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Quadratic Forms with Applications to Algebraic Geometry and Topology

Albrecht Pfister

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Preface

This book grew out of a graduate course that I gave at the University of Cambridge in the Easter Term of 1993. The idea of publishing a somewhat enlarged and polished version of my lectures came from Professor J.W.S. Cassels, who, in addition, made it possible for me to spend my sabbatical at Cambridge supported by a Research Grant of the SERC and an appointment as a "Visiting Fellow Commoner" of Trinity College. I thank these institutions for their help in making my stay in Cambridge a very pleasant one.

I should point out in this connection that a great deal of my research on quadratic forms began in the year 1963 when I attented a colloquium talk given by Cassels on "Sums of Squares of Rational Functions" at the University of Göttingen. Later, our connections intensified during the Academic Year 1966/67 when I studied and lectured in Cambridge. My early Lecture Notes [Pfister 1967_1] give an idea of the status of the algebraic theory of quadratic forms in those days. Thus, much of my previous work as well as the present book owe their existence to the constant encouragement and interest of Cassels over many years. For this reason, I wish to express my deep gratitude to him.

This book is not a systematic treatise on quadratic forms. Excellent books of this kind are already available, in particular the books of O'Meara [O'M] on the arithmetic theory over number fields and their integer domains and the books of Lam [L] and Scharlau [S] on the algebraic theory over general fields.

The choice of material considered herein reflects my own interests and incorporates a considerable amount of my scientific work over the past 30 years. It starts with some "highlights" about quadratic forms in Chapters 1 and 2. A main theme of the text concerns the field invariants: "level" (Chapter 3), "Pythagoras number" (Chapter 7), and "u-invariant" (Chapter 8). Many people have contributed to the results presented here. Furthermore, I have emphasized the way in which quadratic forms lead to rich interconnections linking algebra, number theory, algebraic geometry, and algebraic topology. Such topics are covered in Chapters 3, 4, 5, 6 and 10. Finally, systems of quadratic forms (Chapter 9) serve as a kind of clue for relating algebraic geometry and topology to quadratic forms. The specific topics of the various sections can best be seen from the table of contents, and so there is no point in repeating them here.

The prerequisites on the part of the reader are fairly modest. Standard knowledge from introductory courses suffices for most parts of the text. In several places where I need more advanced results a precise reference is given. I have tried to make the main body of the book self-contained with full proofs. Side results or more difficult theorems which go far beyond the methods used here are given without proofs. Examples, notes and open questions have been added whenever possible. They can be used by the reader both to clarify his understanding and to extend his knowledge of the concepts.

I hope that the book will prove equally well suited for graduate students, teachers, researchers on quadratic forms, and mathematicians working in other disciplines with an interest in the topics treated here. My special thanks go to Michael Meurer for proof-reading the manuscript and to Mrs Jutta Gonska for preparing an excellent typescript.

Mainz, December 1994 Albrecht Pfister

Chapter 1

The Representation Theorems of Cassels

§1. Preliminaries on Quadratic Forms

1.1 Definition. Let K be a (commutative) field, let n be a natural number. An n-ary *quadratic form* over K is a homogeneous polynomial of degree 2 in n variables with coefficients from K. It has the form

$$\varphi(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} a_{ij} x_i x_j \in K[x_1, \ldots, x_n].$$

In matrix notation this can be written as follows:
Let x be the column vector with components $x_1, \ldots, x_n$, let x' be its transpose which is a row vector and let $A = (a_{ij})$ be the (n by n)-matrix in $M_{n,n}(K)$ which is determined by the coefficients a_{ij} of φ. Then

$$\varphi(x) = x'Ax.$$

1.2 Definition. Two n-ary quadratic forms φ and ψ over K are called *equivalent* if there is a nonsingular linear transformation $T \in GL_n(K)$ such that

$$\psi(x) = \varphi(Tx).$$

Clearly this is an equivalence relation. We write

$$\psi \cong \varphi \quad (\text{over } K).$$

From now on we shall suppose that the characteristic of K is different from 2. The case char $K = 2$ is postponed to section 4.

For char $K \neq 2$ we can replace the coefficients a_{ij} by $\frac{a_{ij}+a_{ji}}{2}$ without changing the quadratic form φ. Then $a_{ij} = a_{ji}$, i.e. A is *symmetric.* Under an equivalence T the symmetric matrix A is replaced by the congruent matrix

$$B = T'AT$$

which is again symmetric. Furthermore, we see that the polynomial

$$\varphi(x) = \sum_{i} a_{ii} x_i^2 + \sum_{i<j} (a_{ij} + a_{ji}) x_i x_j$$

uniquely determines the matrix A if A is symmetric since the a_{ii} and $2a_{ij}$ (for $i < j$) are exactly the coefficients of φ.

1.3 Definition. Let char $K \neq 2$, let $\varphi(x) = x'Ax$ be a quadratic form over K with $A = A'$ and let x, y be independent indeterminate vectors. Put

$$b_\varphi(x,y) = \frac{1}{2}(\varphi(x+y) - \varphi(x) - \varphi(y)) = x'Ay = y'Ax.$$

b_φ is called *"the associated symmetric bilinear form"* of φ.

Conversely, any symmetric bilinear form

$$b(x,y) = x'Ay \qquad \text{with} \quad A = A'$$

determines a quadratic form $\varphi(x) := b(x,x)$, and these two processes are inverse to one another. Therefore the theories of quadratic forms over K and of symmetric bilinear forms over K (in finitely many variables) essentially coincide if char $K \neq 2$.

Every n-ary quadratic form φ over K induces a map Q_φ from the vector-space $V = K^n$ of n-fold column vectors over K to the field K, namely

$$Q_\varphi : V \to K, \quad Q_\varphi(v) := \varphi(v) \quad \text{for } v \in V.$$

Q_φ is a *quadratic map*, i.e. it has the following properties:

(1) $Q_\varphi(av) = a^2 Q_\varphi(v)$ for $a \in K,\ v \in V$.

(2) The map $B_\varphi : V \times V \to K$ given by

$$B_\varphi(v,w) = \frac{1}{2}(Q_\varphi(v+w) - Q_\varphi(v) - Q_\varphi(w))$$

is K-bilinear (and symmetric).

If $\varphi(x) = x'Ax$ is given by the symmetric matrix $A = (a_{ij})$ and if $e_1, \ldots, e_n$ is the standard basis of V then

$$Q_\varphi(e_i) = a_{ii} \qquad \text{and} \qquad B_\varphi(e_i, e_j) = \frac{1}{2}(a_{ij} + a_{ji}) = a_{ij}.$$

This means that A and φ can be reconstructed from the pair (Q_φ, B_φ).

This observation leads to the following definitions and proposition.

1.4 Definition. Let V be an n-dimensional K-vector-space. A map $Q : V \to K$ is called a *quadratic map* and the pair (V, Q) is then called a *quadratic space over* K if Q satisfies the conditions:

(1) $Q(av) = a^2 Q(v)$ for $a \in K, v \in V$.

(2) The map $B : V \times V \to K$ given by

$$B(v,w) := \frac{1}{2}(Q(v+w) - Q(v) - Q(w))$$

is K-bilinear.

1.5 Definition. Two n-dimensional quadratic spaces (V, Q) and (V', Q') over K are called *isometric* if there exists a K-linear isomorphism $T : V \to V'$ such that

$$Q(v) = Q'(Tv) \quad \text{for all } v \in V.$$

We write: $(V, Q) \cong (V', Q')$.

1.6 Proposition. There is a 1-1-correspondence between equivalence classes of n-ary quadratic forms over K and isometry classes of n-dimensional quadratic spaces over K.

PROOF. The correspondence $\varphi \rightsquigarrow Q_\varphi$ constructed above for $V = K^n$ has the desired properties since φ can be regained from Q_φ and since every n-dimensional K-vector-space V is isomorphic to K^n.

This enables us to switch from the more algebraic language of quadratic forms to the more geometric language of quadratic spaces and vice versa. The latter point of view was introduced in the fundamental paper [Witt 1937] of Witt and has been proved very useful. If there is no danger of confusion we will no longer distinguish between the form φ and the map Q_φ, i.e. we write φ instead of Q_φ and b_φ instead of B_φ.

1.7 Orthogonal Sums. Two quadratic spaces (V_1, φ_1) and (V_2, φ_2) over K of dimensions n_1 and n_2 respectively, give rise to a quadratic space (V, φ) of dimension $n = n_1 + n_2$, namely

$$\begin{aligned} V &= V_1 \oplus V_2, \\ \varphi(v) &= \varphi(v_1) + \varphi(v_2), \end{aligned}$$

for $v_1 \in V_1$, $v_2 \in V_2$ and $v = v_1 + v_2 \in V$. This space (V, φ) is called the orthogonal sum of (V_1, φ_1) and (V_2, φ_2). We also write $\varphi = \varphi_1 \oplus \varphi_2$. If φ_i is given by the symmetric matrix A_i $(i = 1, 2)$ then φ has matrix

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

Similarly, the orthogonal sum of r quadratic spaces can be defined for any $r \in \mathbb{N}$. Up to equivalence it depends only on (the equivalence classes of) the summands but not on their order.

Conversely, let (V, φ) be a quadratic space and let V_i $(i = 1, \ldots, r)$ be subspaces of V such that $V = V_1 \oplus \ldots \oplus V_r$ and $b_\varphi(v_i, v_j) = 0$ for $v_i \in V_i$, $v_j \in V_j$, $i \neq j$. Then $\varphi = \varphi_1 \oplus \ldots \oplus \varphi_r$ with $\varphi_i = \varphi|_{V_i}$, i.e. φ is the orthogonal sum of the forms φ_i.

We can now prove

1.8 Theorem. Let char $K \neq 2$. Then every quadratic space (V, φ) over K is isometric to an orthogonal sum of 1-dimensional spaces. In other words: Every n-ary quadratic form φ over K is equivalent to a *diagonal form* ψ with $\psi(x) = \sum_1^n a_i x_i^2$, $a_i \in K$.

PROOF. We use induction on $\dim V = n$. If $\varphi(v) = 0$ for all $v \in V$ then 1.3 shows $b_\varphi(v_1, v_2) = 0$ for any pair $v_1, v_2 \in V$. In this case any basis $\{v_1, \ldots, v_n\}$ of V is an orthogonal basis.

If $\varphi(v_1) = a_1 \neq 0$ for some $v_1 \in V$ we consider the subspace

$$U = (Kv_1)^\perp = \{u \in V : b_\varphi(u, v_1) = 0\}$$

of all vectors u which are orthogonal to v_1 with respect to b_φ. The condition $b_\varphi(u, v_1) = 0$ amounts to one linear equation for u. Since $\varphi(v_1) = b_\varphi(v_1, v_1) \neq 0$ we have $v_1 \notin U$ and $\dim U = n - 1$. This shows $V = Kv_1 \oplus U$ and $\varphi = \varphi_1 \oplus \varphi_2$ with $\varphi_1 = \varphi|_{Kv_1}$, $\varphi_2 = \varphi|_U$. The induction hypothesis for U finishes the proof.

Note. In the case $\varphi \neq 0$ the element $a_1 \in K^\bullet = K \backslash \{0\}$ is any element which has the form $\varphi(v_1)$, $v_1 \in V$.

Notation. The diagonal form $\psi(x) = \sum_1^n a_i x_i^2$ is abbreviated by

$$\psi = \langle a_1, \ldots, a_n \rangle = \langle a_1 \rangle \oplus \ldots \oplus \langle a_n \rangle.$$

1.9 Definition. Let $A = A'$ be a symmetric matrix. Let (V, φ) with $\varphi(x) = x'Ax$ be the corresponding quadratic space.

(1) The subspace rad $V = V^\perp = \{u \in V : b_\varphi(u, v) = 0 \text{ for all } v \in V\}$ is called the *radical of* (V, φ), and is written as rad V.

(2) (V, φ) is called *regular* if rad $V = 0$.

The following observations are immediate:

- rad $V = \{u \in V : u'Av = 0 \text{ for all } v \in V\} = \{u \in V : u'A = 0\}$.
- rad $V = 0 \Longleftrightarrow \det A \neq 0$.
- The terms *radical* and *regular* are invariant under isometry.
- If φ is not regular then $\varphi \cong \langle a_1, \ldots a_n \rangle$ and, say, $a_n = 0$.

This means that φ can be transformed into a quadratic form which actually depends on at most $n - 1$ variables. Since n can be any natural number in our treatment of quadratic forms we can and will henceforth assume that all forms are regular.

Note. Let φ be a quadratic form over K and let $L \supset K$ be any extension field of K. Then φ may also be considered as a quadratic form over L. This "extended" form is usually denoted by φ_L or $\varphi \otimes L$. We have

$$\varphi = \varphi_K \text{ regular } \iff \varphi_L \text{ regular.}$$

1.10 Definition. For an n-ary quadratic form φ over K we introduce the following notions:

(1) For $a \in K$ we say that "φ *represents* a over K", if there is a *nonzero* vector $0 \neq v \in K^n$ such that

$$\varphi(v) = a.$$

(2) $D_K(\varphi) = \{\varphi(v) : 0 \neq v \in K^n\}$ is the set of all those elements of K which are represented by φ over K.

(3) $D_K^\bullet(\varphi) = D_K(\varphi)\backslash\{0\} \subseteq K^\bullet$.

(4) φ is called *universal* (over K) if $D_K(\varphi) = K^\bullet$.

(5) φ is called *isotropic* (over K) if $0 \in D_K(\varphi)$, otherwise φ is called *anisotropic* (over K).

Example. Consider the form $\varphi = \langle 1, 1\rangle$, i.e. $\varphi(x) = x_1^2 + x_2^2$ over the fields $\mathbf{R}$ and $\mathbf{C}$:
Over $\mathbf{R}$ φ does not represent the elements -1 and 0 since $r_1^2 + r_2^2 > 0$ for any pair $(r_1, r_2) \neq (0,0)$ of real numbers.
Over $\mathbf{C}$ φ does represent -1 and 0 since $-1 = i^2$, $0 = 1^2 + i^2$. Furthermore, φ is universal over $\mathbf{C}$.

This shows that the notions of Definition 1.10 depend very much on the field K, not only on φ.

Clearly a 1-dimensional regular space $\varphi = \langle a\rangle, a \in K^\bullet$, can never be isotropic. Let us study the 2-dimensional regular isotropic spaces over K.

1.11 Proposition. Up to equivalence there is just one regular isotropic quadratic form φ of dimension 2, namely $\varphi(x) = 2x_1x_2$. We have

$$\varphi \cong \langle a, -a\rangle$$

for an arbitrary $a \in K^\bullet$. In particular φ is universal.

PROOF. Let $0 \neq v_1 \in V = K^2$ be an isotropic vector. Since φ is regular there exists $u \in V$ such that $b_\varphi(v_1, u) \neq 0$ and by multiplying u by a suitable element of $K^\bullet$ we can arrange $b_\varphi(v_1, u) = 1$. Clearly u is K-linearly independent from v_1 since $b_\varphi(v_1, v_1) = \varphi(v_1) = 0$. For any $\lambda \in K$ the vectors v_1 and $v_2 = u + \lambda v_1$ form a basis of V for which $\varphi(v_1) = 0$ and $b_\varphi(v_1, v_2) = 1$. Finally, $\varphi(v_2) = \varphi(u) + 2\lambda b_\varphi(u, v_1) + \lambda^2\varphi(v_1) = \varphi(u) + 2\lambda$. Choosing $\lambda = -\frac{\varphi(u)}{2}$ we get $\varphi(v_2) = 0$.

For an indeterminate vector $x = x_1v_1 + x_2v_2$ this gives

$$\varphi(x) = x_1^2\varphi(v_1) + 2x_1x_2b_\varphi(v_1, v_2) + x_2^2\varphi(v_2) = 2x_1x_2.$$

For any $a \in K^\bullet$ φ represents a: Take e.g. $x_1 = \frac{1}{2}$, $x_2 = a$. By Theorem 1.8 we get $\varphi \cong \langle a, a_2\rangle$ for some $a_2 \in K^\bullet$. But φ is isotropic, hence $ac_1^2 + a_2c_2^2 = 0$ for some pair $(c_1, c_2) \neq (0,0)$ in K^2. Then $c_1c_2 \neq 0$ and $a_2 = -a(\frac{c_1}{c_2})^2$. Therefore $\varphi \cong \langle a, -a\rangle$ because the coefficients in a diagonal matrix for φ can be multiplied by arbitrary nonzero squares from K without changing the equivalence class of φ.

Notation. The (equivalence class of a) regular isotropic quadratic form of dimension 2 over K is denoted by H. In other words: $H \cong \langle 1, -1\rangle$. H is called the ***hyperbolic plane***.

Proposition 1.11 can be generalized as follows:

1.12 Proposition. Let (V, φ) be a regular isotropic quadratic space over K with $\dim V = n \geq 2$. Then $V = U \oplus W$ with $U \cong H$, $\dim W = n - 2$; $\varphi \cong \langle 1, -1\rangle \oplus \psi$ with $\psi = \varphi|_W$.

PROOF. As in 1.11 we find vectors $v_1, v_2 \in V$ such that the 2-dimensional subspace $U = Kv_1 + Kv_2$ of V together with the quadratic form $\varphi|_U$ is (isometric to) the hyperbolic plane H. Put $W = U^\perp = \{w \in V : b_\varphi(U, w) = 0\}$. Clearly $\dim W \geq n - 2$. On the other hand $U \cap U^\perp = \operatorname{rad} U = 0$ since $(U, \varphi|_U) \cong H$ is regular. Therefore $\dim W = n - 2$ and $V = U \oplus W$ (orthogonal sum). For the form φ this means $\varphi \cong \langle 1, -1\rangle \oplus \psi$.

§2. The Main Theorem

We start with a simple observation. Let $\varphi(x) = \sum_{i,j=1}^n a_{ij}x_ix_j \in K[x_1, \ldots x_n]$ be a quadratic form over a field K. Let $L = K(t)$ be the *rational function field over K* in one variable t. Then we have

2.1 Lemma. φ anisotropic over $K \Rightarrow \varphi_L$ anisotropic over L.

PROOF. Assume $\varphi(f) = 0$ with $0 \neq f = (f_1, \ldots, f_n)$, $f_i \in L$. Choose a common denominator g_0 of the rational functions f_i. Then $f_i = \frac{g_i}{g_0}$ with $g_0, g_1, \ldots, g_n \in K[t]$ and $\varphi(g) = g_0^2\varphi(f) = 0$ for $0 \neq g = (g_1, \ldots, g_n)$. Let now

$0 \neq d \in K[t]$ be the greatest common divisor of the polynomials $g_1, \dots, g_n$. Then $g_i = dh_i$ with $h_i \in K[t]$, and $h_1, \dots, h_n$ are relatively prime. Put $h = (h_1, \dots, h_n)$. Then $\varphi(g) = d^2\varphi(h) = 0$ is an identity in t. Since $K[t]$ is an integral domain and $d = d(t) \neq 0$ we get $\varphi(h) = 0$. Put $c_i = h_i(0) \in K$, $c = (c_1, \dots, c_n)$. The elements are not all zero since otherwise the $h_i(t)$ would all be divisible by t. Hence $0 \neq c \in K^n$ and $\varphi(c) = 0$ by substituting $t \longrightarrow 0$ in the identity $\varphi(h) = 0$ in $K[t]$. This contradicts the anisotropy of φ.

2.2 Theorem. Let $\varphi(x) = \varphi(x_1, \dots, x_n) = \sum_{i,j=1}^n a_{ij}x_ix_j$ be an n-ary quadratic form over the field K, char $K \neq 2$. Let $0 \neq p(t) \in K[t]$ be a polynomial in one variable. Suppose that φ represents $p = p(t)$ over the field $L = K(t)$. Then φ represents p over the ring $K[t]$, i.e. there are polynomials $f_i = f_i(t) \in K[t]$ such that $\varphi(f_1, \dots, f_n) = p$.

PROOF.
1) If φ is not regular we may replace φ by a quadratic form in less than n variables and argue by induction on n. For $n = 1$, $\varphi(x) = a_{11}x_1^2$, $a_{11}f_1^2 = p$ with $f_1 \in K(t)$, the theorem is true since $f_1 \in K[t]$ follows automatically. (Use that $K[t]$ is a unique factorization domain.)
2) Suppose now that φ is regular but isotropic. Then $\varphi \cong H \oplus \psi$ over K by Proposition 1.12, i.e. without loss of generality

$$\varphi(x) = 2x_1x_2 + \psi(x_3, \dots, x_n).$$

Put $x_1 = p(t)$, $x_2 = \frac{1}{2}$, $x_3 = \dots = x_n = 0$. This shows that φ represents p over $K[t]$.
3) From now on φ is (regular and) anisotropic. By assumption we have a representation

$$\varphi\left(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}\right) = p \tag{1}$$

with polynomials $f_0, \dots, f_n \in K[t]$. Without loss of generality the greatest common divisor of $f_0, \dots, f_n$ is 1.

Furthermore we may suppose that under all representations of shape (1) the given one has *minimal* degree $d = \deg f_0 \geq 0$ of the denominator f_0. If $d = 0$ then f_0 is a nonzero constant and we are finished.

Hypothesis: $d > 0$.
Then we have to derive a contradiction. We introduce the $(n+1)$-dimensional quadratic form

$$\psi = \langle -p(t) \rangle \oplus \varphi_L \qquad \text{over} \quad L = K(t). \tag{2}$$

Explicitly: $\psi(x_0, \dots, x_n) = -p(t)x_0^2 + \varphi(x_1, \dots, x_n)$.
(1) implies $\psi(f_0, \dots, f_n) = 0$.

Apply the euclidean algorithm (division by f_0) to the polynomials f_i ($i = 0, \ldots, n$). This gives

$$(3) \qquad f_i = f_0 g_i + r_i \ (i = 0, \ldots, n) \text{ with } g_i, r_i \in K[t],\ \deg r_i < d.$$

In particular, $g_0 = 1$, $r_0 = 0$, $\deg r_0 = -\infty$. Put $f = (f_0, \ldots, f_n)$, $g = (g_0, \ldots, g_n)$. Then $\psi(f) = 0$ and $\psi(g) \neq 0$ by the minimality condition on f_0 since $0 = \deg g_0 < \deg f_0 = d$. In particular, the nonzero vectors f and g are linearly independent over L.

(4) Define $h = \lambda f - \mu g \in L^{n+1}$ with $\lambda = \psi(g)$, $\mu = 2b_\psi(f, g)$.

We have $h = (h_0, \ldots, h_n)$, $h_i \in K[t]$. $\lambda \neq 0$ implies $h \neq 0$. On the other hand we get

$$(5) \quad \psi(h) = \lambda^2 \psi(f) - 2\lambda\mu b_\psi(f, g) + \mu^2 \psi(g) = \lambda^2 \cdot 0 - \lambda\mu^2 + \mu^2\lambda = 0.$$

Actually we must have $h_0 \neq 0$. Otherwise $h = (0, h_1, \ldots, h_n) \neq 0$ would give a nontrivial solution of the equation

$$\psi(h) = \varphi(h_1, \ldots, h_n) = 0 \quad \text{over the field } L = K(t)$$

whereas φ is anisotropic over L by Lemma 2.1. It remains to estimate $\deg h_0$. We have

$$(6) \qquad \begin{aligned} h_0 = \lambda f_0 - \mu &= \psi(g) f_0 - 2b_\psi(f, g) = \frac{1}{f_0} \psi(f_0 g - f) \\ &= \frac{1}{f_0} \sum_{i,j=1}^{n} a_{ij} (f_0 g_i - f_i)(f_0 g_j - f_j). \end{aligned}$$

This implies

$$\deg \psi(f_0 g - f) \leq 2 \max_{i=1,\ldots,n} \deg(f_0 g_i - f_i) = 2 \max_{i=1,\ldots,n} \deg r_i \leq 2(d-1),$$

hence

$$(7) \qquad \deg h_0 = -d + \deg \psi(f_0 g - f) \leq d - 2.$$

Thus h would give a solution of (1) which is "smaller" than f: Contradiction. The proof of 2.2 is finished.

Note. The geometric idea behind the proof of 2.2 is as follows: The equation $\psi = 0$ defines a quadric (hypersurface of degree 2) Q in the projective n-space over L. The "points" f, g are different with $f \in Q$, $g \notin Q$. The "line" joining f and g intersects Q in a second point $h \neq f$. It turns out that the choice (3) for g leads to $\deg h_0 < \deg f_0$.

Theorem 2.2 has the following partial generalization.

2.3 Generalization. Let $\varphi(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ be a quadratic form over $L = K(t)$ such that $a_{ij} \in K[t]$ and $\deg a_{ij} \leq 1$ for all (i, j). Suppose φ is *anisotropic* over L. Let $0 \neq p(t) \in K[t]$ be a polynomial which is represented by φ over L. Then p is already represented over $K[t]$.

PROOF. Part 3) of the above proof carries over verbatim to this slightly more general case. The only change is

$$\deg \psi(f_0 g - f) \leq 1 + 2 \max \deg r_i \leq 2d - 1,$$

hence

$$\deg h_0 \leq d - 1 < d. \tag{7'}$$

This is still enough to derive the contradiction.

Note. The generalization 2.3 is no longer valid if φ is isotropic. Let $\varphi = \langle t, -t \rangle$, $p(t) = 1$. φ is clearly isotropic, hence universal over $L = K(t)$. Thus φ represents $p = 1$ over L. (Derive such a representation explicitly!) But there is clearly no solution of $tf_1^2 - tf_2^2 = 1$ with polynomials $f_1, f_2 \in K[t]$.

Note. At first sight it seems that repeated application of Theorem 2.2 would give the corresponding result for a polynomial $p = p(t_1, \ldots, t_r)$ in several variables. But a closer look reveals that starting from a representation of $p(t_1, t_2)$ over the ring $K(t_2)[t_1]$ the procedure of the above proof with respect to the variable t_2 leads to a representation over $K(t_1)[t_2]$ and not over $K[t_1][t_2]$ since $K[t_1, t_2]$ is no longer a euclidean domain. Actually the *existence* of counter-examples over $\mathbf{R}(t_1, t_2)$ for $\varphi = \langle \underbrace{1, \ldots, 1}_{n} \rangle$ with suitable n goes far back to Hilbert[1888]. Nevertheless the first explicit counter-example (for $n = 4$, $r = 2$) was only found in the year 1967 by Motzkin[1967]. It reads as follows:

2.4 Example. Let $p(x, y) = 1 - 3x^2y^2 + x^4y^2 + x^2y^4 \in \mathbf{R}[x, y]$. Then

(1) p is a sum of four squares in the ring $\mathbf{R}(x)[y]$, hence also in the field $\mathbf{R}(x, y)$.

(2) p is not a sum of (any finite number of) squares in the polynomial ring $\mathbf{R}[x, y]$.

PROOF. 1) Check the following identities:

$$\begin{aligned} p(x, y) &= \frac{(1 - x^2y^2)^2 + x^2(1 - y^2)^2 + x^2(1 - x^2)^2y^2}{1 + x^2} \\ &= \left(\frac{1 + x^2 - 2x^2y^2}{1 + x^2} \right)^2 + \left(\frac{x(1 - x^2)y^2}{1 + x^2} \right)^2 \\ &\quad + \left(\frac{x(1 - x^2)y}{1 + x^2} \right)^2 + \left(\frac{x^2(1 - x^2)y}{1 + x^2} \right)^2 . \end{aligned}$$