psi'(u) = 0$ are indicated by u_1, u_2 and u_3. Because $\psi'(u)$ is a real cubic polynomial, we always have one real zero. If x is sufficiently large compared to y, we can ensure that the other two zeros of $\psi'(u)$ are also real, and that all three are distinct. Additionally, the elementary theory of equations furnishes us with

$$\sum u_i = 0, \qquad \sum_{i<j} u_iu_j = -\tfrac{1}{2}, \qquad u_1u_2u_3 = -\tfrac{1}{4}yx^{-3/2},$$

from which we deduce that one $u_i < 0$, and the other two are positive. Let us label these so that $u_1 < 0 < u_2 < u_3$.

We mention here that the theory of equations also provides a trigonometric form for the roots u_i, namely,

$$\begin{aligned} u_1 &= -\sqrt{2/3}\cdot\sin(\phi + \tfrac{1}{3}\pi), \\ u_2 &= \sqrt{2/3}\cdot\sin\phi, \\ u_3 &= \sqrt{2/3}\cdot\sin\left(\tfrac{1}{3}\pi - \phi\right), \end{aligned} \tag{1.1.16}$$

where the angle ϕ is given by

$$\sin(3\phi) = y\left(\tfrac{2}{3}x\right)^{3/2} \tag{1.1.17}$$

which, under the hypothesis of $y(\frac{2}{3}x)^{-3/2} < 1$, can be guaranteed to be real. The zeros displayed in (1.1.16) undergo a confluence when the angle ϕ tends to $\frac{1}{6}\pi$. The curve this value of ϕ defines is the so-called caustic in the real plane: $y = (\frac{2}{3}x)^{3/2}$. The saddles u_i are therefore, successively, the locations of a local minimum, a local maximum and a local minimum of $\psi(u)$.

For real x and y, we may rotate the contour of integration in (1.1.15) onto the line from $\infty e^{9\pi i/8}$ to $\infty e^{\pi i/8}$ through an application of Jordan's lemma. Since there are three real saddle points for $(-x, y)$ satisfying $\phi < \frac{1}{6}\pi$, we may further represent $P(-x, y)$ as a sum of three contour integrals,

$$P(-x, y) = x^{\frac{1}{2}} \sum_{j=1}^{3} \int_{\Gamma_j} e^{ix^2\psi(u)} du, \tag{1.1.18}$$

where the contours Γ_j are the steepest descent curves: Γ_1, beginning at $\infty e^{9\pi i/8}$, ending at $\infty e^{5\pi i/8}$ and passing through $u_1 < 0$; Γ_2, beginning at $\infty e^{5\pi i/8}$, ending at $\infty e^{-3\pi i/8}$ and passing through $u_2 > 0$; and Γ_3, beginning at $\infty e^{-3\pi i/8}$, ending at $\infty e^{\pi i/8}$ and passing through $u_3 > u_2$. Along these contours, the phase $i\psi(u)$ is real and decreases to $-\infty$ as we move along the Γ_j away from the saddle points so that each integral is effectively a Gaussian integral. The general situation is depicted in Fig. 1.3.

Let us set

$$d_j = \{(-)^j(1 - 6u_j^2)\}^{\frac{1}{2}} \qquad (j = 1, 2, 3).$$

In accordance with the steepest descent methodology mentioned previously, we set $\psi(u) - \psi(u_j) = (-)^{j+1} d_j^2 v^2$, to find at each saddle point u_j,

$$v = (u - u_j)\left\{1 + \frac{4u_j(u - u_j)}{6u_j^2 - 1} + \frac{(u - u_j)^2}{6u_j^2 - 1}\right\}^{1/2}$$

whence reversion yields the expansion, for each j,

$$u - u_j = \sum_{k=1}^{\infty} b_{k,j} v^k,$$

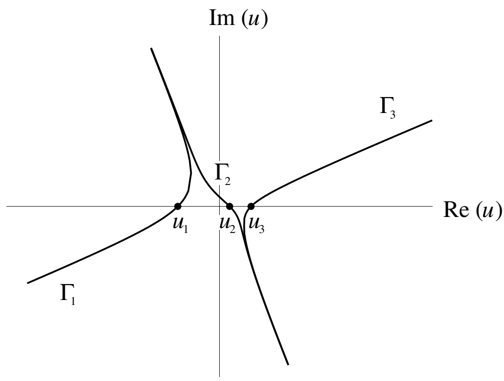


Fig. 1.3. Steepest descent curves through the saddles u_1, u_2 and u_3.

convergent in a neighbourhood of $v = 0$. We observe that $b_{1,j} = 1$ for each $j = 1, 2, 3$. Substitution into each term in (1.1.18) followed by termwise integration will furnish

$$\int_{\Gamma_j} e^{ix^2\psi(u)}du \sim e^{ix^2\psi(u_j)+(-)^{j+1}\pi i/4}\frac{\pi^{\frac{1}{2}}}{xd_j}\, S_j(x,\phi),$$

where $S_j(x,\phi)$ denotes the formal asymptotic sum

$$S_j(x,\phi) = \sum_{k=0}^{\infty}(2k+1)b_{2k+1,j}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(\frac{1}{2})}\left((-)^{j+1}i\right)^k(d_jx)^{-2k}.$$

It then follows that

$$P(-x,y) \sim \sqrt{\frac{\pi}{x}}\sum_{j=1}^{3}\frac{e^{ix^2\psi(u_j)+(-)^{j+1}\pi i/4}}{d_j}\, S_j(x,\phi)$$

for large x. This is evidently a compound asymptotic expansion with each constituent asymptotic series corresponding to a single saddle point of $P(-x, y)$. We shall meet the Pearcey integral again in Chapter 8, in a less restricted setting. □

There also arise situations in which functions depending on parameters other than the asymptotic one may possess asymptotic expansions which not only depend on such auxiliary parameters, but may also undergo discontinuous changes of scale as these parameters vary. Such a discontinuity in the scale can occur, even if the function involved is holomorphic in the control parameter. In more specific terms, let us suppose that a function $F(\lambda;\mu)$ has asymptotic parameter λ and control parameter μ. For $\lambda \to \lambda_0$, and $\mu < \mu_0$, say, one might have an asymptotic form

$$F(\lambda;\mu) \sim A_1(\lambda;\mu)\sum_{n=0}^{\infty}a_{1n}(\mu)\phi_{1n}^{-} + \cdots + A_k(\lambda;\mu)\sum_{n=0}^{\infty}a_{kn}(\mu)\phi_{kn}^{-},$$

where $\{\phi_{jn}^{-}\}$ $(1 \le j \le k)$ are asymptotic scales in the variable λ, while for $\mu > \mu_0$, a different expansion might hold, say

$$F(\lambda;\mu) \sim B_1(\lambda;\mu)\sum_{n=0}^{\infty}b_{1n}(\mu)\phi_{1n}^{+} + \cdots + B_r(\lambda;\mu)\sum_{n=0}^{\infty}b_{rn}(\mu)\phi_{rn}^{+},$$

for different scales $\{\phi_{jn}^{+}\}$ $(1 \le j \le r)$ in λ. For the value $\mu = \mu_0$, a third expansion may hold, involving yet another scale $\{\phi_{jn}\}$ $(1 \le j \le s)$,

$$F(\lambda;\mu_0) \sim C_1(\lambda)\sum_{n=0}^{\infty}c_{1n}\phi_{1n} + \cdots + C_s(\lambda)\sum_{n=0}^{\infty}c_{sn}\phi_{sn} \qquad (\lambda \to \lambda_0).$$

Distinct forms such as these may apply, even if F is analytic in a neighbourhood of μ_0, and the limiting forms of the expansions may not exist as $\mu \to \mu_0^{\pm}$, compounding the difficulty of using such expansions in a neighbourhood of $\mu = \mu_0$.

This setting can be dealt with through the use of a *uniform asymptotic expansion*, a (usually) compound expansion

$$F(\lambda; \mu) \sim D_1(\lambda; \mu) \sum_{n=0}^{\infty} d_{1n}\psi_{1n} + \cdots + D_k(\lambda; \mu) \sum_{n=0}^{\infty} d_{kn}\psi_{kn},$$

where the asymptotic scale $\{\psi_{jn}\}$ $(1 \leq j \leq k)$ is a sequence of functions of the asymptotic parameter, which retains its character as an asymptotic scale for all values of the control parameter in a neighbourhood of $\mu = \mu_0$, i.e., $\psi_{j,n+1} = o(\psi_{jn})$ for $\lambda \to \lambda_0$, for every μ in some neighbourhood of $\mu = \mu_0$.

On first glance, it may appear there is little that is new captured in this account. The essential difference is that the coefficient functions D_j must be continuous in a neighbourhood of $\mu = \mu_0$ for all λ in a neighbourhood of λ_0. Furthermore, for $\mu \neq \mu_0$, each D_j must have expansions for $\lambda \to \lambda_0$ which, when combined with the associated Poincaré expansion $\sum d_{jn}\psi_{jn}$, allows the recovery of either the A- or B-coefficient expansions, and for $\mu = \mu_0$, the recovery of the C-coefficient series. Because the D-series is continuous in μ in a neighbourhood of μ_0, the D-coefficient expansion interpolates continuously from the A-series to the C-series to the B-series as μ varies. This continuous interpolation is possible only through additional complexity in the form of the coefficients D_j.

Example 3. BESSEL FUNCTIONS OF LARGE ORDER. As an illustration, we cite the asymptotic expansion of the Bessel function $J_\nu(\nu x)$ for large positive order and argument in the form

$$J_\nu(\nu x) \sim \begin{cases} \dfrac{e^{\nu(\tanh\alpha - \alpha)}}{(2\pi\nu\tanh\alpha)^{\frac{1}{2}}} \displaystyle\sum_{k=0}^{\infty} \frac{c_k(\coth\alpha)}{\nu^k} \\[2ex] \left(\dfrac{2}{\pi\nu\tan\beta}\right)^{\frac{1}{2}} \left\{\cos\Psi \displaystyle\sum_{k=0}^{\infty} \frac{c_{2k}(i\cot\beta)}{\nu^{2k}}\right. \\[2ex] \qquad\qquad \left. -i\sin\Psi \displaystyle\sum_{k=0}^{\infty} \frac{c_{2k+1}(i\cot\beta)}{\nu^{2k+1}}\right\}, \end{cases} \tag{1.1.19}$$

where in the first expansion $0 < x < 1$ with $x = \operatorname{sech}\alpha$ and in the second expansion $x > 1$ with $x = \sec\beta$. The coefficients $c_k(t)$ are polynomials in t of degree $3k$, with $c_0(t) = 1$, $c_1(t) = \frac{1}{24}(3t - 5t^3), \ldots$ and $\Psi = \nu(\tan\beta - \beta) - \frac{1}{4}\pi$; see Abramowitz & Stegun (1965, p. 366). These expansions describe the asymptotic structure of $J_\nu(\nu x)$ on either side of the transition point $x = 1$. When $0 < x < 1$, $J_\nu(\nu x)$ decays exponentially *away* from the point $x = 1$ while when $x > 1$, $J_\nu(\nu x)$ changes to an oscillatory form with an amplitude that eventually decays like $x^{-\frac{1}{2}}$ as $x \to +\infty$. Both these expansions break down in the neighbourhood of $x = 1$ and so cannot describe uniformly the behaviour of $J_\nu(\nu x)$ for $x > 0$.

A uniformly valid expansion which incorporates both the expansions in (1.1.19) is given by [Abramowitz & Stegun (1965, p. 368)]

$$J_\nu(\nu x) \sim \left(\frac{4\zeta}{1-x^2}\right)^{\frac{1}{4}} \left\{ \frac{\mathrm{Ai}(\nu^{2/3}\zeta)}{\nu^{1/3}} \sum_{k=0}^{\infty} \frac{a_k(\zeta)}{\nu^{2k}} + \frac{\mathrm{Ai}'(\nu^{2/3}\zeta)}{\nu^{5/3}} \sum_{k=0}^{\infty} \frac{b_k(\zeta)}{\nu^{2k}} \right\}, \tag{1.1.20}$$

where $\mathrm{Ai}(z)$ denotes the Airy function. This expansion holds for $\nu \to +\infty$ uniformly with respect to x in the sector $|\arg x| \leq \pi - \epsilon$, $\epsilon > 0$. The variable ζ is defined by

$$\tfrac{2}{3}\zeta^{3/2} = \log\left\{(1 + \sqrt{1-x^2})/x\right\} - \sqrt{1-x^2}$$

the branches being chosen so that ζ is real when $x > 0$. The coefficients $a_k(\zeta)$, $b_k(\zeta)$ are complicated functions of ζ and are expressed in terms of finite sums of the coefficients $c_k((1-x^2)^{-\frac{1}{2}})$, with

$$a_0(\zeta) = 1, \qquad b_0(\zeta) = -\tfrac{5}{48}\zeta^{-2} + \zeta^{-\frac{1}{2}}\left\{\tfrac{5}{24}(1-x^2)^{-\frac{3}{2}} - \tfrac{1}{8}(1-x^2)^{-\frac{1}{2}}\right\}.$$

Although the coefficient functions $a_k(\zeta)$ $(k \geq 1)$ and $b_k(\zeta)$ $(k \geq 0)$ are analytic in the neighbourhood of the transition point $x = 1$ $(\zeta = 0)$, they are, in common with many uniform expansions, expressed in a form that possesses a removable singularity at this point.

The asymptotic forms (1.1.19) can be obtained from (1.1.20) by insertion of the expansion for the Airy function and its derivative; see below for the leading-order terms. When $0 < x < 1$, ζ is bounded away from zero and the arguments of the Airy functions in (1.1.20) are large and positive. These functions are therefore exponential in character and the expansion (1.1.20) reduces to the first form in (1.1.19). On the other hand, when $x > 1$, $\zeta < 0$ and is bounded away from zero, so that the arguments of the Airy functions are large and negative and consequently produce oscillatory terms. In this case the expansion (1.1.20) reduces to the second form in (1.1.19).

At the transition point $x = 1$ $(\zeta = 0)$, we employ the evaluations $\mathrm{Ai}(0) = \Gamma(\frac{1}{3})/(2 \cdot 3^{1/6}\pi)$, $\mathrm{Ai}'(0) = -3^{1/6}\Gamma(\frac{2}{3})/(2\pi)$ together with the limiting value

$\{4\zeta/(1-x^2)\}^{\frac{1}{4}} = 2^{\frac{1}{3}}$ to find the expansion†

$$J_\nu(\nu) \sim \frac{2^{1/3}\nu^{-1/3}}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} \sum_{k=0}^{\infty} \frac{a_k(0)}{\nu^{2k}} - \frac{2^{1/3}\nu^{-5/3}}{3^{1/3}\Gamma\left(\frac{1}{3}\right)} \sum_{k=0}^{\infty} \frac{b_k(0)}{\nu^{2k}}.$$

Example 4. THE PEARCEY INTEGRAL REVISITED. As a further illustration, let us again consider the Pearcey integral. Because of the additional complexity involved, we shall only consider asymptotic behaviour to leading order; the character of the uniform expansion will still be apparent in our terse account.

To leading order, for $y(\frac{2}{3}x)^{-3/2} < 1$ to ensure that the angle ϕ in (1.1.17) satisfies $\phi < \frac{1}{6}\pi$, the Pearcey integral has the asymptotic form

$$P(-x, \mu x^{3/2}) \sim \sum_{j=1}^{3} \frac{e^{ix^2\psi(u_j)+(-)^{j+1}\pi i/4}}{d_j} \sqrt{\frac{\pi}{x}} \tag{1.1.21}$$

where we have set $\mu = y/x^{3/2}$. When $\mu = (2/3)^{3/2}$, so that $\phi = \frac{1}{6}\pi$, the saddle points u_2 and u_3 coalesce into a single saddle of order 2, i.e., a saddle point at which, additionally, the phase function $\psi(u)$ has a vanishing second derivative. A modification of the steepest descent method then allows us to deduce the approximation [Bleistein & Handelsman (1986, pp. 263–265)]

$$\begin{aligned} P(-x, (2/3)^{3/2}x^{3/2}) \sim{}& \frac{e^{ix^2\psi(u_1)+\pi i/4}}{d_1} \sqrt{\frac{\pi}{x}} \\ &+ \frac{e^{ix^2/12}}{2^{1/2}3^{1/3}x^{1/6}} \left\{ \Gamma\left(\tfrac{1}{3}\right) - \frac{i\Gamma\left(\frac{2}{3}\right)}{2 \cdot 3^{1/3}x^{2/3}} \right\} \end{aligned} \tag{1.1.22}$$

for $x \to \infty$; observe that $\psi(u_1) = -\frac{2}{3}$ when $\mu = (2/3)^{3/2}$, and that $d_2 = d_3 = 0$ of Example 2. For $\mu > (2/3)^{3/2}$, the asymptotic behaviour of the Pearcey integral is dominated by the contribution from the saddle point u_1, for which we find

$$P(-x, \mu x^{3/2}) \sim \frac{e^{ix^2\psi(u_1)+\pi i/4}}{d_1} \sqrt{\frac{\pi}{x}} \quad (x \to \infty). \tag{1.1.23}$$

As μ increases from below $(2/3)^{3/2}$, to $(2/3)^{3/2}$ and then beyond, we see a discontinuous change in the asymptotic scales used in (1.1.21) and in (1.1.22), with the complete disappearance of the last two terms in (1.1.22) as we move to $\mu > (2/3)^{3/2}$. The uniform asymptotic approximation that interpolates continuously between these disparate forms in a neighbourhood of $\mu = (2/3)^{3/2}$ results from an application of the cubic transformation introduced by Chester *et al.* (1957). This transformation captures the essential features of the circumstance of a

† We note that the leading term of this expansion yields the well-known approximation due to Cauchy given by

$$J_\nu(\nu) \sim \frac{2^{1/3}\nu^{-1/3}}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} \qquad (\nu \to +\infty).$$

Laplace-type integral undergoing a confluence of two neighbouring simple saddle points. Applied to the Pearcey integral, this method yields an approximation of the form [Kaminski (1989)]

$$
\begin{aligned}
P(-x, \mu x^{3/2}) \sim \frac{e^{ix^2\psi(u_1)+\pi i/4}}{d_1}\sqrt{\frac{\pi}{x}} \\
+ \frac{2\pi e^{ix^2\eta}}{x^{1/6}}\left\{p_0(\mu)\mathrm{Ai}(-x^{4/3}\zeta) + \frac{iq_0(\mu)}{x^{2/3}}\mathrm{Ai}'(-x^{4/3}\zeta)\right\},
\end{aligned}
\tag{1.1.24}
$$

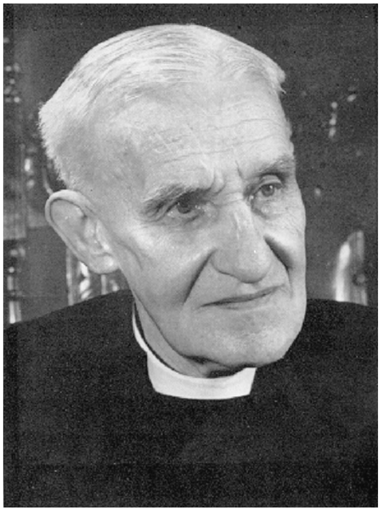
where the quantities η and ζ are given by $\eta = \frac{1}{2}\{\psi(u_2) + \psi(u_3)\}$ and $\zeta^{3/2} = \frac{3}{4}\{\psi(u_2) - \psi(u_3)\}$. The coefficients $p_0(\mu)$ and $q_0(\mu)$ are continuous functions in a neighbourhood of $\mu = (2/3)^{3/2}$ and satisfy $p_0((2/3)^{3/2}) = 2^{-1/2}3^{-1/6}$ and $q_0((2/3)^{3/2}) = 2^{-3/2}3^{-5/6}$. We note that $\zeta < 0$ for $\mu > (2/3)^{3/2}$ and vice versa, and that $\psi(u_2) = \psi(u_3) = \frac{1}{12}$ when $\mu = (2/3)^{3/2}$, at which point $u_2 = u_3$.

The original asymptotic forms can be recovered from (1.1.24) by applying the asymptotic forms of the Airy function and its derivative for large $|z|$, namely [Abramowitz & Stegun (1965, pp. 448–449)]

$$
\begin{aligned}
\mathrm{Ai}(z) &\sim \frac{e^{-2z^{3/2}/3}}{2\sqrt{\pi}\, z^{1/4}} \quad (|\arg z| < \pi), \\
\mathrm{Ai}(-z) &\sim \frac{1}{\sqrt{\pi}\, z^{1/4}}\left\{\sin\left(\tfrac{1}{4}\pi + \tfrac{2}{3}z^{3/2}\right) - \tfrac{5}{48}z^{-3/2}\cos\left(\tfrac{1}{4}\pi + \tfrac{2}{3}z^{3/2}\right)\right\} \\
&\qquad (|\arg z| < \tfrac{2}{3}\pi), \\
\mathrm{Ai}'(z) &\sim -\frac{z^{1/4}e^{-2z^{3/2}/3}}{2\sqrt{\pi}} \quad (|\arg z| < \pi), \\
\mathrm{Ai}'(-z) &\sim -\frac{z^{1/4}}{\sqrt{\pi}}\left\{\cos\left(\tfrac{1}{4}\pi + \tfrac{2}{3}z^{3/2}\right) - \tfrac{7}{48}z^{-3/2}\sin\left(\tfrac{1}{4}\pi + \tfrac{2}{3}z^{3/2}\right)\right\} \\
&\qquad (|\arg z| < \tfrac{2}{3}\pi).
\end{aligned}
$$

For $\mu < (2/3)^{3/2}$ and bounded away from $(2/3)^{3/2}$, the arguments of Ai and Ai$'$ in (1.1.24) are negative, so the preceding asymptotic forms for the Airy function and its derivative produce oscillatory terms, whence the approximation (1.1.24) reduces to (1.1.21). Conversely, if $\mu > (2/3)^{3/2}$, we find $\zeta^{3/2}$ is pure imaginary, in which case the exponentially decaying asymptotic forms for the Airy function and its derivative apply. In this event, the leading term in the asymptotic approximation of $P(-x, \mu x^{3/2})$ is the one arising from the saddle u_1, evident in (1.1.23).

Finally, at the point of confluence when $\mu = (2/3)^{3/2}$, we employ the values of Ai(0) and Ai$'$(0) given earlier along with the values $u_2 = u_3 = 1/\sqrt{6}$ and $u_1 = -2/\sqrt{6}$ to recover (1.1.22) from (1.1.24).



(*a*) (*b*)

Fig. 1.4. Portraits of (*a*) R.H. Mellin (1854–1933) and (*b*) E.W. Barnes (1874–1953) (*reproduced with permission*).

1.2 Biographies of Mellin and Barnes

The names of Mellin and Barnes (Fig. 1.4) are intimately linked with, and were the main exponents of, the asymptotic procedure discussed in this book. We give below a brief biographical account of these two eponymous mathematicians, together with a description of their main mathematical contributions. These accounts are based on Lindelöf (1933) and Elfving (1981) (for Mellin) and Whittaker (1954), Rawlinson (1954) and the Obituary Notices of *The Times* in November 1953 (for Barnes).

Robert Hjalmar Mellin

Robert Hjalmar Mellin, the son of a clergyman, was born in Liminka, northern Ostrobothnia, in Finland on 19 June 1854. He grew up and received his schooling in Hämeenlinna (about 100 km north of Helsinki) and undertook his university studies in Helsinki, where his teacher was the Swedish mathematician G. Mittag-Leffler. In the autumn of 1881 Mellin defended his doctoral dissertation on algebraic functions of a single complex variable. He made two sojourns in Berlin in 1881 and 1882 to study under K. Weierstrass and in 1883–84 he returned to continue his studies with Mittag-Leffler in Stockholm.

Mellin was appointed as a docent at the University of Stockholm from 1884–91 but never actually gave any lectures. Also in 1884 he was appointed a senior lecturer in mathematics at the recently founded Polytechnic Institute which was later (in 1908) to become the Technical University of Finland. In 1901 Mellin withdrew his application for the vacant chair of mathematics at the University of Helsinki in favour of his illustrious (and younger) fellow countryman E. Lindelöf (1870–1946). During the period 1904–07 Mellin was Director of the Polytechnic Institute and in 1908 he became the first professor of mathematics at the new university. He remained at the university for a total of 42 years, retiring in 1926 at the age of 72.

With regard to the ever-burning language question, Mellin was a fervent fennoman with an apparently fiery temperament. It must be recalled, at this juncture, that Finland had for a long time been part of the kingdom of Sweden and had consequently been subjected to its language and culture.† Mellin was one of the founders of the Finnish Academy of Sciences in 1908 as a purely Finnish alternative to the predominantly Swedish-speaking Society of Sciences. From 1908 until his death on 5 April 1933, at the age of 78, he represented his country on the editorial board of *Acta Mathematica*.

Mellin's research work was principally in the area of the theory of functions which resulted from the influence of his teachers Mittag-Leffler and Weierstrass. He studied the transform which now bears his name‡ and established its reciprocal properties. He applied this technique systematically in a long series of papers to the study of the gamma function, hypergeometric functions, Dirichlet series, the Riemann zeta function and related number-theoretic functions. He also extended his transform to several variables and applied it to the solution of partial differential equations. The use of the inverse form of the transform, expressed as an integral along a path parallel to the imaginary axis of the complex plane of integration, was developed by Mellin as a powerful tool for the generation of asymptotic expansions. In this theory, he included the possibility of higher-order poles (thereby leading to the inclusion of logarithmic terms in the expansion) and to several sequences of poles yielding sums of asymptotic expansions of very general form.

During the last decade of his life Mellin was, rather curiously for an analyst, preoccupied by Einstein's theory of relativity and he wrote no less than 10 papers on this topic. In these papers, where he was largely concerned with general philosophical problems of time and space, he adopted a quixotic standpoint in his attempt to refute the theory as being logically untenable.

Ernest William Barnes

Ernest William Barnes was born in Birmingham on 1 April 1874, the eldest of four sons of John Starkie Barnes and Jane Elizabeth Kerry, both elementary

† After the Napoleonic wars Finland became an autonomous Grand Duchy under Russia, to finally emerge as an independent republic in the aftermath of the First World War.

‡ We point out that similar studies of an incomplete nature had been carried out earlier by Pincherle; for references, see Watson (1966, p. 190).