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Cohen-Macaulay rings

WINFRIED BRUNS
& JÜRGEN HERZOG

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Revised edition

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For our wives,

Ulrike and Maja

Contents

Preface to the revised edition

The main change in the revised edition is the new Chapter 10 on tight closure. This theory was created by Mel Hochster and Craig Huneke about ten years ago and is still strongly expanding. We treat the basic ideas, F-regular rings, and F-rational rings, including Smith's theorem by which F-rationality implies pseudo-rationality. Among the numerous applications of tight closure we have selected the Briançon–Skoda theorem and the theorem of Hochster and Huneke saying that equicharacteristic direct summands of regular rings are Cohen–Macaulay. To cover these applications, Section 8.4, which develops the technique of reduction to characteristic p, had to be rewritten. The title of Part III, no longer appropriate, has been changed.

Another noteworthy addition are the theorems of Gotzmann in the new Section 4.3. We believe that Chapter 4 now treats all the basic theorems on Hilbert functions. Moreover, this chapter has been slightly reorganized.

The new Section 5.5 contains a proof of Hochster's formula for the Betti numbers of a Stanley–Reisner ring since the free resolutions of such rings have recently received much attention. In the first edition the formula was used without proof.

We are grateful to all the readers of the first edition who have suggested corrections and improvements. Our special thanks go to L. Avramov, A. Conca, S. Iyengar, R. Y. Sharp, B. Ulrich, and K.-i. Watanabe.

Osnabrück and Essen,
October 1997

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Preface to the first edition

The notion of a Cohen–Macaulay ring marks the cross-roads of two powerful lines of research in present-day commutative algebra. While its main development belongs to the homological theory of commutative rings, it finds surprising and fruitful applications in the realm of algebraic combinatorics. Consequently this book is an introduction to the homological and combinatorial aspects of commutative algebra.

We have tried to keep the text self-contained. However, it has not proved possible, and would perhaps not have been appropriate, to develop commutative ring theory from scratch. Instead we assume the reader has acquired some fluency in the language of rings, ideals, and modules by working through an introductory text like Atiyah and Macdonald [15] or Sharp [344]. Nevertheless, to ease the access for the non-expert, the essentials of dimension theory have been collected in an appendix.

As exemplified by Matsumura's standard textbook [270], it is natural to have the notions of grade and depth follow dimension theory, and so Chapter 1 opens with the introduction of regular sequences on which their definition is based. From the very beginning we stress their connection with homological and linear algebra, and in particular with the Koszul complex.

Chapter 2 introduces Cohen–Macaulay rings and modules, our main subjects. Next we study regular local rings. They form the most special class of Cohen–Macaulay rings; their theory culminates in the Auslander–Buchsbaum–Serre and Auslander–Buchsbaum–Nagata theorems. Unlike the Cohen–Macaulay property in general, regularity has a very clear geometric interpretation: it is the algebraic counterpart of the notion of a non-singular point. Similarly the third class of rings introduced in Chapter 2, that of complete intersections, is of geometric significance.

In Chapter 3 a new homological aspect determines the development of the theory, namely the existence of injective resolutions. It leads us to the study of Gorenstein rings which in several respects are distinguished by their duality properties. When a Cohen–Macaulay local ring is not Gorenstein, then (almost always) it has at least a canonical module which, so to speak, acts as its natural partner in duality theorems, a decisive fact for many combinatorial applications. We then introduce local cohomology and prove Grothendieck's vanishing and local duality theorems.

Chapter 4 contains the combinatorial theory of commutative rings which mainly consists in the study of the Hilbert function of a graded module and the numerical invariants derived from it. A central point is Macaulay's theorem describing all possible Hilbert functions of homogeneous rings by a numerical condition. The intimate connection between homological and combinatorial data is displayed by several theorems, among them Stanley's characterization of Gorenstein domains. In the second part of this chapter the method of associated rings and modules is developed and used for assigning numerical invariants to modules over local rings.

Chapters 1–4 form the first part of the book. We consider this material as basic. The second part consists of Chapters 5–7 each of which is devoted to a special class of rings.

Chapter 5 contains the theory of Stanley–Reisner rings of simplicial complexes. Its main goal is the proof of Stanley's upper bound theorem for simplicial spheres. The transformation of this topological notion into an algebraic condition is through Hochster's theorem which relates simplicial homology and local cohomology. Furthermore we study the Gorenstein property for simplicial complexes and their canonical modules.

In Chapter 6 we investigate normal semigroup rings. The combinatorial object represented by a normal semigroup ring is the set of lattice points within a convex cone. According to a theorem of Hochster, normal semigroup rings are Cohen–Macaulay. Again the crucial point is the interplay between cellular homology on the geometric side and local cohomology on the algebraic. The fact that the ring of invariants of a linear torus action on a polynomial ring is a normal semigroup ring leads us naturally to the study of invariant rings, in particular those of finite groups. The chapter closes with the Hochster–Roberts theorem by which a ring of invariants of a linearly reductive group is Cohen–Macaulay.

Chapter 7 is devoted to determinantal rings. They are discussed in the framework of Hodge algebras and algebras with straightening laws. We establish the straightening laws of Hodge and of Doubilet, Rota, and Stein, prove that determinantal rings are Cohen–Macaulay, compute their canonical module, and determine the Gorenstein rings among them. In view of the extensive treatment available in [61], we have restricted this chapter to the absolutely essential.

The third part of the book is constituted by Chapters 8 and 9. They owe their existence to the fact that a Noetherian local ring is in general not Cohen–Macaulay. But Hochster has shown that such a ring possesses a (not necessarily finite) Cohen–Macaulay module, at least when it contains a field. The construction of these 'big' Cohen–Macaulay modules in Chapter 8 is a paradigm of characteristic p methods in commutative algebra, and we hope that it will prepare the reader for the more recent developments in this area which are centered around the notion of tight closure introduced by Hochster and Huneke [190].

In Chapter 9 we deduce the consequences of the existence of big Cohen–Macaulay modules, for example the intersection theorems of Peskine and Szpiro and Roberts, the Evans–Griffith syzygy theorem, and bounds for the Bass numbers of a module.

Chapters 8 and 9 are completely independent of Chapters 4–7, and the reader who is only interested in the homological theory may proceed from the end of Section 3.5 directly to Chapter 8.

It is only to be expected that the basic notions of homological algebra are ubiquitous in our book. But most of the time we will only use the long exact sequences for Ext and Tor, and the behaviour of these functors under flat extensions. Where we go beyond that, we have inserted a reference to Rotman [318]. One may regard it as paradoxical that we freely use the Ext functors while Chapter 3 contains a complete treatment of injective modules. However, their theory has several peculiar aspects so that we thought such a treatment would be welcomed by many readers.

The book contains numerous exercises. Some of them will be used in the main text. For these we have provided hints or even references to the literature, unless their solutions are completely straightforward. A reference of type A.n points to a result in the appendix.

Parts of this book were planned while we were guests of the Mathematisches Forschungsinstitut Oberwolfach. We thank the Forschungsinstitut for its generous hospitality.

We are grateful to all our friends, colleagues, and students, among them L. Avramov, C. Baeţica, M. Barile, A. Conca, H.–B. Foxby, C. Huneke, D. Popescu, P. Schenzel, and W. Vasconcelos who helped us by providing valuable information and by pointing out mistakes in preliminary versions. Our sincere thanks go to H. Matsumura and R. Sharp for their support in the early stages of this project.

We are deeply indebted to our friend Udo Vetter for reading a large part of the manuscript and for his unfailing criticism.

Vechta and Essen, February 1993

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Part I

Basic concepts

1 Regular sequences and depth

After dimension, depth is the most fundamental numerical invariant of a Noetherian local ring R or a finite R-module M. While depth is defined in terms of regular sequences, it can be measured by the (non-)vanishing of certain Ext modules. This connection opens commutative algebra to the application of homological methods. Depth is connected with projective dimension and several notions of linear algebra over Noetherian rings.

Equally important is the description of depth (and its global relative grade) in terms of the Koszul complex which, in a sense, holds an intermediate position between arithmetic and homological algebra.

This introductory chapter also contains a section on graded rings and modules. These allow a decomposition of their elements into homogeneous components and therefore have a more accessible structure than rings and modules in general.

1.1 Regular sequences

Let M be a module over a ring R. We say that $x \in R$ is an *M-regular element* if $xz = 0$ for $z \in M$ implies $z = 0$, in other words, if x is not a zero-divisor on M. Regular sequences are composed of successively regular elements:

Definition 1.1.1. A sequence $\boldsymbol{x} = x_1, \dots, x_n$ of elements of R is called an *M-regular sequence* or simply an *M-sequence* if the following conditions are satisfied: (i) x_i is an $M/(x_1, \dots, x_{i-1})M$-regular element for $i = 1, \dots, n$, and (ii) $M/\boldsymbol{x}M \neq 0$.

In this situation we shall sometimes say that M is an *$\boldsymbol{x}$-regular module.* A *regular sequence* is an R-sequence.

A *weak M-sequence* is only required to satisfy condition (i).

Very often R will be a local ring with maximal ideal $\mathfrak{m}$, and $M \neq 0$ a finite R-module. If $\boldsymbol{x} \subset \mathfrak{m}$, then condition (ii) is satisfied automatically because of Nakayama's lemma.

The classical example of a regular sequence is the sequence $X_1, \dots, X_n$ of indeterminates in a polynomial ring $R = S[X_1, \dots, X_n]$. Conversely we shall see below that an M-sequence behaves to some extent like a sequence of indeterminates; this will be made precise in 1.1.8.

The next proposition contains a condition under which a regular sequence stays regular when the module or the ring is extended.

Proposition 1.1.2. *Let R be a ring, M an R-module, and $\boldsymbol{x} \subset R$ a weak M-sequence. Suppose $\varphi : R \to S$ is a ring homomorphism, and N an R-flat S-module. Then $\boldsymbol{x} \subset R$ and $\varphi(\boldsymbol{x}) \subset S$ are weak $(M \otimes_R N)$-sequences. If $\boldsymbol{x}(M \otimes_R N) \neq M \otimes_R N$, then $\boldsymbol{x}$ and $\varphi(\boldsymbol{x})$ are $(M \otimes_R N)$-sequences.*

PROOF. Multiplication by x_i is the same operation on $M \otimes N$ as multiplication by $\varphi(x_i)$; so it suffices to consider $\boldsymbol{x}$. The homothety $x_1 : M \to M$ is injective, and $x_1 \otimes N$ is injective too, because N is flat. Now $x_1 \otimes N$ is just multiplication by x_1 on $M \otimes N$. So x_1 is an $(M \otimes N)$-regular element. Next we have $(M \otimes N)/x_1(M \otimes N) \cong (M/x_1M) \otimes N$; an inductive argument will therefore complete the proof. □

The most important special cases of 1.1.2 are given in the following corollary. In its part (b) we use $\hat{M}$ to denote the $\mathfrak{m}$-adic completion of a module M over a local ring $(R, \mathfrak{m}, k)$ (by this notation we indicate that R has maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$).

Corollary 1.1.3. *Let R be a Noetherian ring, M a finite R-module, and $\boldsymbol{x}$ an M-sequence.*
(a) *Suppose that a prime ideal $\mathfrak{p} \in \operatorname{Supp} M$ contains $\boldsymbol{x}$. Then $\boldsymbol{x}$ (as a sequence in $R_\mathfrak{p}$) is an $M_\mathfrak{p}$-sequence.*
(b) *Suppose that R is local with maximal ideal $\mathfrak{m}$. Then $\boldsymbol{x}$ (as a sequence in $\hat{R}$) is an $\hat{M}$-sequence.*

PROOF. Both the extensions $R \to R_\mathfrak{p}$ and $R \to \hat{R}$ are flat. (a) By hypothesis $M_\mathfrak{p} \neq 0$, and Nakayama's lemma implies $M_\mathfrak{p} \neq \mathfrak{p}M_\mathfrak{p}$. *A fortiori* we have $\boldsymbol{x}M_\mathfrak{p} \neq M_\mathfrak{p}$. (b) It suffices to note that $\hat{M} = M \otimes \hat{R}$ is a finite $\hat{R}$-module. □

The interplay between regular sequences and homological invariants is a major theme of this book, and numerous arguments will be based on the next proposition.

Proposition 1.1.4. *Let R be a ring, M an R-module, and $\boldsymbol{x}$ a weak M-sequence. Then an exact sequence*

$$N_2 \xrightarrow{\varphi_2} N_1 \xrightarrow{\varphi_1} N_0 \xrightarrow{\varphi_0} M \longrightarrow 0$$

of R-modules induces an exact sequence

$$N_2/\boldsymbol{x}N_2 \longrightarrow N_1/\boldsymbol{x}N_1 \longrightarrow N_0/\boldsymbol{x}N_0 \longrightarrow M/\boldsymbol{x}M \longrightarrow 0.$$

PROOF. By induction it is enough to consider the case in which $\boldsymbol{x}$ consists of a single M-regular element x. We obtain the induced sequence if we tensor the original one by $R/(x)$. Since tensor product is a right exact functor, we only need to verify exactness at N_1/xN_1. Let $\bar{}$ denote residue classes modulo x. If $\bar{\varphi}_1(\bar{y}) = 0$, then $\varphi_1(y) = xz$ for some $z \in N_0$ and

$x\varphi_0(z) = 0$. By hypothesis we have $\varphi_0(z) = 0$; hence there is $y' \in N_1$ with $z = \varphi_1(y')$. It follows that $\varphi_1(y - xy') = 0$. So $y - xy' \in \varphi_2(N_2)$, and $\bar{y} \in \bar{\varphi}_2(\bar{N}_2)$ as desired. □

If we want to preserve the exactness of a longer sequence, then we need a stronger hypothesis.

Proposition 1.1.5. *Let R be a ring and*

$$N_\bullet : \cdots \longrightarrow N_m \xrightarrow{\varphi_m} N_{m-1} \longrightarrow \cdots \longrightarrow N_0 \xrightarrow{\varphi_0} N_{-1} \longrightarrow 0$$

an exact complex of R-modules. If $\boldsymbol{x}$ is weakly N_i-regular for all i, then $N_\bullet \otimes R/(\boldsymbol{x})$ is exact again.

PROOF. Once more one uses induction on the length of the sequence $\boldsymbol{x}$. So it is enough to treat the case $\boldsymbol{x} = x$. Since x is regular on N_i, it is regular on $\operatorname{Im}\varphi_{i+1}$ too. Therefore we can apply 1.1.4 to each exact sequence $N_{i+3} \to N_{i+2} \to N_{i+1} \to \operatorname{Im}\varphi_{i+1} \to 0$. □

Easy examples show that a permutation of a regular sequence need not be a regular sequence; see 1.1.13. Nevertheless there are natural conditions under which regular sequences can be permuted.

Let x_1, x_2 be an M-sequence, and denote the kernel of the multiplication by x_2 on M by K. Suppose that $z \in K$. Then we must have $z \in x_1M$, $z = x_1z'$, and $x_1(x_2z') = 0$, whence $x_2z' = 0$ and $z' \in K$, too. This shows $K = x_1K$ so that $K = 0$ if Nakayama's lemma is applicable. Somewhat surprisingly, x_1 is always regular on M/x_2M; the reader may check this easily.

Proposition 1.1.6. *Let R be a Noetherian local ring, M a finite R-module, and $\boldsymbol{x} = x_1, \dots, x_n$ an M-sequence. Then every permutation of $\boldsymbol{x}$ is an M-sequence.*

PROOF. Every permutation is a product of transpositions of adjacent elements. Therefore it is enough to show that $x_1, \dots, x_{i+1}, x_i, \dots, x_n$ is an M-sequence. The hypothesis of the proposition is satisfied for $\bar{M} = M/(x_1, \dots, x_{i-1})M$ and the $\bar{M}$-sequence $x_i, \dots, x_n$. So it suffices to treat the case $i = 1$ and to show that x_2, x_1 is an M-sequence. In view of the discussion above we only need to appeal to Nakayama's lemma. □

Quasi-regular sequences. Let R be a ring, M an R-module, and $\boldsymbol{X} = X_1, \dots, X_n$ be indeterminates over R. Then we write $M[\boldsymbol{X}]$ for $M \otimes R[\boldsymbol{X}]$ and call its elements *polynomials with coefficients in M*. If $\boldsymbol{x} = x_1, \dots, x_n$ is a sequence of elements of R, then the substitution $X_i \mapsto x_i$ induces an R-algebra homomorphism $R[\boldsymbol{X}] \to R$ and also an R-module homomorphism $M[\boldsymbol{X}] \to M$. We write $F(\boldsymbol{x})$ for the image of $F \in M[\boldsymbol{X}]$ under this map. (Since the monomials form a basis of the free R-module $R[\boldsymbol{X}]$, we may speak of the coefficients and the degree of an element of $M[\boldsymbol{X}]$.)

Theorem 1.1.7 (Rees). *Let R be a ring, M an R-module, $\boldsymbol{x} = x_1, \ldots, x_n$ an M-sequence, and $I = (x_1, \ldots, x_n)$. Let $X = X_1, \ldots, X_n$ be indeterminates over R. If $F \in M[X]$ is homogeneous of (total) degree d and $F(\boldsymbol{x}) \in I^{d+1}M$, then the coefficients of F are in IM.*

PROOF. We use induction on n. The case $n = 1$ is easy. Let $n > 1$ and suppose that the theorem holds for regular sequences of length at most $n-1$. We must first prove an auxiliary result which is an interesting fact in itself: *let $J = (x_1, \ldots, x_{n-1})$; then x_n is regular on M/J^jM for all $j \geq 1$.*

In fact, suppose that $x_n y \in J^jM$ for some $j > 1$. Arguing by induction we have $y \in J^{j-1}M$; so $y = G(x_1, \ldots, x_{n-1})$ where $G \in M[X_1, \ldots, X_{n-1}]$ is homogeneous of degree $j-1$. Set $G' = x_n G$. Then the theorem applied to $G' \in M[X_1, \ldots, X_{n-1}]$ yields that the coefficients of G' are in JM. Since x_n is regular modulo JM, it follows that the coefficients of G are in JM too, and therefore $y \in J^jM$.

The proof of the theorem for sequences of length n requires induction on d. The case $d = 0$ is trivial. Assume that $d > 0$. First we reduce to the case in which $F(\boldsymbol{x}) = 0$. Since $F(\boldsymbol{x}) \in I^{d+1}M$, one has $F(\boldsymbol{x}) = G(\boldsymbol{x})$ with G homogeneous of degree $d+1$. Then $G = \sum_{i=1}^n X_i G_i$ with G_i homogeneous of degree d. Set $G'_i = x_i G_i$ and $G' = \sum_{i=1}^n G'_i$. So $F - G'$ is homogeneous of degree d, and $(F - G')(\boldsymbol{x}) = 0$. Furthermore, $F - G'$ has coefficients in IM if and only if this holds for F.

Thus assume that $F(\boldsymbol{x}) = 0$. Then we write $F = G + X_n H$ with $G \in M[X_1, \ldots, X_{n-1}]$. The auxiliary claim above implies that $H(\boldsymbol{x}) \in J^dM \subset I^dM$. By induction on d the coefficients of H are in IM. On the other hand $H(\boldsymbol{x}) = H'(x_1, \ldots, x_{n-1})$ with $H' \in M[X_1, \ldots, X_{n-1}]$ homogeneous of degree d. As

$$(G + x_n H')(x_1, \ldots, x_{n-1}) = F(\boldsymbol{x}) = 0,$$

it follows by induction on n that $G + x_n H'$ has coefficients in JM. Since $x_n H'$ has its coefficients in IM, the coefficients of G must be in IM too. $\square$

Let I be an ideal in R. One defines the *associated graded ring of R with respect to I* by

$$\operatorname{gr}_I(R) = \bigoplus_{i=0}^{\infty} I^i/I^{i+1}.$$

The multiplication in $\operatorname{gr}_I(R)$ is induced by the multiplication $I^i \times I^j \to I^{i+j}$, and $\operatorname{gr}_I(R)$ is a graded ring with $(\operatorname{gr}_I(R))_0 = R/I$. If M is an R-module, one similarly constructs the *associated graded module*

$$\operatorname{gr}_I(M) = \bigoplus_{i=0}^{\infty} I^iM/I^{i+1}M.$$

It is straightforward to verify that $\mathrm{gr}_I(M)$ is a graded $\mathrm{gr}_I(R)$-module. (Graded rings and modules will be discussed in Section 1.5. The reader not familiar with the basic terminology may wish to consult 1.5.) Let I be generated by $x_1,\dots,x_n$. Then one has a natural surjection $R[X] = R[X_1,\dots,X_n] \to \mathrm{gr}_I(R)$ which is induced by the natural homomorphism $R \to R/I$ and the substitution $X_i \mapsto \bar{x}_i \in I/I^2$. Similarly there is an epimorphism $\psi : M[X] \to \mathrm{gr}_I(M)$. One first defines ψ on the homogeneous components by assigning to a homogeneous polynomial $F \in M[X]$ of degree d the residue class of $F(\boldsymbol{x})$ in $I^dM/I^{d+1}M$; then ψ is extended additively. As the reader may check, ψ is an epimorphism of graded $R[X]$-modules. Obviously $IM[X] \subset \operatorname{Ker}\psi$; via the identification $M[X]/IM[X] \cong (M/IM)[X]$, we therefore get an induced epimorphism $\varphi : (M/IM)[X] \to \mathrm{gr}_I(M)$. The kernel of ψ is generated by the homogeneous polynomials $F \in M[X]$ of degree d, $d \in \mathbb{N}$, such that $F(\boldsymbol{x}) \in I^{d+1}M$. So we obtain as a reformulation of 1.1.7

Theorem 1.1.8. *Let R be a ring, M an R-module, $\boldsymbol{x} = x_1,\dots,x_n$ an M-sequence, and $I = (\boldsymbol{x})$. Then the map $(M/IM)[X_1,\dots,X_n] \to \mathrm{gr}_I(M)$ induced by the substitution $X_i \mapsto \bar{x}_i \in I/I^2$ is an isomorphism.*

This theorem says very precisely to what extent a regular sequence resembles a sequence of indeterminates: the residue classes $\bar{x}_i \in I/I^2$ operate on $\mathrm{gr}_I(M)$ exactly like indeterminates. Since a regular sequence may lose regularity under a permutation, whereas 1.1.8 is independent of the order in which $\boldsymbol{x}$ is given, it is not possible to reverse 1.1.8; see however 1.1.15. Later on it will be useful to have a name for sequences $\boldsymbol{x}$ satisfying the conclusion of 1.1.8; we call them *M-quasi-regular* if, in addition, $\boldsymbol{x}M \neq M$.

Exercises

1.1.9. Let $0 \to U \to M \to N \to 0$ be an exact sequence of R-modules, and $\boldsymbol{x}$ a sequence which is weakly U-regular and (weakly) N-regular. Prove that $\boldsymbol{x}$ is (weakly) M-regular too.

1.1.10. (a) Let $x_1,\dots,x_i,\dots,x_n$ and $x_1,\dots,x_i',\dots,x_n$ be (weakly) M-regular. Show that $x_1,\dots,x_ix_i',\dots,x_n$ is (weakly) M-regular. (Hint: In the essential case $i = 1$ one finds an exact sequence as in 1.1.9 with $M/x_1x_1'M$ as the middle term.)
(b) Prove that $x_1^{e_1},\dots,x_n^{e_n}$ is (weakly) M-regular for all $e_i \geq 1$.

1.1.11. Prove that the converse of 1.1.2 holds if, in the situation of 1.1.2, N is faithfully flat over R.

1.1.12. (a) Prove that if $\boldsymbol{x}$ is a weak M-sequence, then $\mathrm{Tor}_1^R(M, R/(\boldsymbol{x})) = 0$.
(b) Prove that if, in addition, $\boldsymbol{x}$ is a weak R-sequence, then $\mathrm{Tor}_i^R(M, R/(\boldsymbol{x})) = 0$ for all $i \geq 1$.

1.1.13. Let $R = K[X,Y,Z]$, k a field. Show that X, $Y(1-X)$, $Z(1-X)$ is an R-sequence, but $Y(1-X)$, $Z(1-X)$, X is not.

1.1.14. Prove that $x_1,\dots,x_n$ is M-quasi-regular if and only if $\bar{x}_1,\dots,\bar{x}_n \in I/I^2$ is a $\mathrm{gr}_I(M)$-regular sequence where $I = (x_1,\dots,x_n)$.

1.1.15. Suppose that $\boldsymbol{x}$ is M-quasi-regular, and let $I = (x_1,\dots,x_n)$. Prove
(a) if $x_1 z \in I^i M$ for $z \in M$, then $z \in I^{i-1}M$,
(b) $x_2,\dots,x_n$ is (M/x_1M)-quasi-regular,
(c) if R is Noetherian local and M is finite, then $\boldsymbol{x}$ is an M-sequence.

1.2 Grade and depth

Let R be a Noetherian ring and M an R-module. If $\boldsymbol{x} = x_1,\dots,x_n$ is an M-sequence, then the sequence $(x_1) \subset (x_1,x_2) \subset \cdots \subset (x_1,\dots,x_n)$ ascends strictly for obvious reasons. Therefore an M-sequence can be extended to a maximal such sequence: an M-sequence $\boldsymbol{x}$ (contained in an ideal I) is *maximal* (in I), if $x_1,\dots,x_{n+1}$ is not an M-sequence for any $x_{n+1} \in R$ ($x_{n+1} \in I$). We will prove that all maximal M-sequences in an ideal I with $IM \neq M$ have the same length if M is finite. This allows us to introduce the fundamental notions of grade and depth.

In connection with regular sequences, finite modules over Noetherian rings are distinguished for two reasons: first, every zero-divisor of M is contained in an associated prime ideal, and, second, the number of these prime ideals is finite. Both facts together imply the following proposition that is 'among the most useful in the theory of commutative rings' (Kaplansky [231], p. 56).

Proposition 1.2.1. *Let R be a Noetherian ring, and M a finite R-module. If an ideal $I \subset R$ consists of zero-divisors of M, then $I \subset \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass} M$.*

PROOF. If $I \not\subset \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M$, then there exists $a \in I$ with $a \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} M$. This follows immediately from 1.2.2. □

The following lemma, which we have just used in its simplest form, is the standard argument of 'prime avoidance'.

Lemma 1.2.2. *Let R be a ring, $\mathfrak{p}_1,\dots,\mathfrak{p}_m$ prime ideals, M an R-module, and $x_1,\dots,x_n \in M$. Set $N = \sum_{i=1}^n Rx_i$. If $N_{\mathfrak{p}_j} \not\subset \mathfrak{p}_j M_{\mathfrak{p}_j}$ for $j = 1,\dots,m$, then there exist $a_2,\dots,a_n \in R$ such that $x_1 + \sum_{i=2}^n a_i x_i \notin \mathfrak{p}_j M_{\mathfrak{p}_j}$ for $j = 1,\dots,m$.*

PROOF. We use induction on m, and so suppose that there are $a_2',\dots,a_n' \in R$ for which $x_1' = x_1 + \sum_{i=2}^n a_i' x_i \notin \mathfrak{p}_j M_{\mathfrak{p}_j}$ for $j = 1,\dots,m-1$. Moreover, it is no restriction to assume that the $\mathfrak{p}_i$ are pairwise distinct and that $\mathfrak{p}_m$ is a minimal member of $\mathfrak{p}_1,\dots,\mathfrak{p}_m$. So there exists $r \in (\bigcap_{j=1}^{m-1} \mathfrak{p}_j) \setminus \mathfrak{p}_m$. Put $x_i' = rx_i$ for $i = 2,\dots,n$ and $N' = \sum_{i=1}^n Rx_i'$. Since $r \notin \mathfrak{p}_m$ we have $N'_{\mathfrak{p}_m} = N_{\mathfrak{p}_m}$. On the other hand, as $r \in \mathfrak{p}_j$ for $j = 1,\dots,m-1$, it follows that $x_1' + x_i' \notin \mathfrak{p}_j M_{\mathfrak{p}_j}$ for $i = 2,\dots,n$ and $j = 1,\dots,m-1$. If $x_1' \notin \mathfrak{p}_m M_{\mathfrak{p}_m}$, then x_1' is the element desired; otherwise $x_1' + x_i' \notin \mathfrak{p}_m M_{\mathfrak{p}_m}$ for some $i \in \{2,\dots,n\}$, and we choose $x_1' + x_i'$. □

Note that if $M = R$ and $N = I \subset R$, then the condition $N_{\mathfrak{p}_j} \not\subset \mathfrak{p}_j M_{\mathfrak{p}_j}$ simplifies to $I \not\subset \mathfrak{p}_j$.

Suppose that an ideal I is contained in $\mathfrak{p} \in \operatorname{Ass} M$. By definition, there exists $z \in M$ with $\mathfrak{p} = \operatorname{Ann} z$. Hence the assignment $1 \mapsto z$ induces a monomorphism $\varphi' : R/\mathfrak{p} \to M$, and thus a non-zero homomorphism $\varphi : R/I \to M$. This simple observation allows us to describe in homological terms that a certain ideal consists of zero-divisors:

Proposition 1.2.3. *Let R be a ring, and M, N R-modules. Set $I = \operatorname{Ann} N$.*
(a) *If I contains an M-regular element, then* $\operatorname{Hom}_R(N, M) = 0$.
(b) *Conversely, if R is Noetherian, and M, N are finite,* $\operatorname{Hom}_R(N, M) = 0$ *implies that I contains an M-regular element.*

Proof. (a) is evident. (b) Assume that I consists of zero-divisors of M, and apply 1.2.1 to find a $\mathfrak{p} \in \operatorname{Ass} M$ such that $I \subset \mathfrak{p}$. By hypothesis, $\mathfrak{p} \in \operatorname{Supp} N$; so $N_\mathfrak{p} \otimes k(\mathfrak{p}) \neq 0$ by Nakayama's lemma, and since $N_\mathfrak{p} \otimes k(\mathfrak{p})$ is just a direct sum of copies of $k(\mathfrak{p})$, one has an epimorphism $N_\mathfrak{p} \to k(\mathfrak{p})$. (By $k(\mathfrak{p})$ we denote the residue class field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ of $R_\mathfrak{p}$.) Note that $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass} M_\mathfrak{p}$. Hence the observation above yields a non-zero $\varphi' \in \operatorname{Hom}_{R_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p})$. Since $\operatorname{Hom}_{R_\mathfrak{p}}(N_\mathfrak{p}, M_\mathfrak{p}) \cong \operatorname{Hom}_R(N, M)_\mathfrak{p}$, it follows that $\operatorname{Hom}_R(N, M) \neq 0$. (See [318], Theorem 3.84 for the isomorphism just applied.) $\square$

Lemma 1.2.4. *Let R be a ring, M, N be R-modules, and $\boldsymbol{x} = x_1, \dots, x_n$ a weak M-sequence in* $\operatorname{Ann} N$. *Then*

$$\operatorname{Hom}_R(N, M/\boldsymbol{x}M) \cong \operatorname{Ext}_R^n(N, M).$$

Proof. We use induction on n, starting from the vacuous case $n = 0$. Let $n \geq 1$, and set $\boldsymbol{x}' = x_1, \dots, x_{n-1}$. Then the induction hypothesis implies that $\operatorname{Ext}_R^{n-1}(N, M) \cong \operatorname{Hom}_R(N, M/\boldsymbol{x}'M)$. As x_n is $(M/\boldsymbol{x}'M)$-regular, $\operatorname{Ext}_R^{n-1}(N, M) = 0$ by 1.2.3. Therefore the exact sequence

$$0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$$

yields an exact sequence

$$0 \longrightarrow \operatorname{Ext}_R^{n-1}(N, M/xM) \xrightarrow{\psi} \operatorname{Ext}_R^n(N, M) \xrightarrow{\varphi} \operatorname{Ext}_R^n(N, M).$$

The map φ is multiplication by x_1 inherited from M, but multiplication by x_1 on N also induces φ; see [318], Theorem 7.16. Since $x_1 \in \operatorname{Ann} N$, one has $\varphi = 0$. Hence ψ is an isomorphism, and a second application of the induction hypothesis yields the assertion. $\square$

Let R be Noetherian, I an ideal, M a finite R-module with $M \neq IM$, and $\boldsymbol{x} = x_1, \dots, x_n$ a maximal M-sequence in I. From 1.2.3 and 1.2.4

we have, since I contains an $(M/(x_1,\dots,x_{i-1})M)$-regular element for $i = 1,\dots,n$,

$$\operatorname{Ext}_R^{i-1}(R/I,M) \cong \operatorname{Hom}_R\big(R/I, M/(x_1,\dots,x_{i-1})M\big) = 0.$$

On the other hand, since $IM \neq M$ and $\boldsymbol{x}$ is a maximal M-sequence in I, then I must consist of zero-divisors of $M/\boldsymbol{x}M$, whence

$$\operatorname{Ext}_R^n(R/I,M) \cong \operatorname{Hom}_R(R/I, M/\boldsymbol{x}M) \neq 0.$$

We have therefore proved

Theorem 1.2.5 (Rees). *Let R be a Noetherian ring, M a finite R-module, and I an ideal such that $IM \neq M$. Then all maximal M-sequences in I have the same length n given by*

$$n = \min\{i : \operatorname{Ext}_R^i(R/I,M) \neq 0\}.$$

Definition 1.2.6. Let R be a Noetherian ring, M a finite R-module, and I an ideal such that $IM \neq M$. Then the common length of the maximal M-sequences in I is called the *grade of I on M*, denoted by

$$\operatorname{grade}(I,M).$$

We complement this definition by setting $\operatorname{grade}(I,M) = \infty$ if $IM = M$. This is consistent with 1.2.5:

$$\operatorname{grade}(I,M) = \infty \quad \Longleftrightarrow \quad \operatorname{Ext}_R^i(R/I,M) = 0 \text{ for all } i.$$

For, if $IM = M$, then $\operatorname{Supp} M \cap \operatorname{Supp} R/I = \emptyset$ by Nakayama's lemma, hence

$$\text{(1)} \qquad \operatorname{Supp}\operatorname{Ext}_R^i(R/I,M) \subset \operatorname{Supp} M \cap \operatorname{Supp} R/I = \emptyset;$$

conversely, if $\operatorname{Ext}_R^i(R/I,M) = 0$ for all i, then 1.2.5 gives $IM = M$.

The inclusion in (1) results from the natural isomorphism

$$\operatorname{Ext}_{R_\mathfrak{p}}^i(N_\mathfrak{p}, M_\mathfrak{p}) \cong \operatorname{Ext}_R^i(N,M)_\mathfrak{p}$$

which holds if R is Noetherian, N a finite R-module, M an arbitrary R-module, and $\mathfrak{p} \in \operatorname{Spec} R$; see [318], Theorem 9.50.

A special situation will occur so often that it merits a special notation:

Definition 1.2.7. Let $(R,\mathfrak{m},k)$ be a Noetherian local ring, and M a finite R-module. Then the grade of $\mathfrak{m}$ on M is called the *depth of M*, denoted

$$\operatorname{depth} M.$$

Because of its importance we repeat the most often used special case of 1.2.5:

Theorem 1.2.8. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and M a finite non-zero R-module. Then* $\operatorname{depth} M = \min\{i : \operatorname{Ext}_R^i(k, M) \neq 0\}$.

Some formulas for grade. We now study the behaviour of $\operatorname{grade}(I, M)$ along exact sequences.

Proposition 1.2.9. *Let R be a Noetherian ring, $I \subset R$ an ideal, and $0 \to U \to M \to N \to 0$ an exact sequence of finite R-modules. Then*

$$\begin{aligned}\operatorname{grade}(I, M) &\geq \min\{\operatorname{grade}(I, U), \operatorname{grade}(I, N)\},\\ \operatorname{grade}(I, U) &\geq \min\{\operatorname{grade}(I, M), \operatorname{grade}(I, N) + 1\},\\ \operatorname{grade}(I, N) &\geq \min\{\operatorname{grade}(I, U) - 1, \operatorname{grade}(I, M)\}.\end{aligned}$$

PROOF. The given exact sequence induces a long exact sequence

$$\begin{aligned}\cdots \to \operatorname{Ext}_R^{i-1}(R/I, N) \to \operatorname{Ext}_R^i(R/I, U) \to \operatorname{Ext}_R^i(R/I, M)&\\ \to \operatorname{Ext}_R^i(R/I, N) \to \operatorname{Ext}_R^{i+1}(R/I, U) \to \cdots&\end{aligned}$$

One observes that $\operatorname{Ext}_R^i(R/I, M) = 0$ if $\operatorname{Ext}_R^i(R/I, U)$ and $\operatorname{Ext}_R^i(R/I, N)$ both vanish. Therefore the first inequality follows from 1.2.5 and our discussion of the case $\operatorname{grade}(I, \_) = \infty$. Completely analogous arguments show the second and the third inequality. □

The next proposition collects some formulas which are useful in the computation of grades. (In the sequel $V(I)$ denotes the set of prime ideals containing I.)

Proposition 1.2.10. *Let R be a Noetherian ring, I, J ideals of R, and M a finite R-module. Then*

(a) $\operatorname{grade}(I, M) = \inf\{\operatorname{depth} M_\mathfrak{p} : \mathfrak{p} \in V(I)\}$,

(b) $\operatorname{grade}(I, M) = \operatorname{grade}(\operatorname{Rad} I, M)$,

(c) $\operatorname{grade}(I \cap J, M) = \min\{\operatorname{grade}(I, M), \operatorname{grade}(J, M)\}$,

(d) *if $\boldsymbol{x} = x_1, \ldots, x_n$ is an M-sequence in I, then* $\operatorname{grade}(I/(\boldsymbol{x}), M/\boldsymbol{x}M) = \operatorname{grade}(I, M/\boldsymbol{x}M) = \operatorname{grade}(I, M) - n$,

(e) *if N is a finite R-module with* $\operatorname{Supp} N = V(I)$, *then*

$$\operatorname{grade}(I, M) = \inf\{i : \operatorname{Ext}_R^i(N, M) \neq 0\}.$$

PROOF. (a) It is evident from the definition that $\operatorname{grade}(I, M) \leq \operatorname{grade}(\mathfrak{p}, M)$ for $\mathfrak{p} \in V(I)$, and it follows from 1.1.3 that $\operatorname{grade}(\mathfrak{p}, M) \leq \operatorname{depth} M_\mathfrak{p}$. Furthermore, if $\operatorname{grade}(I, M) = \infty$, then $\operatorname{Supp} M \cap V(I) = \emptyset$ so that $\operatorname{depth} M_\mathfrak{p} = \infty$ for all $\mathfrak{p} \in V(I)$. Thus suppose $IM \neq M$ and choose a maximal M-sequence $\boldsymbol{x}$ in I. By 1.2.1 there exists $\mathfrak{p} \in \operatorname{Ass}(M/\boldsymbol{x}M)$ with $I \subset \mathfrak{p}$. Since $\mathfrak{p}R_\mathfrak{p} \in \operatorname{Ass}(M/\boldsymbol{x}M)_\mathfrak{p}$ and $(M/\boldsymbol{x}M)_\mathfrak{p} \cong M_\mathfrak{p}/\boldsymbol{x}M_\mathfrak{p}$, the ideal

$\mathfrak{p}R_\mathfrak{p}$ consists of zero-divisors of $M_\mathfrak{p}/\boldsymbol{x}M_\mathfrak{p}$, and $\boldsymbol{x}$ (as a sequence in $R_\mathfrak{p}$) is a maximal $M_\mathfrak{p}$-sequence.

(b) and (c) follow easily from (a).

(d) Set $\bar{R} = R/(\boldsymbol{x})$, $\bar{I} = I/(\boldsymbol{x})$, and $\bar{M} = M/\boldsymbol{x}M$. Elementary arguments show that $IM = M \iff I\bar{M} = \bar{M} \iff \bar{I}\bar{M} = \bar{M}$. Furthermore $y_1, \dots, y_n \in I$ form an $\bar{M}$-sequence if and only if $\bar{y}_1, \dots, \bar{y}_n \in \bar{I}$ form such a sequence. This proves the first equation. The second equation results from 1.2.5.

(e) The hypothesis entails that $\operatorname{Rad}\operatorname{Ann} N = \operatorname{Rad} I$. By (b) we may therefore assume that $I = \operatorname{Ann} N$. Now one repeats the proof of 1.2.5 (and the discussion of the case $IM = M$) with R/I replaced by N. □

The name 'grade' was originally used by Rees [303] for a different, though related invariant:

Definition 1.2.11. Let R be a Noetherian ring and $M \neq 0$ a finite R-module. Then the *grade of M* is given by

$$\operatorname{grade} M = \min\{i : \operatorname{Ext}_R^i(M, R) \neq 0\}.$$

For systematic reasons the grade of the zero-module is infinity.

It follows directly from 1.2.10(e) that $\operatorname{grade} M = \operatorname{grade}(\operatorname{Ann} M, R)$. It is customary to set

$$\operatorname{grade} I = \operatorname{grade} R/I = \operatorname{grade}(I, R),$$

for an ideal $I \subset R$, and we follow this convention. (Of course, $\operatorname{grade} I$ has two different meanings now, but we will never use it to denote the grade of the module I.)

Depth and dimension. Let $(R, \mathfrak{m})$ be Noetherian local and M a finite R-module. All the minimal elements of $\operatorname{Supp} M$ belong to $\operatorname{Ass} M$. Therefore, if $x \in \mathfrak{m}$ is an M-regular element, then $x \notin \mathfrak{p}$ for all minimal elements of $\operatorname{Supp} M$, and induction yields $\dim M/\boldsymbol{x}M = \dim M - n$ if $\boldsymbol{x} = x_1, \dots, x_n$ is an M-sequence. (Note that $\dim M/\boldsymbol{x}M \geq \dim M - n$ is automatic; see A.4.) We have proved:

Proposition 1.2.12. *Let $(R, \mathfrak{m})$ be a Noetherian local ring and $M \neq 0$ a finite R-module. Then every M-sequence is part of a system of parameters of M. In particular* $\operatorname{depth} M \leq \dim M$.

The inequality in 1.2.12 can be somewhat refined:

Proposition 1.2.13. *With the notation of* 1.2.12 *one has* $\operatorname{depth} M \leq \dim R/\mathfrak{p}$ *for all* $\mathfrak{p} \in \operatorname{Ass} M$.

PROOF. We use induction on $\operatorname{depth} M$. There is nothing to prove for $\operatorname{depth} M = 0$. If $\operatorname{depth} M > 0$, then there exists an M-regular $x \in \mathfrak{m}$. For

$\mathfrak{p} \in \operatorname{Ass} M$ we choose $z \in M$ such that Rz is maximal among the cyclic submodules of M annihilated by $\mathfrak{p}$. If $z \in xM$, then $z = xy$ with $y \in M$, and $\mathfrak{p}y = 0$ since x is M-regular; moreover, Rz is a proper submodule of Ry, contrary to the choice of z. Therefore $\mathfrak{p}$ consists of zero-divisors of M/xM, and is contained in some $\mathfrak{q} \in \operatorname{Ass}(M/xM)$. As $x \notin \mathfrak{p}$, we have $\mathfrak{p} \notin \operatorname{Supp}(M/xM)$, and thus $\mathfrak{p} \neq \mathfrak{q}$. Now $\operatorname{depth}(M/xM) = \operatorname{depth} M - 1$ by 1.2.10, whence, by induction,

$$\dim R/\mathfrak{p} > \dim R/\mathfrak{q} \geq \operatorname{depth}(M/xM) = \operatorname{depth} M - 1. \qquad \square$$

A global variant of 1.2.12 says that height bounds grade.

Proposition 1.2.14. *Let R be a Noetherian ring and $I \subset R$ an ideal. Then* $\operatorname{grade} I \leq \operatorname{height} I$.

PROOF. Since $\operatorname{grade} I = \inf\{\operatorname{depth} R_\mathfrak{p} : \mathfrak{p} \in V(I)\}$ by 1.2.10, and $\operatorname{height} I = \inf\{\dim R_\mathfrak{p} : \mathfrak{p} \in V(I)\}$, the assertion follows from 1.2.12. $\square$

Depth, type, and flat extensions. Finally we investigate how depth behaves under flat local extensions. As a by-product we obtain a result on the behaviour of the type of a module under such extensions. This is an invariant which refines the information given by the depth:

Definition 1.2.15. Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, and M a finite non-zero R-module of depth t. The number $r(M) = \dim_k \operatorname{Ext}^t_R(k, M)$ is called the *type of M*.

Proposition 1.2.16. *Let $\varphi : (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a homomorphism of Noetherian local rings. Suppose M is a finite R-module, and N is a finite S-module which is flat over R. Then*
(a) $\operatorname{depth}_S M \otimes_R N = \operatorname{depth}_R M + \operatorname{depth}_S N/\mathfrak{m}N$,
(b) $r_S(M \otimes_R N) = r_R(M) \cdot r_S(N/\mathfrak{m}N)$.

The proof of the proposition is by reduction to the case of depth 0. We collect the essential arguments in a lemma.

Lemma 1.2.17. *Under the hypotheses of* 1.2.16 *the following hold:*

(a) $\dim_l \operatorname{Hom}_S(l, M \otimes N) = \dim_k \operatorname{Hom}_R(k, M) \cdot \dim_l \operatorname{Hom}_S(l, N/\mathfrak{m}N)$,

(b) *if $\mathbf{y}$ is an $(N/\mathfrak{m}N)$-sequence in S, then $\mathbf{y}$ is an $(M \otimes_R N)$-sequence, and $N/\mathbf{y}N$ is flat over R.*

PROOF. (a) Set $T = S/\mathfrak{m}S$. There is a natural isomorphism

$$\operatorname{Hom}_S\big(l, \operatorname{Hom}_S(T, M \otimes N)\big) \cong \operatorname{Hom}_S(l, M \otimes N), \tag{2}$$

since the modules on both sides can be identified with the submodule $U = \{z \in M \otimes N : \mathfrak{n}z = 0\}$ of $M \otimes N$. As N is flat over R, we have a natural isomorphism

$$\operatorname{Hom}_S(T, M \otimes N) = \operatorname{Hom}_S(k \otimes S, M \otimes N) \cong \operatorname{Hom}_R(k, M) \otimes N.$$

(see [318], 3.82 and 3.83). Now $\operatorname{Hom}_R(k, M) \cong k^s$ for some $s \geq 0$, and so $\operatorname{Hom}_R(k, M) \otimes N \cong (N/\mathfrak{m}N)^s$. In conjunction with (2), this yields the equation asserted.

(b) One has a natural isomorphism $(M \otimes N)/J(M \otimes N) \cong M \otimes (N/JN)$ for an arbitrary ideal $J \subset S$. Therefore we may use induction on the length n of $\boldsymbol{y}$, and only the case $n = 1$, $\boldsymbol{y} = y$ needs justification.

By Krull's intersection theorem one has $\bigcap_{i=0}^{\infty} \mathfrak{m}^i(M \otimes N) = 0$. Suppose that $yz = 0$ for some $z \in M \otimes N$. If $z \neq 0$, then there exists i such that $z \in \mathfrak{m}^i(M \otimes N) \setminus \mathfrak{m}^{i+1}(M \otimes N)$, and y would be a zero-divisor on $\mathfrak{m}^i(M \otimes N)/\mathfrak{m}^{i+1}(M \otimes N)$. However, consider the embedding $\mathfrak{m}^i M \to M$. Since N is flat, the induced map $\mathfrak{m}^i M \otimes N \to M \otimes N$ is also injective, and its image is $\mathfrak{m}^i(M \otimes N)$. The same reasoning for $\mathfrak{m}^{i+1}$ and flatness again then yield an isomorphism

$$\mathfrak{m}^i(M \otimes N)/\mathfrak{m}^{i+1}(M \otimes N) \cong (\mathfrak{m}^i M/\mathfrak{m}^{i+1} M) \otimes N \cong k^t \otimes N \cong (N/\mathfrak{m}N)^t$$

for some $t \geq 0$. Since y is regular on $N/\mathfrak{m}N$, it must be regular on $\mathfrak{m}^i(M \otimes N)/\mathfrak{m}^{i+1}(M \otimes N)$.

In order to test flatness of N/yN it suffices to consider exact sequences

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

of finite R-modules ([318], Theorem 3.53). By hypothesis

$$0 \longrightarrow M_1 \otimes N \longrightarrow M_2 \otimes N \longrightarrow M_3 \otimes N \longrightarrow 0$$

is also exact. As has been shown previously, y is regular on $M_3 \otimes N$, and $(M_3 \otimes N)/y(M_3 \otimes N) \cong M_3 \otimes N/yN$. Therefore 1.1.4 yields the exactness of

$$0 \longrightarrow M_1 \otimes N/yN \longrightarrow M_2 \otimes N/yN \longrightarrow M_3 \otimes N/yN \longrightarrow 0. \qquad \square$$

Proof of 1.2.16. Let $\boldsymbol{x} = x_1, \dots, x_m$ be a maximal M-sequence, and $\boldsymbol{y} = y_1, \dots, y_n$ a maximal $(N/\mathfrak{m}N)$-sequence. First, $\varphi(\boldsymbol{x}) = \varphi(x_1), \dots, \varphi(x_m)$ is an $(M \otimes N)$-sequence; see 1.1.2. Second, by 1.2.17, $\boldsymbol{y}$ is an $(\bar{M} \otimes N)$-sequence where $\bar{M} = M/\boldsymbol{x}M$. Since $\bar{M} \otimes N \cong (M \otimes N)/\varphi(\boldsymbol{x})(M \otimes N)$, it follows that $\varphi(\boldsymbol{x}), \boldsymbol{y}$ is an $M \otimes N$-sequence.

Set $N' = N/\boldsymbol{y}N$. Then $N'/\mathfrak{m}N' \cong (N/\mathfrak{m}N)/\boldsymbol{y}(N/\mathfrak{m}N)$, and

$$(M \otimes N)/(\varphi(\boldsymbol{x}), \boldsymbol{y})(M \otimes N) \cong \bar{M} \otimes N'.$$

An application of 1.2.4 therefore gives the isomorphisms

$$\operatorname{Hom}_R(k, \bar{M}) \cong \operatorname{Ext}_R^m(k, M), \quad \operatorname{Hom}_S(l, N'/\mathfrak{m}N') \cong \operatorname{Ext}_S^n(l, N/\mathfrak{m}N),$$
$$\operatorname{Hom}_S(l, \bar{M} \otimes N') \cong \operatorname{Ext}_S^{m+n}(l, M \otimes N).$$

Part (a) of 1.2.17 implies that $\dim_l \operatorname{Ext}_S^{m+n}(l, M \otimes N)$ has the dimension required for (b), and in particular is non-zero. Together with the fact that $\varphi(\boldsymbol{x}), \boldsymbol{y}$ is an $(M \otimes N)$-sequence this proves $\operatorname{depth}(M \otimes N) = m + n$. $\square$

The type of a module of depth 0 is the dimension of its socle:

Definition 1.2.18. Let M be a module over a local ring $(R, \mathfrak{m}, k)$. Then

$$\operatorname{Soc} M = (0 : \mathfrak{m})_M \cong \operatorname{Hom}_R(k, M)$$

is called *the socle of* M.

For ease of reference we formulate the following lemma which was already verified in the proof of 1.2.16.

Lemma 1.2.19. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, M a finite R-module and $\boldsymbol{x}$ a maximal M-sequence. Then $r(M) = \dim_k \operatorname{Soc}(M/\boldsymbol{x}M)$.*

Exercises

1.2.20. Let k be a field and $R = k[[X]][Y]$. Deduce that X, Y and $1 - XY$ are maximal R-sequences. (This example shows that the condition $IM \neq M$ in 1.2.5 is relevant.)

1.2.21. Let R be a Noetherian ring, $I \subset R$ an ideal, $I = (x_1, \ldots, x_n)$, and M a finite R-module with $IM \neq M$. Set $g = \operatorname{grade}(I, M)$. Prove
(a) I can be generated by elements $y_1, \ldots, y_n$ such that $y_{i_1}, \ldots, y_{i_h}$ form an M-sequence for all $i_1, \ldots, i_h$ with $1 \leq i_1 < \cdots < i_h \leq n$, $h \leq g$,
(b) if $y_1, \ldots, y_n$ satisfies (a), then, in fact, every permutation of $y_{i_1}, \ldots, y_{i_h}$ is an M-sequence.
Hint: It is possible to choose $y_i = x_i + \sum_{j \neq i} a_j x_j$. Use the discussion above 1.1.6 for (b).

1.2.22. Let R be a Noetherian ring, $I \subset R$ an ideal, and M a finite R-module with $IM \neq M$. Set $\bar{R} = R/\operatorname{Ann} M$.
(a) Prove that $\operatorname{grade}(I, M) \leq \operatorname{height} I\bar{R}$.
(b) Give an example where $\operatorname{grade}(I, M) > \operatorname{height} I$.
(c) Show that if $I = (x_1, \ldots, x_n)$, then $\operatorname{grade}(I, M) \leq n$.

1.2.23. Let R be a Noetherian local ring, and $I \subset R$ an ideal. Show $\operatorname{grade} I \geq \operatorname{depth} R - \dim R/I$. (Hint: Use 1.2.13.)

1.2.24. Let R be a Noetherian ring, M a finite R-module, and I an ideal of R. Show that $\operatorname{grade}(I, M) \geq 2$ if and only if the natural homomorphism $M \to \operatorname{Hom}_R(I, M)$ is an isomorphism.

1.2.25. Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a homomorphism of local rings, and N an R-flat S-module such that $N/\mathfrak{m}N$ has finite length over S. Show that for every finite length R-module M, $\ell_S(M \otimes N) = \ell_R(M) \cdot \ell_S(N/\mathfrak{m}N)$. (The symbol ℓ denotes length). Hint: use induction on $\ell(M)$.

1.2.26. Let $\varphi : (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a homomorphism of Noetherian local rings, and M an S-module which is finite as an R-module.
(a) Suppose $\mathfrak{p} \in \operatorname{Ass}_S M$, and let $x \in M$ with $\operatorname{Ann}_S x = \mathfrak{p}$. Prove that φ induces an embedding $R/(\mathfrak{p} \cap R) \to S/\mathfrak{p} \cong Sx$ which makes $S/\mathfrak{p}$ a finite $R/(\mathfrak{p} \cap R)$-module. Conclude that $\mathfrak{p} \cap R \neq \mathfrak{m}$, if $\mathfrak{p} \neq \mathfrak{n}$.
(b) Show that $\operatorname{depth}_R M = \operatorname{depth}_S M$.
(c) Suppose in addition that φ is surjective. Prove $r_R(M) = r_S(M)$.

1.2.27. Let R be a Noetherian ring, M a finite R-module, and N an arbitrary R-module. Deduce that $\operatorname{Ass}\operatorname{Hom}_R(M,N) = \operatorname{Supp} M \cap \operatorname{Ass} N$.

1.3 Depth and projective dimension

Let R be a ring, and M an R-module; M has an augmented projective resolution

$$P_\bullet : \cdots \longrightarrow P_n \xrightarrow{\varphi_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \longrightarrow 0.$$

(By definition a projective resolution is non-augmented, i.e. M is replaced by 0; for the most part it is clear from the context whether one uses a non-augmented resolution or an augmented one, so that one need not mention the attribute 'augmented' explicitly.) Set $M_0 = M$ and $M_i = \operatorname{Ker}\varphi_{i-1}$ for $i \geq 1$. The modules M_i depend obviously on $P_\bullet$. However, M determines M_i up to projective equivalence ([318], Theorem 9.4), and therefore it is justified to call M_i the *i-th syzygy of M*. The *projective dimension of M*, abbreviated $\operatorname{proj}\dim M$, is infinity if none of the modules M_i is projective. Otherwise $\operatorname{proj}\dim M$ is the least integer n for which M_n is projective; replacing P_n by M_n one gets a projective resolution of M of length n:

$$0 \longrightarrow M_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

For a finite module M over a Noetherian local ring $(R, \mathfrak{m}, k)$ there is a very natural condition which, if satisfied by $P_\bullet$, determines $P_\bullet$ uniquely. It is a consequence of Nakayama's lemma that $x_1, \dots, x_m \in M$ form a minimal system of generators of M if and only if the residue classes $\bar{x}_1, \dots, \bar{x}_m \in M/\mathfrak{m}M \cong M \otimes k$ are a k-basis of $M \otimes k$. Therefore $m = \dim_k M \otimes k$, and

$$\mu(M) = \dim_k M \otimes k$$

is the *minimal number of generators of M*. Set $\beta_0 = \mu(M)$. We choose a minimal system $x_1, \dots, x_{\beta_0}$ of generators of M and specify an epimorphism $\varphi_0 : R^{\beta_0} \to M$ by $\varphi_0(e_i) = x_i$ where $e_1, \dots, e_{\beta_0}$ is the canonical basis of R^{β_0}. Next we set $\beta_1 = \mu(\operatorname{Ker}\varphi_0)$ and define similarly an epimorphism $R^{\beta_1} \to \operatorname{Ker}\varphi_0$. Proceeding in this manner we construct a *minimal free resolution*

$$F_\bullet : \cdots \longrightarrow R^{\beta_n} \xrightarrow{\varphi_n} R^{\beta_{n-1}} \longrightarrow \cdots \longrightarrow R^{\beta_1} \xrightarrow{\varphi_1} R^{\beta_0} \xrightarrow{\varphi_0} M \longrightarrow 0.$$

It is left as an exercise for the reader to prove that $F_\bullet$ is determined by M up to an isomorphism of complexes. The number $\beta_i(M) = \beta_i$ is called the i-th *Betti number of M*.

Proposition 1.3.1. *Let $(R, \mathfrak{m}, k)$ be a Noetherian local ring, M a finite R-module, and*

$$F_\bullet : \cdots \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

a free resolution of M. Then the following are equivalent:
(a) *$F_\bullet$ is minimal;*
(b) *$\varphi_i(F_i) \subset \mathfrak{m}F_{i-1}$ for all $i \geq 1$;*
(c) *$\operatorname{rank} F_i = \dim_k \operatorname{Tor}_i^R(M,k)$ for all $i \geq 0$,*
(d) *$\operatorname{rank} F_i = \dim_k \operatorname{Ext}_R^i(M,k)$ for all $i \geq 0$.*

Proof. The equivalence of (a) and (b) follows easily from Nakayama's lemma. Since $\operatorname{Tor}_i^R(M,k) = H_i(F_\bullet \otimes k)$, (c) holds if and only if $\varphi_i \otimes k = 0$ for all $i \geq 0$. The latter condition is evidently equivalent to (b). To relate (b) to (d) one uses that $\operatorname{Ext}_R^i(M,k) = H^i(\operatorname{Hom}_R(F_\bullet,k)$. □

Corollary 1.3.2. *Let $(R,\mathfrak{m},k)$ be a Noetherian local ring, and M a finite R-module. Then $\beta_i(M) = \dim_k \operatorname{Tor}_i^R(M,k)$ for all i and*

$$\operatorname{proj\,dim} M = \sup\{i\colon \operatorname{Tor}_i^R(M,k) \neq 0\}.$$

The following theorem, the 'Auslander–Buchsbaum formula', is not only of theoretical importance, but also an effective instrument for the computation of the depth of a module.

Theorem 1.3.3 (Auslander–Buchsbaum). *Let $(R,\mathfrak{m})$ be a Noetherian local ring, and $M \neq 0$ a finite R-module. If* $\operatorname{proj\,dim} M < \infty$, *then*

$$\operatorname{proj\,dim} M + \operatorname{depth} M = \operatorname{depth} R.$$

The proof is by induction on $\operatorname{depth} R$. We isolate the main arguments in two lemmas, the first of which, in view of a later application, is more general than needed presently.

Lemma 1.3.4. *Let $(R,\mathfrak{m},k)$ be a local ring, and $\varphi\colon F \to G$ a homomorphism of finite R-modules. Suppose that F is free, and let M be an R-module with $\mathfrak{m} \in \operatorname{Ass} M$. Suppose that $\varphi \otimes M$ is injective. Then*
(a) *$\varphi \otimes k$ is injective;*
(b) *if G is a free R-module, then φ is injective, and $\varphi(F)$ is a free direct summand of G.*

Proof. Since $\mathfrak{m} \in \operatorname{Ass} M$, there exists an embedding $\iota\colon k \to M$. As F is a free R-module, the map $F \otimes \iota$ is also injective. Furthermore we have a commutative diagram

$$\begin{tikzcd} F \otimes k \arrow[r, "F\otimes\iota"] \arrow[d, "\varphi\otimes k"'] & F \otimes M \arrow[d, "\varphi\otimes M"] \\ G \otimes k \arrow[r] & G \otimes M \end{tikzcd}$$

If $\varphi \otimes M$ is injective, then $\varphi \otimes k$ is injective too. This proves (a).

For (b) one notes that its conclusion is equivalent to the injectivity of $\varphi \otimes k$. This is an easy consequence of Nakayama's lemma. □

Lemma 1.3.5. *Let $(R, \mathfrak{m})$ be a Noetherian local ring, and M a finite R-module. If $x \in \mathfrak{m}$ is R-regular and M-regular, then*

$$\operatorname{proj\,dim}_R M = \operatorname{proj\,dim}_{R/(x)} M/xM.$$

PROOF. Choose an augmented minimal free resolution $F_\bullet$ of M. Then $F_\bullet \otimes R/(x)$ is exact by 1.1.5, and therefore it is a minimal free resolution of M/xM over $R/(x)$. Now apply 1.3.2. □

PROOF OF 1.3.3. Let $\operatorname{depth} R = 0$ first. By hypothesis M has a (minimal) free resolution

$$F_\bullet : 0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

with $n = \operatorname{proj\,dim} M$. Since $\operatorname{depth} R = 0$, the maximal ideal $\mathfrak{m}$ is in $\operatorname{Ass} R$. If $n \geq 1$, i.e. if φ_n is really present, then, as shown in 1.3.4, φ_n maps F_n isomorphically onto a free direct summand of F_{n-1}, in contradiction to $\operatorname{proj\,dim} M = n$. Therefore $n = 0$, and furthermore $\operatorname{depth} M = \operatorname{depth} R = 0$ since M is a free R-module.

Let now $\operatorname{depth} R > 0$. Suppose first that $\operatorname{depth} M = 0$. Then 1.2.9 yields $\operatorname{depth} M_1 = 1$ for a first syzygy M_1 of M. Since $\operatorname{proj\,dim} M_1 = \operatorname{proj\,dim} M - 1$, it is enough to prove the desired formula for M_1. Thus we may assume $\operatorname{depth} M > 0$. Then $\mathfrak{m} \notin \operatorname{Ass} R$ and $\mathfrak{m} \notin \operatorname{Ass} M$. So $\mathfrak{m}$ contains an element x which is both R-regular and M-regular. The formulas for the passage from M to M/xM in 1.2.10 and 1.3.5 yield

$$\operatorname{depth}_{R/(x)} R/(x) = \operatorname{depth} R - 1, \quad \operatorname{depth}_{R/(x)} M/xM = \operatorname{depth}_R M - 1,$$
$$\operatorname{proj\,dim}_{R/(x)} M/xM = \operatorname{proj\,dim} M.$$

Therefore induction completes the proof. □

Exercises

1.3.6. Let R be a Noetherian local ring, M a finite R-module, and $\boldsymbol{x}$ an M-sequence of length n. Show $\operatorname{proj\,dim}(M/\boldsymbol{x}M) = \operatorname{proj\,dim} M + n$.

1.3.7. Let R be a Noetherian local ring, and N an n-th syzygy of a finite R-module in a finite free resolution. Prove that $\operatorname{depth} N \geq \min(n, \operatorname{depth} R)$.

1.4 Some linear algebra

In this section we collect several notions and results which may be classified as 'linear algebra': torsion-free and reflexive modules, the rank of a module, the acyclicity criterion of Buchsbaum and Eisenbud, and perfect modules.

Torsion-free and reflexive modules. Let R be a ring, and M an R-module. If the natural map $M \to M \otimes Q$, where Q is the total ring of fractions of R, is injective, then M is *torsion-free*; it is a *torsion module* if $M \otimes Q = 0$. The *dual of* M is the module $\operatorname{Hom}_R(M, R)$, which we usually denote by M^*; the *bidual* then is M^{**}, and analogous conventions apply to homomorphisms. The bilinear map $M \times M^* \to R$, $(x, \varphi) \mapsto \varphi(x)$, induces a natural homomorphism $h\colon M \to M^{**}$. We say that M is *torsionless* if h is injective, and that M is *reflexive* if h is bijective. Some relations between the notions just introduced are given in the exercises. Here we note a useful criterion:

Proposition 1.4.1. *Let R be a Noetherian ring, and M a finite R-module. Then:*

(a) *M is torsionless if and only if*
  (i) *$M_\mathfrak{p}$ is torsionless for all $\mathfrak{p} \in \operatorname{Ass} R$, and*
  (ii) $\operatorname{depth} M_\mathfrak{p} \geq 1$ *for* $\mathfrak{p} \in \operatorname{Spec} R$ *with* $\operatorname{depth} R_\mathfrak{p} \geq 1$;

(b) *M is reflexive if and only if*
  (i) *$M_\mathfrak{p}$ is reflexive for all $\mathfrak{p}$ with* $\operatorname{depth} R_\mathfrak{p} \leq 1$, *and*
  (ii) $\operatorname{depth} M_\mathfrak{p} \geq 2$ *for* $\mathfrak{p} \in \operatorname{Spec} R$ *with* $\operatorname{depth} R_\mathfrak{p} \geq 2$.

PROOF. Consider the natural map $h\colon M \to M^{**}$ and set $U = \operatorname{Ker} h$, $C = \operatorname{Coker} h$. Note that the construction of h commutes with localization in the situation considered. Therefore the necessity of conditions (i) in (a) and (b) is obvious. Next Exercise 1.4.19 implies

$$\operatorname{depth} M_\mathfrak{p}^{**} \geq \min(2, \operatorname{depth} R_\mathfrak{p})$$

for all $\mathfrak{p} \in \operatorname{Spec} R$. That (b)(ii) is necessary for reflexivity follows directly from this inequality. If M is torsionless, then $M_\mathfrak{p}$ is isomorphic to a submodule of $M_\mathfrak{p}^{**}$, and we get $\operatorname{depth} M_\mathfrak{p} \geq \min(1, \operatorname{depth} R_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$. So (a)(ii) is necessary for M to be torsionless.

As to the sufficiency of (a)(i) and (ii), note that $U_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \operatorname{Ass} R$ by (i), and, by (ii), $\operatorname{depth} U_\mathfrak{p} \geq 1$ if $\operatorname{depth} R_\mathfrak{p} \geq 1$. It follows that $\operatorname{Ass} U = \emptyset$, hence $U = 0$.

For the sufficiency of (b)(i) and (ii) we may now use that (a) gives us an exact sequence $0 \to M \to M^{**} \to C \to 0$. If $\operatorname{depth} R_\mathfrak{p} \leq 1$, then $C_\mathfrak{p} = 0$ by (i). If $\operatorname{depth} R_\mathfrak{p} \geq 2$, then $\operatorname{depth} M_\mathfrak{p} \geq 2$ by (ii), and $\operatorname{depth} M_\mathfrak{p}^{**} \geq 2$ by the inequality above. Therefore $\operatorname{depth} C_\mathfrak{p} \geq 1$, and it follows that $\operatorname{Ass} C = \emptyset$. $\square$

Rank. The dimension of a finite dimensional vector space over a field is given either by the minimal number of generators or by the maximal number of linearly independent elements. The second aspect of 'dimension' is generalized in the notion of 'rank':

Definition 1.4.2. Let R be a ring, M an R-module, and Q be the total ring of fractions of R. Then M has *rank* r if $M \otimes Q$ is a free Q-module of rank r. If $\varphi : M \to N$ is a homomorphism of R-modules, then φ has *rank* r if $\operatorname{Im}\varphi$ has rank r.

Proposition 1.4.3. *Let R be a Noetherian ring, and M an R-module with a finite free presentation $F_1 \xrightarrow{\varphi} F_0 \longrightarrow M \longrightarrow 0$. Then the following are equivalent:*
(a) *M has rank r;*
(b) *M has a free submodule N of rank r such that M/N is a torsion module;*
(c) *for all prime ideals $\mathfrak{p} \in \operatorname{Ass} R$ the $R_\mathfrak{p}$-module $M_\mathfrak{p}$ is free of rank r;*
(d) $\operatorname{rank}\varphi = \operatorname{rank} F_0 - r$.

Proof. (a) $\Rightarrow$ (b): A free basis $x_1, \ldots, x_r$ of $M \otimes Q$ can be formed from elements $x_i \in M$ (multiply by a suitable common denominator). Now take $N = \sum Rx_i$.

(b) $\Rightarrow$ (a): This is trivial.

(a) $\Rightarrow$ (c): $M_\mathfrak{p}$ is a localization of $M \otimes Q$.

(c) $\Rightarrow$ (a): Q is a semi-local ring. Its localizations with respect to its maximal ideals are just the localizations of R with respect to the maximal elements of $\operatorname{Ass} R$. By hypothesis M is therefore a projective module over Q, and moreover the localizations with respect to the maximal ideals of Q have the same rank r. Such a module is free; see Lemma 1.4.4 below.

(c) $\Longleftrightarrow$ (d): In view of the equivalence of (a) and (c) we can replace (d) by the condition that $(\operatorname{Im}\varphi)_\mathfrak{p}$ is free and $\operatorname{rank}(\operatorname{Im}\varphi)_\mathfrak{p} = \operatorname{rank} F_0 - r$ for all $\mathfrak{p} \in \operatorname{Ass} R$. Now consider the exact sequence

$$0 \longrightarrow (\operatorname{Im}\varphi)_\mathfrak{p} \longrightarrow (F_0)_\mathfrak{p} \longrightarrow M_\mathfrak{p} \longrightarrow 0.$$

If $M_\mathfrak{p}$ is free, then $(\operatorname{Im}\varphi)_\mathfrak{p}$ must be free. Since $\mathfrak{p} \in \operatorname{Ass} R$, the converse is also true; see 1.3.4. $\square$

Lemma 1.4.4. *Let R be a semi-local ring, and M a finite projective R-module. Then M is free if the localizations $M_\mathfrak{m}$ have the same rank r for all maximal ideals $\mathfrak{m}$ of R.*

Proof. We use induction on r. The case $r = 0$ is trivial. Suppose that $r > 0$. Then 1.2.2 (with $N = M$ and $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ denoting the maximal ideals of R) yields an element $x \in M$ such that $x \notin \mathfrak{m}M_\mathfrak{m}$ for all maximal ideals of M. Thus x is a member of a minimal system of generators of $M_\mathfrak{m}$. Since every such system is a basis of the free module $M_\mathfrak{m}$, one concludes that $(M/Rx)_\mathfrak{m}$ is free of rank $r-1$. By the induction hypothesis M/Rx is free of rank $r-1$. Therefore $M \cong Rx \oplus M/Rx$. In particular Rx is a projective R-module. But Rx is also free: the natural epimorphism $\varphi : R \to Rx$ yields an isomorphism $\varphi_\mathfrak{m} : R_\mathfrak{m} \to (Rx)_\mathfrak{m}$ for every maximal ideal $\mathfrak{m}$. Since $(\operatorname{Ker}\varphi)_\mathfrak{m} = \operatorname{Ker}\varphi_\mathfrak{m}$ it follows that $\operatorname{Ker}\varphi = 0$. $\square$

Rank is additive along exact sequences.

Proposition 1.4.5. *Let R be a Noetherian ring, and $0 \to U \to M \to N \to 0$ an exact sequence of finite R-modules. If two of U, M, N have a rank, then so does the third, and* $\operatorname{rank} M = \operatorname{rank} U + \operatorname{rank} N$.

PROOF. In view of 1.4.3 we may assume that R is local and of depth 0. Then two of U, M, N are free. If U and N are free, then so is M. Thus M is always free (after the reduction to depth 0), and the result follows from the equivalence of 1.4.3(a) and (d). □

Corollary 1.4.6. *Let R be a Noetherian ring, and M an R-module with a finite free resolution $F_\bullet : 0 \to F_s \to F_{s-1} \to \cdots \to F_1 \to F_0$. Then* $\operatorname{rank} M = \sum_{j=0}^{s}(-1)^j \operatorname{rank} F_j$.

PROOF. Observe 1.4.5 and use induction on s. □

Corollary 1.4.7. *Let R be a Noetherian ring, and $I \neq 0$ an ideal with a finite free resolution. Then I contains an R-regular element.*

PROOF. By 1.4.6 I has a rank, and that $\operatorname{rank} I + \operatorname{rank} R/I = \operatorname{rank} R = 1$ follows immediately from 1.4.5. Since I is torsion-free and non-zero, the only possibility is $\operatorname{rank} I = 1$, whence $\operatorname{rank} R/I = 0$. Thus R/I is annihilated by an R-regular element. □

Ideals of minors and Fitting invariants. Let U be an $m \times n$ matrix over R where $m, n \geq 0$. For $t = 1, \dots, \min(m, n)$ we then denote by $I_t(U)$ the ideal generated by the t-minors of U (the determinants of $t \times t$ submatrices). For systematic reasons one sets $I_t(U) = R$ for $t \leq 0$ and $I_t(U) = 0$ for $t > \min(m, n)$. If $\varphi : F \to G$ is a homomorphism of finite free R-modules, then φ is given by a matrix U with respect to bases of F and G. It is an elementary exercise to verify that the ideals $I_t(U)$ only depend on φ. Therefore we may put $I_t(\varphi) = I_t(U)$. It is just as easy to show that $I_t(\varphi)$ is already determined by the submodule $\operatorname{Im} \varphi$ of G. As proved by Fitting in 1936, these ideals are even invariants of $\operatorname{Coker} \varphi$ (when counted properly), and therefore called the *Fitting invariants* of $\operatorname{Coker} \varphi$: let

$$F_1 \xrightarrow{\varphi} F_0 \longrightarrow M \longrightarrow 0 \quad \text{and} \quad G_1 \xrightarrow{\psi} G_0 \longrightarrow M \longrightarrow 0$$

be finite free presentations of the R-module M, and $n = \operatorname{rank} F_0$, $p = \operatorname{rank} G_0$; then $I_{n-u}(\varphi) = I_{p-u}(\psi)$ for all $u \geq 0$. (The proof is left as an exercise for the reader.) This justifies the term *u-th Fitting invariant of M* for $I_{n-u}(\varphi)$.

It is an important property of the ideals $I_t(\varphi)$ that their formation commutes with ring extensions: if S is an R-algebra, then $I_t(\varphi \otimes S) = I_t(\varphi)S$. (Simply consider φ as given by a matrix.)

The ideals $I_t(\varphi)$ determine the minimal number $\mu(M_\mathfrak{p})$ of generators of a localization in the same way that they control the vector space dimension of M if R is a field.

Lemma 1.4.8. *Let R be a ring, M an R-module with a finite free presentation $F_1 \xrightarrow{\varphi} F_0 \longrightarrow M \longrightarrow 0$, and $\mathfrak{p}$ a prime ideal. Then the following are equivalent:*
(a) $I_t(\varphi) \not\subset \mathfrak{p}$;
(b) $(\operatorname{Im}\varphi)_\mathfrak{p}$ *contains a (free) direct summand of* $(F_0)_\mathfrak{p}$ *of rank* t;
(c) $\mu(M_\mathfrak{p}) \leq \operatorname{rank} F_0 - t$.

Proof. It is no restriction to assume that $R = R_\mathfrak{p}$. Nakayama's lemma entails that $\mu(M) = \mu(M/\mathfrak{p}M)$. Similarly it implies that $\operatorname{Im}\varphi$ contains a (free) direct summand of F_0 of rank t if and only if there are elements $x_1, \dots, x_t \in \operatorname{Im}\varphi$ which are linearly independent modulo $\mathfrak{p}F_0$. (Note that every direct summand of a finite free module over a local ring is free itself – again an application of Nakayama's lemma.) After these observations we may replace R by the field $R/\mathfrak{p}$. For vector spaces over fields the equivalence of (a), (b) and (c) is an elementary fact. □

Lemma 1.4.9. *With the notation of* 1.4.8, *the following are equivalent:*
(a) $I_t(\varphi) \not\subset \mathfrak{p}$ *and* $I_{t+1}(\varphi)_\mathfrak{p} = 0$;
(b) $(\operatorname{Im}\varphi)_\mathfrak{p}$ *is a free direct summand of* $(F_0)_\mathfrak{p}$ *of rank* t;
(c) $M_\mathfrak{p}$ *is free and* $\operatorname{rank} M_\mathfrak{p} = \operatorname{rank} F_0 - t$.

Proof. We may assume that $R = R_\mathfrak{p}$. Then each of (b) and (c) is equivalent to the split exactness of the sequence $0 \to \operatorname{Im}\varphi \to F_0 \to M \to 0$.

If (a) holds, then, with respect to suitable bases of F_1 and F_0, the matrix of φ has the form

$$\begin{pmatrix} \mathrm{id}_t & 0 \\ 0 & 0 \end{pmatrix}$$

where id_t is the $t \times t$ identity matrix. This implies (b). The converse is seen similarly. □

Let M be a finite module over a Noetherian ring R. Then M is a projective module (of rank r) if and only if $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module (of rank r) for all $\mathfrak{p} \in \operatorname{Spec} R$. Combining this fact with 1.4.9 we obtain the global version of 1.4.9:

Proposition 1.4.10. *Let R be a Noetherian ring, and M a finite R-module with a finite free presentation $F_1 \xrightarrow{\varphi} F_0 \longrightarrow M \longrightarrow 0$. Then the following are equivalent:*
(a) $I_r(\varphi) = R$ *and* $I_{r+1}(\varphi) = 0$;
(b) M *is projective and* $\operatorname{rank} M = \operatorname{rank} F_0 - r$.

The rank of a homomorphism $\varphi\colon F \to G$ is determined by the ideal $I_t(\varphi)$, just as in elementary linear algebra:

Proposition 1.4.11. *Let R be a Noetherian ring, and let $\varphi: F \to G$ be a homomorphism of finite free R-modules. Then* $\operatorname{rank}\varphi = r$ *if and only if* $\operatorname{grade} I_r(\varphi) \geq 1$ *and* $I_{r+1}(\varphi) = 0$.

The easy proof is left as an exercise for the reader.

The Buchsbaum–Eisenbud acyclicity criterion. Let R be a ring. A complex

$$G_{\bullet}: \cdots \longrightarrow G_m \xrightarrow{\psi_m} G_{m-1} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{\psi_1} G_0 \longrightarrow 0$$

of R-modules is called *acyclic* if $H_i(G_{\bullet}) = 0$ for all $i > 0$, and *split acyclic* if it is acyclic and $\psi_{i+1}(G_{i+1})$ is a direct summand of G_i for $i \geq 0$.

Let R be a Noetherian ring, and

$$F_{\bullet}: 0 \longrightarrow F_s \xrightarrow{\varphi_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

a complex of finite free R-modules. We want to develop a criterion for $F_{\bullet}$ to be acyclic. This criterion will involve ideals generated by certain minors of the homomorphisms φ_i. A first relation between the ideals $I_t(\varphi)$ and the acyclicity of complexes is given in the next proposition.

Proposition 1.4.12. *Let R be a ring, M an R-module,*

$$F_{\bullet}: 0 \longrightarrow F_s \xrightarrow{\varphi_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

be a complex of finite free R-modules, and $\mathfrak{p} \subset R$ be a prime ideal. Set $r_i = \sum_{j=i}^{s} (-1)^{j-i} \operatorname{rank} F_j$ for $i = 1, \ldots, s$. Then the following are equivalent:

(a) *$F_{\bullet} \otimes R_{\mathfrak{p}}$ is split acyclic;*

(b) *$I_{r_i}(\varphi_i) \not\subset \mathfrak{p}$ for $i = 1, \ldots, s$.*

Furthermore, $I_t(\varphi_i)_{\mathfrak{p}} = 0$ for all $i = 1, \ldots, s$ and $t > r_i$, if one of these conditions holds.

If $\mathfrak{p} \in \operatorname{Ass} M$, then (a) *and* (b) *are equivalent to*

(c) *$F_{\bullet} \otimes M_{\mathfrak{p}}$ is acyclic.*

Proof. We may suppose that $R = R_{\mathfrak{p}}$.

(a) $\Rightarrow$ (b): If $F_{\bullet}$ is split acyclic, then $F_{\bullet} \otimes R/\mathfrak{p}$ is a (split) acyclic complex of vector spaces over $R/\mathfrak{p}$; so we can refer to elementary linear algebra.

(b) $\Rightarrow$ (a): We again use induction, and may assume that $\operatorname{Coker}\varphi_2$ is a free R-module of rank r_1. According to 1.4.8, $\operatorname{Im}\varphi_1$ contains a free direct summand U of F_0 of rank r_1. So we get an induced epimorphism $\operatorname{Coker}\varphi_2 \to U$ of free R-modules, both of which have rank r_1. Such a map must be an isomorphism. One easily concludes that $\operatorname{Im}\varphi_1 = U$. Hence $F_{\bullet}$ is split acyclic.

That $I_t(\varphi_i) = 0$ for $t > r_i$, follows most easily from (a) in conjunction with 1.4.9.

(c) $\Rightarrow$ (a): Let $F'_\bullet$ be the truncation $0 \to F_s \to \cdots \to F_1 \to 0$. Then $F'_\bullet \otimes M$ is acyclic; arguing inductively, we may therefore suppose that $F'_\bullet$ is split acyclic. Then $F'_1 = \operatorname{Coker}\varphi_2$ is free, and the induced map $F'_1 \otimes M \to F_0 \otimes M$ is injective by hypothesis. By virtue of 1.3.4, F'_1 is mapped isomorphically onto a free direct summand of F_0.

(a) $\Rightarrow$ (c): This is evident. $\square$

We have completed our preparations for the following important and extremely useful acyclicity criterion.

Theorem 1.4.13 (Buchsbaum–Eisenbud). *Let R be a Noetherian ring, and*

$$F_\bullet : 0 \longrightarrow F_s \xrightarrow{\varphi_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

a complex of finite free R-modules. Set $r_i = \sum_{j=i}^{s}(-1)^{j-i} \operatorname{rank} F_j$. Then the following are equivalent:

(a) *$F_\bullet$ is acyclic;*

(b) $\operatorname{grade} I_{r_i}(\varphi_i) \ge i$ *for* $i = 1, \ldots, s$.

Before we prove the theorem the reader should note that $r_i = \operatorname{rank}\varphi_i \ge 0$ when $F_\bullet$ is acyclic; this is just a restatement of 1.4.6. Conversely, it is not necessary to require that $r_i \ge 0$ for the implication (b) $\Rightarrow$ (a); if $r_i < 0$, then $r_{i+1} > \operatorname{rank} F_i$, and $I_{r_{i+1}}(\varphi_{i+1}) = 0$ in contradiction with (b). In the situation of 1.4.13 we call r_i the *expected rank of* φ_i.

Proof. (a) $\Rightarrow$ (b): By what has just been said and 1.4.11, we see that $\operatorname{grade} I_{r_i}(\varphi_i) \ge 1$ for $i = 1, \ldots, s$. In particular there is an R-regular element x contained in the product of the ideals $I_{r_i}(\varphi_i)$. If x is a unit, then $I_{r_i}(\varphi_i) = R$ for all i, and we are done. Otherwise we use induction. Let $^-$ denote residue classes modulo x. It follows immediately from 1.1.5 that the induced complex $0 \to \bar F_s \to \bar F_{s-1} \to \cdots \to \bar F_2 \to \bar F_1 \to 0$ is acyclic. Furthermore $I_{r_i}(\varphi_i)^- = I_{r_i}(\bar\varphi_i)$, and $\operatorname{grade} I_{r_i}(\bar\varphi_i) \ge i - 1$ by induction. Then $\operatorname{grade} I_{r_i}(\varphi_i) \ge i$ for $i = 2, \ldots, s$.

The reader may have noticed that this implication follows immediately from the Auslander–Buchsbaum formula 1.3.3. In view of the generalization 9.1.6 an independent proof is useful, however.

(b) $\Rightarrow$ (a): Using induction again we may assume that $F'_\bullet : 0 \to F_s \to F_{s-1} \to \cdots \to F_2 \to F_1 \to 0$ is acyclic. We set $M_i = \operatorname{Coker}\varphi_{i+1}$ for $i = 1, \ldots, s$, and show by descending induction that $\operatorname{depth}(M_i)_\mathfrak{p} \ge \min\{i, \operatorname{depth} R_\mathfrak{p}\}$ for all $\mathfrak{p} \in \operatorname{Spec} R$ and $i = 1, \ldots, s$.

As $M_s = F_s$, this is trivial for $i = s$. Let $i < s$ and consider the exact sequence

$$0 \longrightarrow M_{i+1} \longrightarrow F_i \longrightarrow M_i \longrightarrow 0.$$

If $\operatorname{depth} R_\mathfrak{p} \ge i+1$, then $\operatorname{depth}(M_{i+1})_\mathfrak{p} \ge i+1$, and we get $\operatorname{depth}(M_i)_\mathfrak{p} \ge i$ from 1.2.9. If $\operatorname{depth} R_\mathfrak{p} \le i$, then $I_{r_{i+1}}(\varphi_{i+1}) \not\subset \mathfrak{p}$ by hypothesis; on the other hand $\operatorname{rank} M_{i+1} = \operatorname{rank} \varphi_{i+1} = r_{i+1}$, and therefore $I_t(\varphi_{i+1}) = 0$ for $t > r_{i+1}$. So 1.4.9 yields that $(M_i)_\mathfrak{p}$ is free, hence $\operatorname{depth}(M_i)_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p}$.

We still have to show that the induced map $\varphi'_1 : M_1 \to F_0$ is injective. Let $N = \operatorname{Ker} \varphi'_1$. In order to get $N = 0$, we derive that $\operatorname{Ass} N = \emptyset$. If $\operatorname{depth} R_\mathfrak{p} \geq 1$, then $\operatorname{depth}(M_1)_\mathfrak{p} \geq 1$ as seen above; therefore $\mathfrak{p} \notin \operatorname{Ass} M_1 \supset \operatorname{Ass} N$. If $\operatorname{depth} R_\mathfrak{p} = 0$, then $I_{r_i}(\varphi_i) \not\subset \mathfrak{p}$ for $i = 1, \dots, s$, and $F_\bullet \otimes R_\mathfrak{p}$ is even split acyclic by 1.4.12. It follows that $N_\mathfrak{p} = 0$ since $N_\mathfrak{p} \cong H_1(F_\bullet \otimes R_\mathfrak{p})$. □

Often one only needs the following consequence of 1.4.13.

Corollary 1.4.14. *Let R be a Noetherian ring, and $F_\bullet$ be a complex as in 1.4.13. If $F_\bullet \otimes R_\mathfrak{p}$ is acyclic for all prime ideals $\mathfrak{p}$ with* $\operatorname{depth} R_\mathfrak{p} < s$, *then $F_\bullet$ is acyclic.*

PROOF. Let $\mathfrak{p}$ be a prime ideal with $\operatorname{depth} R_\mathfrak{p} < i \leq s$. The implication (a) $\Rightarrow$ (b) of the theorem applied to $F_\bullet \otimes R_\mathfrak{p}$ yields $\operatorname{grade} I_{r_i}(\varphi_i)_\mathfrak{p} \geq i$, which is only possible if $I_{r_i}(\varphi_i) \not\subset \mathfrak{p}$. This shows $\operatorname{grade} I_{r_i}(\varphi_i) \geq i$, and the acyclicity of $F_\bullet$ follows from the implication (b) $\Rightarrow$ (a) of the theorem. □

Theorem 1.4.13 is the most important case of the acyclicity criterion of Buchsbaum and Eisenbud. Its general form will be discussed in Chapter 9.

Perfect modules. Let R be a Noetherian ring, and M a finite R-module. Since one can compute $\operatorname{Ext}^i_R(M, R)$ from a projective resolution of M, one obviously has $\operatorname{grade} M \leq \operatorname{proj\,dim} M$. Modules for which equality is attained have especially good properties.

Definition 1.4.15. Let R be a Noetherian ring. A non-zero finite R-module M is *perfect* if $\operatorname{proj\,dim} M = \operatorname{grade} M$. An ideal I is called *perfect* if R/I is a perfect module.

Perfect modules are 'grade unmixed':

Proposition 1.4.16. *Let R be a Noetherian ring, and M a perfect R-module. For a prime ideal $\mathfrak{p} \in \operatorname{Supp} M$ the following are equivalent:*
(a) $\mathfrak{p} \in \operatorname{Ass} M$;
(b) $\operatorname{depth} R_\mathfrak{p} = \operatorname{grade} M$.
Furthermore $\operatorname{grade} \mathfrak{p} = \operatorname{grade} M$ *for all prime ideals* $\mathfrak{p} \in \operatorname{Ass} M$.

PROOF. For all finite R-modules M and $\mathfrak{p} \in \operatorname{Supp} M$ one has the inequalities

$$\operatorname{grade} M \leq \operatorname{grade} M_\mathfrak{p} \leq \operatorname{proj\,dim} M_\mathfrak{p} \leq \operatorname{proj\,dim} M,$$

and moreover $\operatorname{proj\,dim} M_\mathfrak{p} + \operatorname{depth} M_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p}$ by the Auslander–Buchsbaum formula 1.3.3. If M is perfect, then the inequalities become equations, and $\operatorname{depth} M_{\mathfrak{p}} = 0$ if and only if $\operatorname{depth} R_{\mathfrak{p}} = \operatorname{grade} M$. This shows the equivalence of (a) and (b).

If $\mathfrak{p} \in \operatorname{Ass} M$, then $\mathfrak{p} \supset \operatorname{Ann} M$, and so $\operatorname{grade} \mathfrak{p} \geq \operatorname{grade} M$. For perfect M the converse results from (b) and the inequality $\operatorname{grade} \mathfrak{p} \leq \operatorname{depth} R_{\mathfrak{p}}$. □

It follows easily from 1.3.6 that an ideal generated by a regular sequence in a Noetherian ring R is perfect. Some more examples are described in the following celebrated theorem:

Theorem 1.4.17 (Hilbert–Burch). *Let R be a Noetherian ring, and I an ideal with a free resolution*

$$F_\bullet : 0 \longrightarrow R^n \xrightarrow{\varphi} R^{n+1} \longrightarrow I \longrightarrow 0.$$

Then there exists an R-regular element a such that $I = aI_n(\varphi)$. If I is projective, then $I = (a)$, and if $\operatorname{proj\,dim} I = 1$, *then $I_n(\varphi)$ is perfect of grade* 2.

Conversely, if $\varphi : R^n \to R^{n+1}$ is an R-linear map with $\operatorname{grade} I_n(\varphi) \geq 2$, *then $I = I_n(\varphi)$ has the free resolution $F_\bullet$.*

PROOF. First we prove the converse part. Let $\varphi : R^n \to R^{n+1}$ be a map with $\operatorname{grade} I_n(\varphi) \geq 2$. Then φ is given by an $(n+1) \times n$ matrix U. Let δ_i denote the i-minor of U with the i-th row deleted, and consider the homomorphism $\pi : R^{n+1} \to R$ which sends e_i to $(-1)^i \delta_i$. Laplace expansion shows that we have a complex

$$0 \longrightarrow R^n \xrightarrow{\varphi} R^{n+1} \xrightarrow{\pi} I \longrightarrow 0,$$

which in fact is exact by 1.4.13.

Suppose now that an ideal I with free resolution $F_\bullet$ is given. Then 1.4.13 yields $\operatorname{grade} I_n(\varphi) \geq 2$, and we can apply the first part of the proof to obtain $I \cong \operatorname{Coker} \varphi \cong I_n(\varphi)$; equivalently, there exists an injective linear map $\alpha : I_n(\varphi) \to R$ with $I = \operatorname{Im} \alpha$. According to 1.2.24, α is just multiplication by some $a \in R$. Because of 1.4.7 (or 1.4.13) a cannot be a zero-divisor.

If I is projective, then $I_n(\varphi) = R$ by 1.4.10, and thus $I = (a)$. If $\operatorname{proj\,dim} I = 1$, then $\operatorname{proj\,dim}(R/I_n(\varphi)) = \operatorname{proj\,dim} R/I = 2$, and $R/I_n(\varphi)$ is perfect of grade 2. □

Exercises

1.4.18. Let R be a ring, and M a finite torsion-free module. Prove that if M has a rank, then M is isomorphic to a submodule of a finite free R-module of the same rank.

1.4.19. Let R be a Noetherian ring, I an ideal, and M, N finite modules. Prove $\operatorname{grade}(I, \operatorname{Hom}_R(M, N)) \geq \min(2, \operatorname{grade}(I, N))$.

1.4.20. Let R be a Noetherian ring, and M a finite R-module. Prove
(a) if M is torsionless, then it is torsion-free,
(b) M is torsionless if and only if it is a submodule of a finite free module,
(c) if M is reflexive, then it is a second syzygy, i.e. there is an exact sequence $0 \to M \to F_1 \to F_0$ with F_i finite and free.

1.4.21. Let R be a Noetherian ring, and M a finite R-module. Suppose $\varphi : G \to F$ is a homomorphism of finite free R-modules with $M = \operatorname{Coker} \varphi$. Then $D(M) = \operatorname{Coker} \varphi^*$ is the *transpose* of M. (It is unique up to projective equivalence.) Show that $\operatorname{Ker} h = \operatorname{Ext}^1_R(D(M), R)$ and $\operatorname{Coker} h = \operatorname{Ext}^2_R(D(M), R)$ where $h : M \to M^{**}$ is the natural homomorphism.

1.4.22. Let R be a Noetherian ring, and M a finite R-module such that M^* has finite projective dimension. Prove
(a) if $\operatorname{depth} M_\mathfrak{p} \geq \min(1, \operatorname{depth} R_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$, then M is torsionless,
(b) if $\operatorname{depth} M_\mathfrak{p} \geq \min(2, \operatorname{depth} R_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$, then M is reflexive.
Hint: $\operatorname{proj\,dim} M^* < \infty \Rightarrow \operatorname{proj\,dim} D(M) < \infty$.

1.4.23. Let R be a Noetherian ring, and M a finite R-module. Show that M has a rank if and only if M^* has a rank (and both ranks coincide). Hint: It is enough to consider the case $R = R_\mathfrak{p}$, $\operatorname{depth} R_\mathfrak{p} = 0$. Apply 1.4.22.

1.4.24. Let R be a Noetherian local ring, and $0 \to L_s \to L_{s-1} \to \cdots \to L_1 \to L_0 \to 0$ a complex of finite R-modules. Suppose that the following hold for $i > 0$: (i) $\operatorname{depth} L_i \geq i$, and (ii) $\operatorname{depth} H_i(L_\bullet) = 0$ or $H_i(L_\bullet) = 0$. Show that $L_\bullet$ is acyclic. (This is Peskine and Szpiro's 'lemme d'acyclicité' [297].)
Hint: Set $C_i = \operatorname{Coker}(L_{i+1} \to L_i)$, and show by descending induction that $\operatorname{depth} C_i \geq i$ and $H_i(L_\bullet) = 0$ for $i > 0$.

1.4.25. Let R be a Noetherian ring, I an ideal of finite projective dimension, and M a finite R/I-module. Prove the following inequality of Avramov and Foxby [29]:

$$\operatorname{grade}_{R/I} M + \operatorname{grade}_R R/I \leq \operatorname{grade}_R M \leq \operatorname{grade}_{R/I} M + \operatorname{proj\,dim}_R R/I\,;$$

if I is perfect, then equality is attained. (Use the Auslander–Buchsbaum formula.)

1.4.26. Let R be a Noetherian ring, and M a perfect R-module of grade n. Suppose $P_\bullet$ is a projective resolution of M of length n and set $M' = \operatorname{Ext}^n_R(M, R)$. Prove
(a) $P_\bullet^*$ is acyclic and resolves M',
(b) M' is perfect of grade n, and $M'' = M$,
(c) $\operatorname{Ass} M' = \operatorname{Ass} M$.

1.4.27. Let R be a Noetherian ring, $\boldsymbol{x}$ an R-sequence of length n, and $I = (\boldsymbol{x})$. Show that R/I^m is perfect of grade n for all $m \geq 1$. (Theorem 1.1.8 is useful.)

1.5 Graded rings and modules

In this section we investigate rings and modules which, like a polynomial ring, admit a decomposition of their elements into homogeneous components.

Definition 1.5.1. A *graded ring* is a ring R together with a decomposition $R = \bigoplus_{i \in \mathbb{Z}} R_i$ (as a $\mathbb{Z}$-module) such that $R_i R_j \subset R_{i+j}$ for all $i, j \in \mathbb{Z}$.

A *graded R-module* is an R-module M together with a decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ (as a $\mathbb{Z}$-module) such that $R_i M_j \subset M_{i+j}$ for all $i, j \in \mathbb{Z}$. One calls M_i the i-th *homogeneous* (or *graded*) *component of M*.

The elements $x \in M_i$ are called *homogeneous* (*of degree i*); those of R_i are also called *i-forms*. According to this definition the zero element is homogeneous of arbitrary degree. The degree of x is denoted by $\deg x$. An arbitrary element $x \in M$ has a unique presentation $x = \sum_i x_i$ as a sum of homogeneous elements $x_i \in M_i$. The elements x_i are called the *homogeneous components* of x.

Note that R_0 is a ring with $1 \in R_0$, that all summands M_i are R_0-modules, and that $M = \bigoplus_{i \in \mathbb{Z}} M_i$ is a direct sum decomposition of M as an R_0-module.

Definition 1.5.2. Let R be a graded ring. The *category of graded R-modules*, denoted $\mathcal{M}_0(R)$, has as objects the graded R-modules. A *morphism* $\varphi : M \to N$ in $\mathcal{M}_0(R)$ is an R-module homomorphism satisfying $\varphi(M_i) \subset N_i$ for all $i \in \mathbb{Z}$. An R-module homomorphism which is a morphism in $\mathcal{M}_0(R)$ will be called *homogeneous*.

Let M be a graded R-module and N a submodule of M. N is called a *graded submodule* if it is a graded module such that the inclusion map is a morphism in $\mathcal{M}_0(R)$. This is equivalent to the condition $N_i = N \cap M_i$ for all $i \in \mathbb{Z}$. In other words, N is a graded submodule of M if and only if N is generated by the homogeneous elements of M which belong to N. In particular, if $x \in N$, then all homogeneous components of x belong to N. Furthermore, M/N is graded in a natural way. If φ is a morphism in $\mathcal{M}_0(R)$, then $\operatorname{Ker} \varphi$ and $\operatorname{Im} \varphi$ are graded.

A (not necessarily commutative) R-algebra A is *graded* if, in addition to the definition, $A_i A_j \subset A_{i+j}$.

The graded submodules of R are called *graded ideals*. Let I be an arbitrary ideal of R. Then the graded ideal I^* is defined to be the ideal generated by all homogeneous elements $a \in I$. It is clear that I^* is the largest graded ideal contained in I, and that R/I^* inherits a natural structure as a graded ring.

Examples 1.5.3. (a) Let S be a ring, and $R = S[X_1, \ldots, X_n]$ a polynomial ring over S. Then for every choice of integers $d_1, \ldots, d_n$ there exists a unique grading on R such that $\deg X_i = d_i$ and $\deg a = 0$ for all $a \in S$: the m-th graded component is the S-module generated by all monomials $X_1^{e_1} \cdots X_n^{e_n}$ such that $\sum e_i d_i = m$. If one chooses $d_i = 1$ for all i, then one obtains the grading of the polynomial ring corresponding to the total degree of a monomial. Unless indicated otherwise we will always consider R to be graded in this way.

(b) Every ring R has the *trivial grading* given by $R_0 = R$ and $R_i = 0$ for $i \neq 0$. A typical example of a graded module over R is a complex

$$C_\bullet : \cdots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \cdots$$

of R-modules. Such a complex may be equivalently described as a graded module $C_\bullet = \bigoplus_{i=-\infty}^{\infty} C_i$ together with an R-endomorphism ∂ such that $\partial^2 = 0$ and $\partial(C_i) \subset C_{i-1}$ for all i. (In the terminology to be introduced below, ∂ is a homogeneous endomorphism of degree -1.)

The most important graded rings arise in algebraic geometry as the coordinate rings of projective varieties. They have the form $R = k[X_1, \dots, X_n]/I$ where k is a field and I is an ideal generated by homogeneous polynomials (in the usual sense). Then R is generated as a k-algebra by elements of degree 1, namely the residue classes of the indeterminates. Graded rings R which as R_0-algebras are generated by 1-forms will be called *homogeneous* R_0-algebras. More generally, if R is a graded R_0-algebra generated by elements of positive degree, then we say that R is a *positively graded* R_0-algebra.

We want to clarify which graded rings are Noetherian. Let us first consider positively graded rings.

Proposition 1.5.4. *Let R be a positively graded R_0-algebra, and $x_1, \dots, x_n$ homogeneous elements of positive degree. Then the following are equivalent:*
(a) *$x_1, \dots, x_n$ generate the ideal $\mathfrak{m} = \bigoplus_{i=1}^{\infty} R_i$;*
(b) *$x_1, \dots, x_n$ generate R as an R_0-algebra.*
In particular R is Noetherian if and only if R_0 is Noetherian and R is a finitely generated R_0-algebra.

PROOF. For the implication (a) $\Rightarrow$ (b) it is enough to write every homogeneous element $y \in R$ as a polynomial in $x_1, \dots, x_n$ with coefficients in R_0, and this is very easy by induction on $\deg y$. The rest is evident. □

The last assertion of 1.5.4 holds for graded rings in general.

Theorem 1.5.5. *Let R be a graded ring. Then the following are equivalent:*
(a) *every graded ideal of R is finitely generated;*
(b) *R is a Noetherian ring;*
(c) *R_0 is Noetherian, and R is a finitely generated R_0-algebra;*
(d) *R_0 is Noetherian, and both $S_1 = \bigoplus_{i=0}^{\infty} R_i$ and $S_2 = \bigoplus_{i=0}^{\infty} R_{-i}$ are finitely generated R_0-algebras.*

PROOF. The implications (d) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are obvious. For (a) $\Rightarrow$ (d) we first note that R_0 is a direct summand of R as an R_0-module. It follows that $IR \cap R_0 = I$ for every ideal I of R_0, and thus (a) implies that R_0 is Noetherian. (Extend an ascending chain of ideals of R_0 to R, and contract the extension back to R_0.) A similar argument shows that R_i is a finite R_0-module for every $i \in \mathbb{Z}$.

Let $\mathfrak{m} = \bigoplus_{i=1}^{\infty} R_i$. We claim that $\mathfrak{m}$ is a finitely generated ideal of S_1. By hypothesis $\mathfrak{m}R$ has a finite system of generators $x_1, \dots, x_m$, which may certainly be chosen to be homogeneous of positive degrees d_i. Let d be the maximum of $d_1, \dots, d_m$. Then a homogeneous element $y \in \mathfrak{m}$ with $\deg y \geq d$ can be written as a linear combination of $x_1, \dots, x_m$ with coefficients from S_1. Thus $x_1, \dots, x_m$ together with a finite set of homogeneous elements spanning $R_1, \dots, R_{d-1}$ over R_0 generate $\mathfrak{m}$ as an ideal of S_1. According to 1.5.4, S_1 is a finitely generated R_0-algebra, and the claim for S_2 follows by symmetry. $\square$

Very often we shall derive properties of a graded ring or module from its localizations with respect to graded prime ideals. The following lemma is basic for such arguments.

Lemma 1.5.6. *Let R be a graded ring.*
(a) *For every prime ideal $\mathfrak{p}$ the ideal $\mathfrak{p}^*$ is a prime ideal.*
(b) *Let M be a graded R-module.*
  (i) *If $\mathfrak{p} \in \operatorname{Supp} M$, then $\mathfrak{p}^* \in \operatorname{Supp} M$.*
  (ii) *If $\mathfrak{p} \in \operatorname{Ass} M$, then $\mathfrak{p}$ is graded; furthermore $\mathfrak{p}$ is the annihilator of a homogeneous element.*

Proof. (a) Let $a, b \in R$ such that $ab \in \mathfrak{p}^*$. We write $a = \sum_i a_i$, $a_i \in R_i$, and $b = \sum_j b_j$, $b_j \in R_j$. Assume that $a \notin \mathfrak{p}^*$ and $b \notin \mathfrak{p}^*$. Then there exist integers p, q such that $a_p \notin \mathfrak{p}^*$, but $a_i \in \mathfrak{p}^*$ for $i < p$, and $b_q \notin \mathfrak{p}^*$, but $b_j \in \mathfrak{p}^*$ for $j < q$. The $(p+q)$-th homogeneous component of ab is $\sum_{i+j=p+q} a_i b_j$. Thus $\sum_{i+j=p+q} a_i b_j \in \mathfrak{p}^*$, since $\mathfrak{p}^*$ is graded. All summands of this sum, except possibly $a_p b_q$, belong to $\mathfrak{p}^*$, and so it follows that $a_p b_q \in \mathfrak{p}^*$ as well. Since $\mathfrak{p}^* \subset \mathfrak{p}$, and since $\mathfrak{p}$ is a prime ideal we conclude that $a_p \in \mathfrak{p}$ or $b_q \in \mathfrak{p}$. But a_p and b_q are homogeneous, and so $a_p \in \mathfrak{p}^*$ or $b_q \in \mathfrak{p}^*$, a contradiction.

(b) For (i) assume $\mathfrak{p}^* \notin \operatorname{Supp} M$; then $M_{\mathfrak{p}^*} = 0$. Let $x \in M$ be a homogeneous element. Then there exists an element $a \in R \setminus \mathfrak{p}^*$ such that $ax = 0$. It follows that $a_i x = 0$ for all homogeneous components a_i of a. Since $a \in R \setminus \mathfrak{p}^*$, there exists an integer i such that $a_i \notin \mathfrak{p}^*$. Since a_i is homogeneous, we even have $a_i \notin \mathfrak{p}$. Hence $x/1 = 0$ in $M_{\mathfrak{p}}$. This holds true for all homogeneous elements of M. Thus we conclude that $M_{\mathfrak{p}} = 0$, a contradiction.

For (ii) we choose an element $x \in M$ with $\mathfrak{p} = \operatorname{Ann} x$. Let $x = x_m + \cdots + x_n$ be its decomposition as a sum of homogeneous elements x_i of degree i. Similarly we decompose an element $a = a_p + \cdots + a_q$ of $\mathfrak{p}$. Since $ax = 0$, we have equations $\sum_{i+j=r} a_i x_j = 0$ for $r = m+p, \dots, n+q$. It follows that $a_p x_m = 0$, and, by induction, $a_p^i x_{m+i-1} = 0$ for all $i \geq 1$.

Thus a_p^{n-m+1} annihilates x. As $\mathfrak{p}$ is a prime ideal, we have $a_p \in \mathfrak{p}$. Iterating this procedure we see that each homogeneous component of a belongs to $\mathfrak{p}$.

In order to prove the second assertion in (ii) one can now use the fact that $\mathfrak{p}$ is generated by homogeneous elements. It follows easily that $\mathfrak{p}$ annihilates all the homogeneous components of x. Set $\mathfrak{a}_i = \operatorname{Ann} x_i$; then, as just seen, $\mathfrak{p} \subset \mathfrak{a}_i$. On the other hand $\bigcap_{i=m}^n \mathfrak{a}_i \subset \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, there exists j with $\mathfrak{a}_j \subset \mathfrak{p}$, and therefore $\mathfrak{a}_j = \mathfrak{p}$. □

Let $\mathfrak{p}$ be a prime ideal of R, and let S be the set of homogeneous elements of R not belonging to $\mathfrak{p}$. The set S is multiplicatively closed, and we put $M_{(\mathfrak{p})} = M_S$ for any graded R-module M. For $x/a \in M_{(\mathfrak{p})}$, x homogeneous, we set $\deg x/a = \deg x - \deg a$. We further define a grading on $M_{(\mathfrak{p})}$ by setting

$$(M_{(\mathfrak{p})})_i = \{x/a \in M_{(\mathfrak{p})} : x \text{ homogeneous, } \deg x/a = i\}.$$

It is easy to see that $R_{(\mathfrak{p})}$ is a graded ring and that $M_{(\mathfrak{p})}$ is a graded $R_{(\mathfrak{p})}$-module; $M_{(\mathfrak{p})}$ is called the *homogeneous localization* of M. The extension ideal $\mathfrak{p}^* R_{(\mathfrak{p})}$ is a graded prime ideal in $R_{(\mathfrak{p})}$, and the factor ring $R_{(\mathfrak{p})}/\mathfrak{p}^* R_{(\mathfrak{p})}$ has the property that every non-zero homogeneous element is invertible.

Lemma 1.5.7. *Let R be a graded ring. The following conditions are equivalent:*
(a) *every non-zero homogeneous element is invertible;*
(b) *$R_0 = k$ is a field, and either $R = k$ or $R = k[t, t^{-1}]$ for some homogeneous element $t \in R$ of positive degree which is transcendental over k.*

Proof. (a) ⇒ (b): $R_0 = k$ is a field. If $R = R_0$, then R is a field. Otherwise $R \neq R_0$, and there exist non-zero homogeneous elements of positive degree. Let t be an element of least positive degree, say $\deg t = d$. As t is invertible there exists a homomorphism $\varphi: k[T, T^{-1}] \to R$ of graded rings where φ maps k identically to R_0 and where $\varphi(T) = t$. (The grading on $k[T, T^{-1}]$ is of course defined by setting $\deg T = d$.)

We claim that φ is an isomorphism. Let $f \in \operatorname{Ker} \varphi$, $f = \sum_{i \in \mathbb{Z}} a_i T^i$, $a_i \in k$; then $0 = \varphi(f) = \sum_{i \in \mathbb{Z}} a_i t^i$, and so $a_i t^i = 0$ for all i. As t is invertible, we get $a_i = (a_i t^i) \cdot t^{-i} = 0$ for all i, which implies that $f = 0$. Hence φ is injective. In order to show that φ is surjective, we pick a non-zero homogeneous element $a \in R$ of degree i. If $i = 0$, then $a \in \operatorname{Im} \varphi$. Thus we may assume that $i \neq 0$. Write $i = jd + r$ with $0 \le r < d$. The element at^{-j} has degree r. As d was the least positive degree, we conclude that $r = 0$. Thus $a = bt^j$ for some $b \in R_0$, and hence $a = \varphi(bT^j) \in \operatorname{Im} \varphi$.

(b) ⇒ (a) is trivial. □

The following theorem contains the dimension theory of graded rings and modules: