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Algebraic Cycles and Motives

Volume 1

Edited by
Jan Nagel and Chris Peters

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Algebraic Cycles and Motives

Volume 1

Edited by

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Preface

These proceedings contain a selection of papers from the EAGER conference "Algebraic Cycles and Motives" that was held at the Lorentz Center in Leiden on the occasion of the 75th birthday of Professor J.P. Murre (Aug 30–Sept 3, 2004). The conference attracted many of the leading experts in the field as well as a number of young researchers. As the papers in this volume cover the main research topics and some interesting new developments, they should give a good indication of the present state of the subject. This volume contains sixteen research papers and six survey papers.

The theory of algebraic cycles deals with the study of subvarieties of a given projective algebraic variety X, starting with the free group $Z^p(X)$ on irreducible subvarieties of X of codimension p. In order to make this very large group manageable, one puts a suitable equivalence relation on it, usually *rational equivalence*. The resulting *Chow group* $CH^p(X)$ in general might still be very big. If X is a smooth variety, the intersection product makes the direct sum of all the Chow groups into a ring, the Chow ring $CH^*(X)$. Hitherto mysterious ring can be studied through its relation to cohomology, the first example of which is the cycle class map: every algebraic cycle defines a class in singular, de Rham, or ℓ-adic cohomology. Ultimately this cohomological approach leads to the theory of motives and motivic cohomology developed by A. Grothendieck, M. Levine, M. Nori, V. Suslin and A. Voevodsky, just to mention a few main actors.

There were about 60 participants at the conference, coming from Europe, the United States, India and Japan. During the conference there were 22 one hour lectures. On the last day there were three special lectures devoted to the scientific work of Murre, in honour of his 75th birthday. The lectures covered a wide range of topics, such as the study of algebraic cycles using Abel–Jacobi/regulator maps and normal functions, motives (Voevodsky's triangulated category of mixed motives, finite-dimensional motives), the conjectures of Bloch–Beilinson and Murre on filtrations on Chow groups and Bloch's conjecture, and results of a more arithmetic flavour for varieties defined over number fields or local fields.

Let us start by discussing the **survey papers**. The first, a paper by J. Ayoub is devoted to the construction of a motivic version of the vanishing cycle formalism. It is followed by a paper by L.Barbieri Viale who presents an overview of the main results of the theory of *mixed motives of level at most one*. In a series of recent papers, M. De Cataldo and L. Migliorini have made a detailed study of the *topological properties of algebraic maps using the theory of perverse sheaves*. Their survey provides an introduction to this work, illustrated by a number of low-dimensional examples. Déglise's paper contains a careful exposition of *Voevodsky's theory of sheaves with transfers over a regular base scheme*, with detailed proofs. The paper of M. Green and P. Griffiths contains an outline of an ambitious research program that centers around the *extension of normal functions over a higher-dimensional base*, and its applications to the Hodge conjecture. (The case where the base space is a curve is known by work of F. El Zein and S. Zucker.) A. Krishna and V. Srinivas discuss the theory of *zero-cycles on singular varieties* and its applications to algebra. The paper of D. Ramakrishnan is a brief survey of results concerning *algebraic cycles on Hilbert modular varieties*.

In discussing the **research papers** we have grouped according to the main research themes, although in the proceedings they are listed alphabetically according to the name of the authors.

One of the leading themes in the theory of algebraic cycles is the study of the conjectural *Bloch–Beilinson filtration* on Chow groups. In the course of his work on motives, J. Murre found an equivalent and more explicit version of this conjecture, which states that the motive of a smooth projective algebraic variety should admit a Chow–Künneth decomposition with a number of specific properties. The paper of B. Kahn, J. Murre and C. Pedrini contains a detailed exposition of these matters with emphasis on the study of the transcendental part of the motive of a surface. The paper of S. Bloch and H. Esnault is devoted to the construction of an algebraic cycle that induces the Künneth projector onto $H^1(U)$ for a quasi-projective variety U, and the paper of Miller *et al.* shows the existence of certain Chow–Künneth projectors for compactified families of abelian threefolds over a certain Picard modular surface studied by Holzapfel. Beauville studies the splitting of the Bloch–Beilinson filtration for certain symplectic projective manifolds. The notion of "finite–dimensionality" of motives, which recently attracted a lot of attention, is studied in the papers of S.-I. Kimura and U. Jannsen.

The latter paper uses this notion to verify Murre's conjectures in a number of examples.

Another important theme is the *study of algebraic cycles using Hodge theory.* The paper of C. Peters and J. Steenbrink deals with the motivic nearby fiber and its relation to the limit mixed Hodge structure of a family of projective varieties. Morihiko Saito constructs the total infinitesimal invariant of a higher Chow cycle, an object that lives in the direct sum of the cohompology of filtered logarithhmic complexes with coefficients. In the papers of M. Asakura, S. Saito and J. Nagel, infitesimal methods are used to study the regulator map on higher Chow groups. M. Asakura and S. Saito use these techniques to verify a conjecture of Beilinson ("Beilinson's Hodge conjecture with coefficients") in certain cases. J. Lewis defines a twisted version of Milnor K-theory and a corresponding twisted version of the regulator, which is shown to have a nontrivial image in certain examples.

The remaining papers deal with a variety of topics. The papers of C. Deninger and A. Werner and S. Brivio and A. Verra deal with *vector bundles.* C. Deninger and A. Werner study the category of degree zero vector bundles with "potentially strongly semistable reduction" on a p-adic curve. S. Brivio and A. Verra investigate the properties of the theta map defined on the moduli space of semistable vector bundles over a curve. T. Shioda studies the structure of the *Mordell–Weil lattice of certain elliptic K3 surfaces*, and the paper of J. Stienstra studies a potential link between the theory of *motives and string theory* using diffraction patterns.

The conference has been financed by the Lorentz Center, EAGER (European Algebraic Geometry Research Training Network), the KNAW (Royal Netherlands Academy of Arts and Sciences), and the Thomas Stieltjes Instituut. We heartily thank these institutions for their financial support.

It is a pleasure to dedicate this volume to Jacob Murre. The study of algebraic cycles and motives has been his life-long passion, and he has made a number of important contributions to the subject.

Chris Peters and Jan Nagel, May 2006.

Program

Day	Hour	Speaker	Title
	10:00–11:00	P. Griffiths	Algebraic cycles and singularities of normal functions
Monday	11:15–12:15	A. Beauville	When does the Bloch–Beilinson filtration split?
Aug 30	13:30–14:30	S. Müller-Stach	Higher Abel–Jacobi maps
	14:30–15:30	F. Déglise	Cycle modules and triangulated mixed motives
	16:00–17:00	O. Tommasi	Rational cohomology of the moduli space of genus 4 curves
	09:30–10:30	H. Esnault	Deligne's integrality theorem in unequal characteristic and rational points over finite fields
Tuesday	11:15–12:15	U. Jannsen	Some remarks on finite dimensional motives
	13:30–14:30	L. Barbieri Viale	Motivic Albanese
August 31	14:45–15:45	B. van Geemen	Some remarks on Brauer groups of elliptic fibrations on K3 surfaces
	16:15–17:15	L. Migliorini	Hodge theory of projective maps
	09:30–10:30	K. Künnemann	Extensions in Arakelov geometry
Wednesday	11:15–12:15	J.-L Colliot-Thélène	Zero-cycles on linear algebraic groups over local fields
September 1	13:30–14:30	C. Deninger	Vector bundles on p-adic curves and parallel transport
	09:30–10:30	S. Saito	Finiteness results for motivic cohomology
	11:15–12:15	T. Shioda	Finding cycles on certain K3 surfaces
Thursday	13:30–14:30	D. Ramakrishnan	Cycles on Hilbert modular fourfolds
	14:45–15:45	A. Verra	Moduli of vector bundles on curves and correspondences: the genus two case
September 2	16:15–17:15	J. Ayoub	Conservation of ϕ and the Bloch conjecture
	10:00–11:00	A. Conte	25 years of joint work with Jaap Murre
Friday	11:30–12:30	V. Srinivas	Zero cycles on singular varieties
	13:30–14:30	F Oort	Geometric aspects of the scientific work of Jaap Murre, I
September 3	14:45–15:45	S. Bloch	Geometric aspects of the scientific work of Jaap Murre, II

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Volume 1

Survey Articles

1

The Motivic Vanishing Cycles and the Conservation Conjecture

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To Jacob Murre for his 75th birthday

1.1 Introduction

Let X be a noetherian scheme. Following Morel and Voevodsky (see [24], [25], [28], [33] and [37]), one can associate to X the motivic stable homotopy category $\mathbf{SH}(X)$. Objects of $\mathbf{SH}(X)$ are T-spectra of simplicial sheaves on the smooth Nisnevich site $(\mathrm{Sm}/X)_{\mathrm{Nis}}$, where T is the pointed quotient sheaf $\mathbb{A}^1_X/\mathbb{G}\mathrm{m}_X$. As in topology, $\mathbf{SH}(X)$ is triangulated in a natural way. There is also a tensor product $-\otimes_X-$ and an "internal hom": $\underline{\mathrm{Hom}}_X$ on $\mathbf{SH}(X)$ (see [20] and [33]). Given a morphism $f : X \longrightarrow Y$ of noetherian schemes, there is a pair of adjoint functors (f^*, f_*) between $\mathbf{SH}(X)$ and $\mathbf{SH}(Y)$. When f is quasi-projective, one can extend the pair (f^*, f_*) to a quadruple $(f^*, f_*, f_!, f^!)$ (see [3] and [8]). In particular we have for $\mathbf{SH}(-)$ the full package of the Grothendieck six operators. It is then natural to ask if we have also the seventh one, that is, if we have a vanishing cycle formalism (analogous to the one in the étale case, developed in [9] and [10]).

In the third chapter of our PhD thesis [3], we have constructed a vanishing cycles formalism for motives. The goal of this paper is to give a detailed account of that construction, to put it in a historical perspective and to discuss some applications and conjectures. In some sense, it is complementary to [3] as it gives a quick introduction to the theory with emphasis on motivations rather than a systematic treatment. The reader will not find all the details here: some proofs will be omitted or quickly sketched, some results will be stated with some additional assumptions (indeed we will be mainly interested in motives with rational coefficients over characteristic zero schemes).

For the full details of the theory, one should consult [3]. Let us mention also that M. Spitzweck has a theory of limiting motives which is closely related to our motivic vanishing cycles formalism. For more information, see [35].

The paper is organized as follows. First we recall the classical pictures: the étale and the Hodge cases. Although this is not achieved here, these classical constructions should be in a precise sense realizations of our motivic construction. In section 1.3 we introduce the notion of a specialization system which encodes some formal properties of the family of nearby cycles functors. We state without proofs some important theorems about specialization systems obtained in [3]. In section 1.4, we give our main construction and prove motivic analogues of some well-known classical results about nearby cycles functors: constructibility, commutation with tensor product and duality, etc. We also construct a monodromy operator on the unipotent part of the nearby cycles which is shown to be nilpotent. Finally, we propose a conservation conjecture which is weaker than the conservation of the classical realizations but strong enough to imply the Schur finiteness of constructible motives‡.

In the literature, the functors Ψ_f have two names: they are called "nearby cycles functors" or "vanishing cycles functors". Here we choose to call them the nearby cycles functors. The properties of these functors form what we call the vanishing cycles formalism (as in [9] and [10]).

1.2 The classical pictures

We briefly recall the construction of the nearby cycles functors $\mathsf{R}\Psi_f$ in étale cohomology. We then explain a construction of Rapoport and Zink which was the starting point of our definition of Ψ_f in the motivic context. After that we shall recall some facts about limits of variations of Hodge structures. A very nice exposition of these matters can be found in [15].

1.2.1 The vanishing cycles formalism in étale cohomology

Let us fix a prime number ℓ and a finite commutative ring Λ such that $\ell^\nu.\Lambda = 0$ for ν large enough. When dealing with étale cohomology, we shall always assume that ℓ is invertible on our schemes. For a reasonable scheme V, we denote by $\mathsf{D}^+(V, \Lambda)$ the derived category of bounded below complexes of étale sheaves on V with values in Λ-modules.

‡ *Constructible motives* means *geometric motives* of [40]. They are also the compact objects in the sense Neeman [30] (see remark 1.3.3).

Let S be the spectrum of a strictly henselian DVR (discrete valuation ring). We denote by η the generic point of S and by s the closed point:

$$\eta \xrightarrow{j} S \xleftarrow{i} s.$$

We also fix a separable closure $\bar{\eta}$ of the point η. From the point of view of étale cohomology, the scheme S plays the role of a small disk so that η is a punctured small disk and $\bar{\eta}$ is a universal cover of that punctured disk. We will also need the normalization $\bar{S}$ of S in $\bar{\eta}$:

$$\bar{\eta} \xrightarrow{\bar{j}} \bar{S} \xleftarrow{\bar{i}} s.$$

Now let $f : X \longrightarrow S$ be a finite type S-scheme. We consider the commutative diagram with cartesian squares

$$\begin{tikzcd} X_\eta \arrow[r, "j"] \arrow[d, "f_\eta"'] & X \arrow[d, "f"] & X_s. \arrow[l, "i"'] \arrow[d, "f_s"] \\ \eta \arrow[r, "j"] & S & s \arrow[l, "i"'] \end{tikzcd}$$

Following Grothendieck (see [10]), we look also at the diagram

$$\begin{tikzcd} X_{\bar{\eta}} \arrow[r, "\bar{j}"] \arrow[d, "f_{\bar{\eta}}"'] & \bar{X} \arrow[d, "\bar{f}"] & X_s \arrow[l, "\bar{i}"'] \arrow[d, "f_s"] \\ \bar{\eta} \arrow[r, "\bar{j}"] & \bar{S} & s \arrow[l, "\bar{i}"'] \end{tikzcd}$$

obtained in the same way by base-changing the morphism f. (This is what we will call the "Grothendieck trick"). We define then the triangulated functor:

$$\mathsf{R}\Psi_f : \mathsf{D}^+(X_\eta, \Lambda) \longrightarrow \mathsf{D}^+(X_s, \Lambda)$$

by the formula: $\mathsf{R}\Psi_f(A) = \bar{i}^*\mathsf{R}\bar{j}_*(A_{X_{\bar{\eta}}})$ for $A \in \mathsf{D}^+(X_\eta, \Lambda)$. By construction, the functor $\mathsf{R}\Psi_f$ comes with an action of the Galois group of $\bar{\eta}/\eta$, but we will not explicitly use this here. The basic properties of these functors concern the relation between $\mathsf{R}\Psi_g$ and $\mathsf{R}\Psi_{g\circ h}$ (see [9]):

Proposition 1.2.1. *Let $g : Y \longrightarrow S$ be an S-scheme and suppose given an S-morphism $h : X \longrightarrow Y$ such that $f = g \circ h$. We form the commutative diagram*

$$\begin{tikzcd} X_\eta \arrow[r, "j"] \arrow[d, "h_\eta"'] & X \arrow[d, "h"] & X_s \arrow[l, "i"'] \arrow[d, "h_s"] \\ Y_\eta \arrow[r, "j"] & Y & Y_s. \arrow[l, "i"'] \end{tikzcd}$$

There exist natural transformations of functors

- $\alpha_h : h_s^*\mathsf{R}\Psi_g \longrightarrow \mathsf{R}\Psi_f h_\eta^*$,
- $\beta_h : \mathsf{R}\Psi_g \mathsf{R}h_{\eta *} \longrightarrow \mathsf{R}h_{s*}\mathsf{R}\Psi_f$.

Furthermore, α_h is an isomorphism when h is smooth and β_h is an isomorphism when h is proper.

The most important case, is maybe when $g = \mathrm{id}_S$ and $f = h$. Using the easy fact that $\mathsf{R}\Psi_{\mathrm{id}_S}\Lambda = \Lambda$, we get that:

- $\mathsf{R}\Psi_f\Lambda = \Lambda$ if f is smooth,
- $\mathsf{R}\Psi_{\mathrm{id}_S}\mathsf{R}f_{\eta *}\Lambda = \mathsf{R}f_{s*}\mathsf{R}\Psi_f\Lambda$ if f is proper.

The last formula can be rewritten in the following more expressive way: $\mathrm{H}^*_{\text{ét}}(X_{\bar{\eta}}, \Lambda) = \mathrm{H}^*_{\text{ét}}(X_s, \mathsf{R}\Psi_f\Lambda)$. In concrete terms, this means that for a proper S-scheme X, the étale cohomology of the constant sheaf on the generic geometric fiber $X_{\bar{\eta}}$ is isomorphic to the étale cohomology of the special fiber X_s with value in the complex of nearby cycles $\mathsf{R}\Psi_f\Lambda$. This is a very useful fact, because usually the scheme X_s is simpler than $X_{\bar{\eta}}$ and the complex $\mathsf{R}\Psi_f\Lambda$ can often be computed using local methods.

1.2.2 The Rapoport-Zink construction

We keep the notations of the previous paragraph. We now suppose that X is a semi-stable S-scheme i.e. locally for the étale topology X is isomorphic to the standard scheme $S[t_1, \dots, t_n]/(t_1 \dots t_r - \pi)$ where π is a uniformizer of S and $r \leq n$ are positive integers. In [32], Rapoport and Zink constructed an important model of the complex $\mathsf{R}\Psi_f(\Lambda)$. Their construction is based on the following two facts:

- There exists a canonical arrow $\theta : \Lambda_\eta \longrightarrow \Lambda_\eta(1)[1]$ in $\mathsf{D}^+(\eta, \Lambda)$ called the fundamental class with the property that the composition $\theta \circ \theta$ is zero,
- The morphism $\theta : i^*\mathsf{R}j_*\Lambda \longrightarrow i^*\mathsf{R}j_*\Lambda(1)[1]$ in $\mathsf{D}^+(X_s, \Lambda)$ has a representative on the level of complexes $\underline{\theta} : \mathcal{M}^\bullet \longrightarrow \mathcal{M}^\bullet(1)[1]$ such that the composition
$$\mathcal{M}^\bullet \longrightarrow \mathcal{M}^\bullet(1)[1] \longrightarrow \mathcal{M}^\bullet(2)[2]$$
is zero as a map of complexes.

Therefore we obtain a double complex

$$\mathcal{RZ}^{\bullet,\bullet} = [\cdots \to 0 \to \mathcal{M}^\bullet(1)[1] \to \mathcal{M}^\bullet(2)[2] \to \mathcal{M}^\bullet(3)[3] \\ \to \cdots \to \mathcal{M}^\bullet(n)[n] \to \dots]$$

where the complex $\mathcal{M}^\bullet(1)[1]$ is placed in degree zero. Furthermore, following Rapoport and Zink, we get a map $\mathsf{R}\Psi_f\Lambda \longrightarrow \mathsf{Tot}(\mathcal{RZ}^{\bullet,\bullet})$ which is an isomorphism in $\mathsf{D}^+(X_s, \Lambda)$ (see [32] for more details). Here $\mathsf{Tot}(-)$ means the simple complex associated to a double complex. In particular, Rapoport and Zink's result says that the nearby cycles complex $\mathsf{R}\Psi_f\Lambda$ can be constructed using two ingredients:

- The complex $i^*\mathsf{R}j_*\Lambda$,
- The fundamental class θ.

Our construction of the nearby cycles functor in the motivic context is inspired by this fact. Indeed, the above ingredients are motivic (see 1.4.1 for a definition of the motivic fundamental class). We will construct in paragraph 1.4.2 a motivic analogue of $\mathcal{RZ}^{\bullet,\bullet}$ based on these two motivic ingredients and then define the (unipotent) "motivic nearby cycles" to be the associated total motive. In fact, for technical reasons, we preferred to use a motivic analogue of the dual version of $\mathcal{RZ}^{\bullet,\bullet}$. By the dual of the Rapoport-Zink complex, we mean the bicomplex

$$\mathcal{Q}^{\bullet,\bullet} = [\cdots \to \mathcal{M}^\bullet(-n)[-n] \to \cdots \to \mathcal{M}^\bullet(-1)[-1] \to \mathcal{M}^\bullet \to 0 \to \dots]$$

where the complex $\mathcal{M}^\bullet$ is placed in degree zero. It is true that by passing to the total complex, the double complex $\mathcal{Q}^{\bullet,\bullet}$ gives in the same way as $\mathcal{RZ}^{\bullet,\bullet}$ the nearby cycles complex.

1.2.3 The limit of a variation of Hodge structures

Let D be a small analytic disk, 0 a point of D and $D^\star = D - 0$. Let $f : X^\star \longrightarrow D^\star$ be an analytic family of smooth projective varieties. For $t \in D^\star$, we denote by X_t the fiber $f^{-1}(t)$ of f. For any integer q, the local system $\mathsf{R}^q f_*\mathbb{C} = (\mathsf{R}^q f_*\mathbb{Z}) \otimes \mathbb{C}$ on $D^\star$ with fibers $(\mathsf{R}^q f_*\mathbb{C})_t = \mathrm{H}^q(X_t, \mathbb{C})$ is the sheaf of horizontal sections of the Gauss-Manin connection ∇ on $\mathsf{R}^q f_*\Omega^{\cdot}_{X^\star/D^\star}$. The decreasing filtration F^k on the de Rham complex $\Omega^{\cdot}_{X^\star/D^\star}$ given by

$$F^k\Omega^{\cdot}_{X^\star/D^\star} = [0 \to \dots 0 \to \Omega^k_{X^\star/D^\star} \to \cdots \to \Omega^n_{X^\star/D^\star}]$$

induces a filtration $F^k\mathsf{R}^q f_*\Omega^{\cdot}_{X^\star/D^\star}$ by locally free $\mathcal{O}_{D^\star}$-submodules on $\mathsf{R}^q f_*\Omega^{\cdot}_{X^\star/D^\star}$.

For any $t \in D^\star$, we get by applying the tensor product $- \otimes_{\mathcal{O}_{D^\star}} \mathbb{C}(t)$ a filtration F^k on $\mathrm{H}^q(X_t, \mathbb{C})$ which is the Hodge filtration. The data:

- The local system $\mathsf{R}^q f_* \mathbb{Z}$,
- The $\mathcal{O}_{D^\star}$-module $(\mathsf{R}^q f_* \mathbb{Z}) \otimes \mathcal{O}_{D^\star} = \mathsf{R}^q f_* \Omega^{\cdot}_{X^\star/D^\star}$ together with the Gauss-Manin connexion,
- The filtration F^k on $(\mathsf{R}^q f_* \mathbb{Z}) \otimes \mathcal{O}_{D^\star}$

satisfy the Griffiths transversality condition and are called a Variation of (pure) Hodge Structures.

Let us suppose for simplicity that f extends to a semi-stable proper analytic morphism: $X \longrightarrow D$. We denote by $\omega^{\cdot}_{X/D}$ the relative de Rham complex with logarithmic poles on $Y = X - X^\star$, that is,

$$\omega^1_{X/D} = \Omega^1_X(\log(Y))/\Omega^1_D(\log(0)).$$

We fix a uniformizer $t : D \to \mathbb{C}$, a universal cover $\bar{D}^\star \to D^\star$ and a logarithm $\log t$ on $\bar{D}^\star$. In [36], Steenbrink constructed an isomorphism $(\omega^{\cdot}_{X/D})_{|Y} \longrightarrow \mathsf{R}\Psi_f \mathbb{C}$ depending on these choices. From this, he deduced a mixed Hodge structure on $\mathrm{H}^q(Y, (\omega^{\cdot}_{X/D})_{|Y})$ which is by definition the limit of the above Variation of Hodge Structures.

1.2.4 The analogy between the situations in étale cohomology and Hodge theory

Let V be a smooth projective variety defined over a field k of characteristic zero. Suppose also given an algebraic closure $\bar{k}/k$ with Galois group G_k and an embedding $\sigma : k \subset \mathbb{C}$. In the étale case, the ℓ-adic cohomology of $V_{\bar{k}}$ is equipped with a structure of a continuous G_k-module. In the complex analytic case, the Betti cohomology of $V(\mathbb{C})$ is equipped with a Hodge structure.

Now let $f : X \longrightarrow C$ be a flat and proper family of smooth varieties over k parametrized by an open k-curve C. Then for any $\bar{k}$-point t of C, we have a continuous Galois module‡ $\mathrm{H}^q(X_t, \mathbb{Q}_\ell)$. These continuous Galois modules can be thought of as a "Variation of Galois Representations" parametrized by C which is the étale analogue of the Variation of Hodge structures $(\mathrm{H}^q(X_t(\mathbb{C}), \mathbb{Q}), F^k)$ that we discussed in the above paragraph.

Now let s be a point of the boundary of C and choose a uniformizer near s. As in the Hodge–theoretic case, the variation of Galois modules above has a "limit" on s which is a "mixed" Galois module given by the following data:

- A monodromy operator N which is nilpotent. This operator induces the monodromy filtration which turns out to be compatible with the weight

‡ In general only an open subgroup of G_k acts on the cohomology, unless t factors trough a k-rational point.

filtration of Steenbrink's mixed Hodge structure on the limit cohomology (see [15]),

- The grading associated to the monodromy filtration is a continuous Galois module of "pure" type.

As in the analytic case, this limit is defined via the nearby cycles complex. Indeed, choose an extension of f to a projective scheme X' over $C' = C \cup \{s\}$. Let Y be the special fiber of X'. The choice of a uniformizer gives us a complex $\mathsf{R}\Psi_{X'/C'}\mathbb{Q}_\ell$ on Y. Then the "limit" of our "Variation of Galois representations" is given by $H^q(Y, \mathsf{R}\Psi_{X'/C'}\mathbb{Q}_\ell)$. The monodromy operator N is induced from the representation on $\mathsf{R}\Psi_{X'/C'}\mathbb{Q}_\ell$ of the étale fundamental group of the punctured henselian neighbourhood of s in C.

1.3 Specialization systems

The goal of this section is to axiomatize some formal properties of the nearby cycles functors that we expect to hold in the motivic context. The result will be the notion of specialization systems. We then state some consequences of these axioms which play an important role in the theory. Before doing that we recall briefly the motivic categories we use.

1.3.1 The motivic categories

Let X be a noetherian scheme. In this paper we will use two triangulated categories associated to X:

(i) The motivic stable homotopy category $\mathbf{SH}(X)$ of Morel and Voevodsky,
(ii) The stable category of mixed motives $\mathbf{DM}(X)$ of Voevodsky.

These categories are respectively obtained by taking the *homotopy category* (in the sense of Quillen [31]) associated to the two *model categories* of $T = (\mathbb{A}^1_X/\mathbb{G}\mathrm{m}_X)$-spectra:

(i) The category $\mathbf{Spect}^T_s(X)$ of T-spectra of simplicial sheaves on the smooth Nisnevich site $(\mathrm{Sm}/X)_{\mathrm{Nis}}$,
(ii) The category $\mathbf{Spect}^T_{\mathrm{tr}}(X)$ of T-spectra of complexes of sheaves with transfers on the smooth Nisnevich site $(\mathrm{Sm}/X)_{\mathrm{Nis}}$.

Recall that a T-spectrum E is a sequence of objects $(E_n)_{n\in\mathbb{N}}$ connected by maps of the form $E_n \longrightarrow \underline{\mathrm{Hom}}(T, E_{n+1})$. We sometimes denote by $\mathbf{Spect}^T(X)$ one of the two categories $\mathbf{Spect}^T_s(X)$ or $\mathbf{Spect}^T_{\mathrm{tr}}(X)$. We do not intend to give the detailed construction of these model categories as this has already been done in several places (cf. [5], [20], [24], [25], [28], [33], [37]). For the reader's convenience, we however give some indications. We focus mainly on the class of weak equivalences; indeed this is enough to define the homotopy category which is obtained by formally inverting the arrows in this class. The weak equivalences in these two categories of T-spectra are called the stable $\mathbb{A}^1$-weak equivalences and are defined in the three steps. We restrict ourself to the case of simplicial sheaves; the case of complexes of sheaves with transfers is completely analogous.

Step 1. We first define simplicial weak equivalences for simplicial sheaves. A map $A_\bullet \longrightarrow B_\bullet$ of simplicial sheaves on $(\mathrm{Sm}/X)_{\mathrm{Nis}}$ is a simplicial weak equivalence if for any smooth X-scheme U and any point $u \in U$, the map of simplicial sets‡ $A_\bullet(\mathrm{Spec}(\mathcal{O}^h_{U,u})) \longrightarrow B_\bullet(\mathrm{Spec}(\mathcal{O}^h_{U,u}))$ is a weak equivalence (i.e. induces isomorphisms on the set of connected components and on the homotopy groups).

Step 2. Next we perform a Bousfield localization of the simplicial model structure on simplicial sheaves in order to invert the projections $\mathbb{A}^1_U \longrightarrow U$ for smooth X-schemes U (see [13] for a general existence theorem on localizations and [28] for this particular case). The model structure thus obtained is the $\mathbb{A}^1$-model structure on simplicial sheaves over $(\mathrm{Sm}/X)_{\mathrm{Nis}}$. We denote $\mathbf{Ho}_{\mathbb{A}^1}(X)$ the associated homotopy category.

Step 3. If A is a pointed simplicial sheaf and $E = (E_n)_n$ is a T-spectrum of simplicial sheaves we define the stable cohomology groups of A with values in E to be the colimit: $\mathsf{Colim}_n \hom_{\mathbf{Ho}_{\mathbb{A}^1}(X)}(T^{\wedge n} \wedge A, E_n)$. We then say that a morphism of spectra $(E_n)_n \longrightarrow (E'_n)_n$ is a stable $\mathbb{A}^1$-weak equivalence if it induces isomorphisms on cohomology groups for every simplicial sheaf A.

By inverting stable $\mathbb{A}^1$-weak equivalences in $\mathbf{Spect}^T_s(X)$ and $\mathbf{Spect}^T_{\mathrm{tr}}(X)$ we get respectively the categories $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$. Let U be a smooth X-scheme. We can associate to U the pointed simplicial sheaf U_+ which is simplicially constant, represented by $U \coprod X$ and pointed by the trivial map $X \longrightarrow U \coprod X$. Then, we can associate to U_+ its infinite T-suspension $\Sigma^\infty_T(U_+)$ given in level n by $T^{\wedge n} \wedge U_+$. This provides a covariant functor $M : \mathrm{Sm}/X \longrightarrow \mathbf{SH}(X)$ which associates to U its motive $M(U)$. Similarly we can associate to U the complex $\mathbb{Z}_{\mathrm{tr}}(U)$, concentrated in degree zero, and then take its infinite suspension given in level n by $\mathbb{Z}_{\mathrm{tr}}(\mathbb{A}^n \times U)/\mathbb{Z}_{\mathrm{tr}}((\mathbb{A}^n - 0) \times$

‡ This map of simplicial sets is the stalk of $A_\bullet \longrightarrow B_\bullet$ at the point $u \in U$ with respect to the Nisnevich topology.

$U) \simeq T^{\overset{\mathrm{L}}{\otimes} n} \overset{\mathrm{L}}{\otimes} U$. This also gives a covariant functor $M : \mathrm{Sm}/X \longrightarrow \mathbf{DM}(X)$. The images in $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$ of the identity X-scheme are respectively denoted by $\mathbb{I}_X$ and $\mathbb{Z}_X$. When there is no confusion we will drop the index X.

Remark 1.3.1. Sometimes it is useful to stop in the middle of the above construction and consider the homotopy category $\mathbf{Ho}_{\mathbb{A}^1}(X)$ of *step 2*. The abelian version with transfers of $\mathbf{Ho}_{\mathbb{A}^1}(X)$ is the category $\mathbf{DM}_{\mathrm{eff}}(X)$ which is used at the end of the paper. This is the category of effective motives whose objects are complexes of Nisnevich sheaves with transfers and morphisms obtained by inverting $\mathbb{A}^1$-weak equivalences.

Remark 1.3.2. One can also consider the categories $\mathbf{SH}_{\mathbb{Q}}(X)$ and $\mathbf{DM}_{\mathbb{Q}}(X)$ obtained from $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$ by killing torsion objects (using a Verdier localization) or equivalently by repeating the above three steps using simplicial sheaves and complexes of sheaves with transfers of $\mathbb{Q}$-vector spaces (instead of sets and abelian groups). It is important to note that the categories $\mathbf{SH}_{\mathbb{Q}}(X)$ and $\mathbf{DM}_{\mathbb{Q}}(X)$ are essentially the same at least for X a field. Indeed, an unpublished result of Morel (see however the announcement [27]) claims that $\mathbf{SH}_{\mathbb{Q}}(k)$ decomposes into $\mathbf{DM}_{\mathbb{Q}}(k) \oplus ?(k)$ with $?(k)$ a “small part” equivalent to the zero category unless the field k is formally real (i.e., if (-1) is not a sum of squares in k).

Remark 1.3.3. The triangulated categories $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$ have infinite direct sums. It is then possible to speak about compact motives. A motive M is compact if the functor $\hom(M, -)$ commutes with infinite direct sums (see [30]). If U is a smooth X-scheme, then its motive $M(U)$ (in $\mathbf{SH}(X)$ or $\mathbf{DM}(X)$) is known to be compact (see for example [33]). Therefore, the triangulated categories with infinite sums $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$ are compactly generated in the sense of [30]. We shall denote $\mathbf{SH}^{ct}(X)$ and $\mathbf{DM}^{ct}(X)$ the triangulated subcategories of $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$ whose objects are the compact ones. The letters *ct* stand for constructible and we shall call them the categories of constructible motives (by analogy with the notion of constructible sheaves in étale cohomology considered in [2]).

The elementary functorial operators f^*, f_* and $f_\#$ of the categories $\mathbf{SH}(-)$ and $\mathbf{DM}(-)$ are defined by deriving the usual operators f^*, f_* and $f_\#$ on the level of sheaves. For $\mathbf{Ho}_{\mathbb{A}^1}(-)$, the details can be found in [28]. It is possible to extend these operators to spectra (see [34]). For $\mathbf{DM}(-)$ one can follow the same construction. Details will appear in [6]. The tensor product is obtained by using the category of symmetric spectra. The details for $\mathbf{SH}(-)$ can be found in [20]. For $\mathbf{DM}(-)$ this will be included in [6].

Using the elementary functorial operators: f^*, $f_\#$, f_* and $\otimes$, it is possible to fully develop the Grothendieck formalism of the six operators (see chapters I and II of [3]). For example, assuming resolution of singularities one can prove that all the Grothendieck operators preserve constructible motives.

Except for the monodromy triangle, the formalism of motivic vanishing cycles can be developed equally using the categories $\mathbf{SH}(-)$ or $\mathbf{DM}(-)$. In fact, one can more generally work in the context of a *stable homotopical 2-functor*. See [3] for a definition of this notion and for the construction of the functors Ψ in this abstract setting.

1.3.2 Definitions and examples

Let B be a base scheme. We fix a diagram

$$\eta \xrightarrow{j} B \xleftarrow{i} s$$

with j (resp. i) an open (resp. closed) immersion. We do not suppose that B is the spectrum of a DVR or that s is the complement of η. Every time we are given a B-scheme $f : X \longrightarrow B$, we form the commutative diagram with cartesian squares

$$\begin{tikzcd} X_\eta \arrow[r, "j"] \arrow[d, "f_\eta"'] & X \arrow[d, "f"] & X_s \arrow[l, "i"'] \arrow[d, "f_s"] \\ \eta \arrow[r, "j"] & B & s. \arrow[l, "i"'] \end{tikzcd}$$

We recall the following definition from [3], chapter III:

Definition 1.3.4. A *specialization system* *sp* over (B, j, i) is given by the following data:

(1) For a B-scheme $f : X \longrightarrow B$, a triangulated functor:

$$sp_f : \mathbf{SH}(X_\eta) \longrightarrow \mathbf{SH}(X_s)$$

(2) For a morphism $g : Y \longrightarrow X$ a natural transformation of functors:

$$\alpha_g : g_s^* sp_f \longrightarrow sp_{f\circ g} g_\eta^*.$$

These data should satisfy the following three axioms:

- The natural transformations $\alpha_?$ are compatible with the composition of morphisms. More precisely, given a third morphism $h : Z \longrightarrow Y$, the diagram

$$\begin{tikzcd} (g\circ h)_s^* sp_f \arrow[rr] \arrow[d, "\sim"'] & & sp_{fgh}(g\circ h)_\eta^* \arrow[d, "\sim"] \\ h_s^* g_s^* sp_f \arrow[r] & h_s^* sp_{fg} g_\eta^* \arrow[r] & sp_{fgh} h_\eta^* g_\eta^* \end{tikzcd}$$

is commutative,

- The natural transformation α_g is an isomorphism when g is smooth,
- If we define the natural transformation $\beta_g : sp_f g_{\eta*} \longrightarrow g_{s*} sp_{f\circ g}$ by the composition

$$sp_f g_{\eta*} \longrightarrow g_{s*} g_s^* sp_f g_{\eta*} \xrightarrow{\alpha_g} g_{s*} sp_{fg} g_\eta^* g_{\eta*} \longrightarrow g_{s*} sp_{fg}$$

then β_g is an isomorphism when g is projective.

Remark 1.3.5. A morphism $sp \longrightarrow sp'$ of specialization systems is a collection of natural transformations $sp_f \longrightarrow sp'_f$, one for every B-scheme f, commuting with the α_g, i.e., such that the squares

$$\begin{tikzcd} g_s^* sp_f \arrow[r] \arrow[d] & sp_{fg} g_\eta^* \arrow[d] \\ g_s sp'_f \arrow[r] & sp'_{fg} f_\eta^* \end{tikzcd}$$

are commutative.

Remark 1.3.6. Let us keep the notations of the Definition 1.3.4. It is possible to construct from $\alpha_?$ two natural transformations (see chapter III of [3])

$$sp_{f\circ g} g_\eta^! \longrightarrow g_s^! sp_f \quad \text{and} \quad g_{s!} sp_{f\circ g} \longrightarrow sp_f g_{\eta!}.$$

These natural transformations are important for the study of the action of the duality operators on the motivic nearby cycles functors in paragraph 1.4.5. However, we will not need them for the rest of the paper.

Remark 1.3.7. The above definition makes sense for any stable homotopical 2-functor from the category of schemes to the 2-category of triangulated categories (see chapter I of [3]). In particular, one can speak about specialization systems in $\mathbf{DM}(-)$, $\mathbf{SH}_{\mathbb{Q}}(-)$ and of course in $\mathsf{D}^+(-,\Lambda)$. For example, the family of nearby cycles functors $\Psi = (\Psi_f)_{f\in \mathsf{Fl}(\mathsf{Sch})}$ of the paragraph 1.2.1 is in a natural way a specialization system in $\mathsf{D}^+(-,\Lambda)$ with base (S, j, i).

Example 1.3.8. It is easy to produce examples of specialization systems. The most simple (but still very interesting) example is what we call in chapter III of [3] the canonical specialization system χ. It is defined by $\chi_f(A) = i^* j_*(A)$.

Example 1.3.9. Given a specialization system sp and an object $E \in \mathbf{SH}(\eta)$, we can define a new specialization system by the formula: $sp'_f(-) = sp_f(- \otimes f_\eta^* E)$. In the same way, given an object F of $\mathbf{SH}(s)$, we define a third specialization system by the formula: $sp''_f(-) = sp_f(-) \otimes f_s^* F$.

1.3.3 The basic results

We state here some (non-trivial) results that follow from the axioms of Definition 1.3.4. For the proofs (which are too long to be included here) the reader can consult chapter III of [3]. For simplicity, we shall stick to the case where B is an affine, smooth and geometrically irreducible curve over a field k of *characteristic zero*, s a closed point of B and η a non-empty open subscheme of $B - s$ or the generic point of B.

We fix a section $\pi \in \Gamma(B, \mathcal{O}_B)$ which we suppose to have a zero of order one on s and to be invertible on η. We then define for $n \in \mathbb{N}$, two simple B-schemes:

- $B_n = B[t]/(t^n - \pi)$ and $e_n : B_n \longrightarrow B$ the obvious morphism,
- $B'_n = B[t, u, u^{-1}]/(t^n - u.\pi)$ and $e'_n : B'_n \longrightarrow B$ the obvious morphism.

Recall that the unit objects of $\mathbf{SH}(X)$ and $\mathbf{DM}(X)$ were respectively denoted by $\mathbb{I} = \mathbb{I}_X$ and $\mathbb{Z} = \mathbb{Z}_X$. We shall also denote by $\mathbb{Q} = \mathbb{Q}_X$ the unit object of $\mathbf{DM}_{\mathbb{Q}}(X)$. The proofs of the following three theorems are in [3], chapter III.

Theorem 1.3.10. ***1-*** *Let sp be a specialization system over (B, j, i) for* $\mathbf{SH}$ *(resp. for* $\mathbf{DM}$*). Suppose that for all $n \in \mathbb{N}$, the objects:*

- $sp_{e_n}(\mathbb{I}) \in \mathrm{Ob}(\mathbf{SH}((B_n)_s))$ *(resp.* $sp_{e_n}(\mathbb{Z}) \in \mathrm{Ob}(\mathbf{DM}((B_n)_s))$*),*
- $sp_{e'_n}(\mathbb{I}) \in \mathrm{Ob}(\mathbf{SH}((B'_n)_s))$ *(resp.* $sp_{e'_n}(\mathbb{Z}) \in \mathrm{Ob}(\mathbf{SH}((B'_n)_s))$*),*

are constructible (see remark 1.3.3). Then for any B-scheme $f : X \longrightarrow B$, and any constructible object A of $\mathbf{SH}(X_\eta)$ *(resp.* $\mathbf{DM}(X_\eta)$*), the object $sp_f(A)$ is constructible.*

2- *Let sp be a specialization system over (B, j, i) for* $\mathbf{DM}_{\mathbb{Q}}(-)$*. Suppose that for all $n \in \mathbb{N}$, the objects $sp_{e_n}(\mathbb{Q}) \in$* $\mathbf{DM}_{\mathbb{Q}}(s)$ *are constructible. Then for any B-scheme $f : X \longrightarrow B$, and any constructible object $A \in$* $\mathbf{DM}_{\mathbb{Q}}(X_\eta)$*, the object $sp_f(A)$ is constructible.*

The following result will play an important role:

Theorem 1.3.11. *1) Let $sp \longrightarrow sp'$ be a morphism between two specialization systems over (B, j, i) for* **SH** *(resp.* **DM***). Suppose that for every $n \in \mathbb{N}$, the induced morphisms:*

- $sp_{e_n}(\mathbb{I}) \longrightarrow sp'_{e_n}(\mathbb{I})$ *(resp.* $sp_{e_n}(\mathbb{Z}) \longrightarrow sp'_{e_n}(\mathbb{Z})$*),*
- $sp_{e'_n}(\mathbb{I}) \longrightarrow sp'_{e'_n}(\mathbb{I})$ *(resp.* $sp_{e'_n}(\mathbb{Z}) \longrightarrow sp'_{e'_n}(\mathbb{Z})$*),*

are isomorphisms.

Then for any B-scheme $f : X \longrightarrow B$, and any constructible object A of $\mathbf{SH}(X_\eta)$ *(resp. of* $\mathbf{DM}(X_\eta)$*) the morphism*

$$sp_f(A) \longrightarrow sp'_f(A)$$

is an isomorphism. When sp_f and sp'_f both commute with infinite sums, the constructibility condition on A can be dropped.

2) If we are working in $\mathbf{DM}_\mathbb{Q}(-)$ *the same conclusions hold under the following weaker condition: For every $n \in \mathbb{N}$ the morphisms $sp_{e_n}(\mathbb{Q}) \longrightarrow sp'_{e_n}(\mathbb{Q})$ are isomorphisms.*

Remark 1.3.12. In part *2* of Theorems 1.3.10 and 1.3.11, we cannot replace $\mathbf{DM}_\mathbb{Q}$ by $\mathbf{SH}_\mathbb{Q}$. Indeed, we use in an essential way the fact that the stable homotopical 2-functor $\mathbf{DM}_\mathbb{Q}$ is separated (like "separated" for presheaves)
(see chapter II of [3]), that is, the functor e^* is conservative for a finite surjective morphism e. This property for $\mathbf{DM}_\mathbb{Q}$ is easily proved by reducing to a finite field extension and using transfers. It fails for $\mathbf{SH}_\mathbb{Q}$ already for the morphism $\mathrm{Spec}(\mathbb{C}) \longrightarrow \mathrm{Spec}(\mathbb{R})$. However, using Morel's result [27], one sees that $\mathbf{SH}_\mathbb{Q}$ is separated when restricted to the category of schemes on which (-1) is a sum of squares.

The previous two theorems are deduced using resolution of singularities from the following result:

Theorem 1.3.13. *Let sp be a specialization system over B. Let $f : X \longrightarrow B$ be a B-scheme. Suppose that X is regular, X_s is a reduced normal crossing divisor in X and fix a smooth branch $D \subset X_s$. We denote by D^0 the smooth locus of f contained in D, i.e., D^0 is the complement in X_s of the union of all the branches that meet D properly. Let us denote by u the closed immersion $D \subset X_s$ and v the open immersion $D^0 \subset D$. The obvious morphism $id \longrightarrow v_* v^*$ induces an isomorphism: $[u^* sp_f f^*_\eta] \longrightarrow v_* v^* [u^* sp_f f^*_\eta]$. Furthermore, if p is the projection of D^0 over s then $v^*[u^* sp_f f^*_\eta] \simeq p^* sp_{id_B}$.*