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Local Analysis for the Odd Order Theorem

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In memory of R. H. Bruck

Contents

Preface

About 30 years ago, Walter Feit and John G. Thompson [8] proved the Odd Order Theorem, which states that all finite groups of odd order are solvable. In the words of Daniel Gorenstein [15, p. 14], "it is not possible to overemphasize the importance of the Feit-Thompson Theorem for simple group theory." Their proof consists of a set of preliminary results followed by three parts–local analysis, characters, and generators and relations–corresponding to Chapters IV, V, and VI of their paper (denoted by **FT** here). Local analysis of a finite group G means the study of the structure of, and the interaction between, the centralizers and normalizers of nonidentity p-subgroups of G. Here Sylow's Theorem is the first main tool. The main purpose of this book is to present a new version of the local analysis of a minimal counterexample G to the Feit-Thompson Theorem, that is, of Chapter IV and its preliminaries. We also include a remarkably short and elegant revision of Chapter VI by Thomas Peterfalvi in Appendix C.

What we would ideally like to prove, but cannot, is that each maximal subgroup M of G has a nonidentity proper normal subgroup M_0 such that

(1) $C_{M_0}(a) = 1$, for all elements $a \in M - M_0$,
(2) $M_0 \cap M_0{}^g = 1$, for all elements $g \in G - M$,
(3) M_0 is nilpotent,
(4) M/M_0 is cyclic,

and such that the totality of these subgroups M_0, with M ranging over all of the maximal subgroups of G, forms a partition of G:

(5) each nonidentity element of G lies in exactly one of the subgroups M_0.

Relating each step in our procedure (as well as the main results, given in Section 16) to this hypothetical goal will help give the reader a sense of direction and motivation: after the normal Hall subgroup M_σ has been introduced in Section 10, it can be read as M_0. (Section 16 is self-contained, except for notation from Section 1, and can be read as a supplement to this introduction.)

In addition, we strongly recommend first studying a theorem of Feit, Thompson, and Marshall Hall [7], the immediate predecessor of **FT**, which proved solvability under the additional CN-condition: the centralizer of every nonidentity element of G is nilpotent. The local analysis part of its proof leads to conditions (1)–(5) for a minimal counterexample G. A guide to reading this miniature model for **FT** and our work is given in Appendix D. This theorem is actually needed in **FT** [8, p. 983], although not for the part covered by this book. Incidently, the conditions (1)–(5) above clearly imply the CN-condition. Furthermore, (1) means that M is a Frobenius group with kernel M_0, and thus implies (3) by a very special case of a theorem of Thompson (Theorem 3.7).

The Odd Order Theorem was originally conjectured in the nineteenth century. The first essential step toward its proof was taken by Michio Suzuki [25] in 1957. He showed that CA-groups of odd order are solvable; here CA means that all centralizers are abelian. In this case it is a routine matter to derive (1)–(5), with all M_0 abelian. Suzuki's contribution, a model for the later CN-paper, was mainly character-theoretic. Conditions (1)–(2) and variations thereof occur in much more general situations as the end result of local analysis, and it is therefore of fundamental importance for finite group theory that they have strong character theoretic implications. See [14, pp. 139-148], [17, pp. 195-205], or [26, pp. 281-294] for details.

It is the purpose of this book to make the Feit-Thompson Theorem more accessible to a reader familiar with some standard topics in finite group theory, such as Chapters 1–8 of Gorenstein's first book [14] (henceforth denoted by **G**). However it is possible to manage comfortably with considerably less reading. We give information about prerequisites in Appendix A. For the convenience of the reader, strictly necessary references to other works appear only in Chapter I, and refer only to **G**. Further information about the influence of the theorem and its proof, together with a detailed description of the proof, may be found in **G**, pp. 450-461, and in [15, pp. 13-39].

As stated above, our main text and Appendix C correspond to Chapters IV and VI of **FT** and the necessary preliminaries. As to the missing link, the necessary character theory, we must refer the reader to Chapter V of **FT** or to some unpublished work of David Sibley, who has obtained very interesting improvements [23, pp. 385-388]. Fortunately, Chapter V of the original paper is somewhat less complicated than Chapter IV.

We hope that in the not too distant future there will be a unified revised proof of the Feit-Thompson Theorem. In addition, we and others have some thoughts now for further improving this work; in this spirit, we include a few results that are not needed for Chapter V of **FT** or for Sibley's work. However, in view of the considerable interest expressed in this work and the improvements and corrections sent to us by readers of preliminary versions, we have decided to publish the work now as a set of lecture notes.

In a sense, the first steps toward the writing of this book were taken in 1962, when the second author began to study a preprint of the Odd Order Paper, with the encouragement and assistance of his Ph. D. advisor, R. H. Bruck. However, the actual writing of a revision started with a class at the University of Chicago in the Winter and Spring Quarters of 1975.

We wish to thank the members of the 1975 class (particularly David Burry, Noboru Itô, Richard Niles, David T. Price and Jeffrey D. Smith) and of a similar class given in Winter, 1986 (particularly Curtis Bennett, Walter Carlip, Diane Herrmann, Arunas Liulevicius, Peter Sin, and Wayne W. Wheeler). In addition, preliminary versions of this work were read by Paul Lescot, Thomas Peterfalvi, and David Sibley, and studied in seminars at the University of Florida and Wayne State University, led by László Héthelyi (of Technical University, Budapest) and by Daniel Frohardt, David Gluck and Kay Magaard, respectively. We thank each of these individuals and the members of these seminars for their corrections and suggestions.

For permission to include unpublished work, we thank David Sibley (Theorem 14.4, Corollary 15.9); I. Martin Isaacs (Appendix B); Walter Carlip and Wayne W. Wheeler (Appendix C); and especially Walter Feit and John G. Thompson (Theorem 15.8, Corollary 15.9, Appendix E). Appendix C is based on a beautiful revision [22] of Chapter VI of **FT**, for which we thank the author, Thomas Peterfalvi.

We are particularly indebted to Professors Feit and Thompson for their help and encouragement throughout the preparation of this work.

We note with great sadness the deaths of two individuals who also played instrumental roles: R. H. Bruck and Daniel Gorenstein. Without them this work might never have been started nor ever have been completed.

As this book has gone through many stages and vicissitudes in twenty years, there is a danger that we have inadvertently overlooked some individuals to whom thanks are due. To them we sincerely apologize.

During the preparation of parts of this work the second author enjoyed the support of the Guggenheim Foundation and the National Science Foundation, and the hospitality of the Mathematical Institute, Oxford; Jesus College, Oxford; Kansas State University; and Universität Kiel. He thanks each of these institutions. He also thanks the members of his family for their helpful patience, forbearance, or nagging.

An earlier, complete version of this work was prepared by the second author with the assistance of Alexandre Turull in 1979. The present version was prepared with the assistance of Walter Carlip. Both have made valuable corrections and improvements in the mathematical content and the wording of the texts, particularly Dr. Carlip, who has also worked assiduously, over the course of many years to put preliminary drafts into TEX and to produce the final camera-ready copy printed here. We thank both for their efforts.

CHAPTER I

Preliminary Results

Here we give general results about finite groups, mainly solvable groups and p-groups, including some special properties of groups of odd order. In Chapters II–IV we will apply the results of this section to a hypothetical minimal counterexample to the Odd Order Theorem. As mentioned in the preface, all necessary references in this chapter are taken from **G**.

1. Elementary Properties of Solvable Groups

Suppose G is a group. We say that a group A *operates* on G, or A is an *operator group* on G, if there is given a homomorphism ϕ from A into $\operatorname{Aut} G$. In this case we usually write x^α instead of $\phi(\alpha)(x)$ for $x \in G$ and $\alpha \in A$. We say that A *fixes* an element x of G, or that x is A-*invariant*, if $x^\alpha = x$ for every $\alpha \in A$. We say that A *fixes* a nonempty subset S of G, or that S is A-*invariant*, if each element of A fixes S as a set. As in **G**, pp. 30, 33, the set (group) of all A-invariant elements of G will be denoted by $C_G(A)$. Similarly, if S is a nonempty subset of G, $C_A(S)$ will denote the set of all elements of A that fix every element of S.

We will frequently use the fact (**G**, p. 18) that if H and K are subgroups of a group G, then

$$[H, K] \lhd \langle H, K\rangle.$$

By applying this fact to the semidirect product of a group G by an operator group A, we see that $[G, A]$ is a normal subgroup of G fixed by A. As in **G**, p. 19, $[G, A, A]$ will denote $[[G, A], A]$. Also, we say A *stabilizes* a normal series

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = 1$$

of G if each G_i is A-invariant and A acts trivially on each factor G_{i-1}/G_i, $1 \le i \le n$.

Suppose that A is an operator group on a group G. As in **FT**, p. 840, we say that A acts in a *prime manner* on G if

$$C_G(\alpha) = C_G(A) \text{ for all } \alpha \in A^{\#}.$$

(Note that this must occur if $|A|$ is prime and that we allow $A = 1$.) We say that A acts *regularly*, or in a *regular manner* on G if

$$C_G(\alpha) = 1 \text{ for all } \alpha \in A^{\#}.$$

(Thus, if A acts regularly, then $A \subseteq \operatorname{Aut} G$ and A acts in a prime manner on G. This disagrees slightly with the definition in **G**, p. 39, which requires also that $A \neq 1$.)

In the subsequent text we will write $H \lhd\lhd G$ to mean that H is a *subnormal* subgroup of G. This means that H is a member of a normal series of G (**G**, Exercise 1.5, p. 13). Equivalently, there exists a series

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G.$$

We use the property that every subgroup of a nilpotent group G is subnormal in G. This follows immediately from the fact that proper subgroups of a nilpotent group are properly contained in their normalizers (**G**, Theorem 2.3.4, p. 22).

All groups considered in this work will be finite except when explicitly stated otherwise.

For later use we make the following definition. Given a prime p and a group G, we say that G has *p-length one* if $G = O_{p',p,p'}(G)$. (This differs slightly from the definition in **G**, p. 227, in that our definition includes groups of p'-order, that is, groups that, according to the usual definition, would have p-length zero.)

A group G is called a *Z-group* if all of its Sylow subgroups are cyclic.

For any subset T of G we define

$$\mathscr{C}_G(T) = \{\, t^g \mid t \in T \text{ and } g \in G \,\}.$$

A nonempty subset X of G is a *TI-subset* of G if $X \cap X^g \subseteq 1$ for all $x \in G - N(X)$. In particular, a nonidentity proper subgroup H of G is a *TI-subgroup* of G if $H \cap H^g = 1$ for all $g \in G - N(H)$.

In the text that follows we will denote by $\mathcal{E}_p(G)$ the set of all elementary abelian p-subgroups of G; $\mathcal{E}_p{}^*(G)$ the set of all maximal elementary abelian p-subgroups of G; and $\mathcal{E}_p{}^i(G)$ the set of all elementary abelian subgroups of order p^i in G (where i is a positive integer). We let $\mathcal{E}(G)$ be the union of the sets $\mathcal{E}_q(G)$ for all primes q. We define $\mathcal{E}^*(G)$ and $\mathcal{E}^i(G)$ analogously.

For a prime p, a p-group R will be called *narrow* if it contains no elementary abelian subgroup of order p^3 or if it contains a subgroup R_0 of order p and a cyclic subgroup R_1 such that $C_R(R_0) = R_0 \times R_1$. (This definition is not standard and is used only in this book. It corresponds to the definition of π^* on p. 845 of **FT**.)

Lemma 1.1. Suppose that M is a minimal normal subgroup of a finite group G. If M is solvable, then $M \subseteq Z(F(G))$ and is elementary abelian.

Proof. Elementary. $\square$

Proposition 1.2 (P. Hall). Suppose that G is a solvable group and that $G^* \lhd G$. Let $\mathscr{D}$ be the set of all chief factors U/V of G. Let $\mathscr{D}^*$ be the set of all chief factors U/V of G for which $U \subseteq F(G^*)$. Then

$$F(G^*) = \bigcap_{U/V \in \mathscr{D}} C_{G^*}(U/V) = \bigcap_{U/V \in \mathscr{D}^*} C_{G^*}(U/V).$$

Proof. Let

$$H = \bigcap_{U/V \in \mathscr{D}} C_{G^*}(U/V) \quad \text{and} \quad H^* = \bigcap_{U/V \in \mathscr{D}^*} C_{G^*}(U/V).$$

Take $U/V \in \mathscr{D}$. Then U/V is a minimal normal subgroup of G/V. By Lemma 1.1,

$$U/V \subseteq Z(F(G/V)).$$

Since $F(G^*)V/V$ is nilpotent and is also normal in G/V, we know that $F(G^*)V/V \subseteq F(G/V)$. Hence $F(G^*)V/V$ centralizes U/V. As U/V was taken arbitrarily, $F(G^*) \subseteq H$.

Clearly $H \subseteq H^*$. To complete the proof, we assume that $H^* \not\subseteq F(G^*)$ and obtain a contradiction. Let K be a normal subgroup of G minimal with respect to the property that $K \subseteq H^*$ and $K \not\subseteq F(G^*)$. Take a chief series for G that includes K, and let

$$K = K_0 \supset K_1 \supset \cdots \supset K_n = 1 \tag{1.1}$$

be the part of the chief series from K to 1. By the choice of K, we have $K_1 \subseteq F(G^*)$. Hence, for $i = 2, \ldots, n$, we have $K_{i-1}/K_i \in \mathscr{D}^*$ and, since $K \subseteq H^*$, we have $[K_{i-1}, K] \subseteq K_i$. Since K is solvable, K/K_1 is abelian and $[K_0, K] = [K, K] \subseteq K_1$. Thus the series (1.1) is a central series for K. Hence K is nilpotent. Therefore $K \subseteq F(G^*)$, a contradiction. $\square$

Proposition 1.3 (P. Hall). Suppose that G is a solvable group. Then $C_G(F(G)) \subseteq F(G)$.

Proof. Let $G^* = G$ in Proposition 1.2. $\square$

Proposition 1.4. Suppose that G is a solvable group, A is a group of automorphisms of G, and $(|A|, |G|) = 1$. Then A acts faithfully on $F(G)$.

Proof. We may assume that A is cyclic. Let X be the semidirect product of G by A. Then X is solvable. We embed A and G in X. Let $\sigma = \pi(A)$ and $F = F(X)$.

Since A is certainly a Hall σ-subgroup of X and $A\mathcal{O}_\sigma(F)$ is a σ-group, $A = A\mathcal{O}_\sigma(GF) \supseteq \mathcal{O}_\sigma(F)$. As $A \subseteq \operatorname{Aut} G$ and

$$[\mathcal{O}_\sigma(F), G] \subset \mathcal{O}_\sigma(F) \cap G = 1,$$

we have $\mathcal{O}_\sigma(F) = 1$. Thus

$$F = \mathcal{O}_\sigma(F) \times \mathcal{O}_{\sigma'}(F) = \mathcal{O}_{\sigma'}(F) \subseteq \mathcal{O}_{\sigma'}(X) = G.$$

Clearly $F = F(G)$. By Proposition 1.3,

$$C_A(F) = A \cap C_X(F(X)) \subseteq A \cap F(X) \subseteq A \cap G = 1. \quad \square$$

Proposition 1.5. Suppose that G is a solvable group, A is an operator group on G, and $(|A|, |G|) = 1$. Let π be a set of primes. Then:

(a) A fixes some Hall π-subgroup of G;
(b) every A-invariant π-subgroup of G is contained in an A-invariant Hall π-subgroup of G;
(c) if H_1 and H_2 are A-invariant Hall π-subgroups of G, then H_1 and H_2 are conjugate by an element of $C_G(A)$;
(d) if H is any A-invariant normal subgroup of G, then $C_{G/H}(A)$ is the image of $C_G(A)$ in G/H; and
(e) if $C_G(A)$ contains a Hall π'-subgroup of G, then $[G, A] \subseteq \mathcal{O}_\pi(G)$.

Proof. Statements (a), (c), and (d) follow from P. Hall's theorem on solvable groups (**G**, Theorem 6.4.1, p. 231) and from the proof of Theorem 6.2.2, pp. 224–5 of **G**.

To prove (b) we proceed by induction on $|G|$. Let K be an A-invariant π-subgroup of G and M a minimal A-invariant normal subgroup of G. If G itself is a π-group, there is nothing to prove, and so we may assume G is not a π-group. Now KM/M is an A-invariant π-subgroup of G/M so, by induction, there exists an A-invariant Hall π-subgroup H/M of G/M that contains KM/M. Thus H is an A-invariant subgroup of G such that $K \subseteq H \subseteq G$ and $|H|_\pi = |G|_\pi$. If $H \neq G$, we can apply induction to H to conclude that K is contained in an A-invariant Hall π-subgroup of H and we are done. If $H = G$, then M is a normal Sylow p-subgroup of G for some prime p outside π. By (a), G has an A-invariant Hall π-subgroup Q and clearly $G = QM$ with $Q \cap M = 1$. Now $|Q \cap KM| = |K|$, and hence K and $Q \cap KM$ are both A-invariant Hall π-subgroups of KM. By (c), there exists an element $x \in C_{KM}(A)$ such that $K = (Q \cap KM)^x \subseteq Q^x$. Clearly Q^x is an A-invariant Hall π-subgroup of G.

To prove (e), let H be an A-invariant Hall π-subgroup and let K be a Hall π'-subgroup of G contained in $C_G(A)$. Then $G = KH$. Therefore

$$[G, A] = \langle\, h^{-1}k^{-1}k^\alpha h^\alpha \mid k \in K,\, h \in H,\, \alpha \in A \,\rangle \subseteq H.$$

Since $[G, A] \lhd G$, we have $[G, A] \subseteq O_\pi(G)$. $\square$

Proposition 1.6. Suppose that G is a solvable group, A is an operator group on G, and $(|A|, |G|) = 1$. Then:

(a) $G = C_G(A)[G, A] = [G, A]C_G(A)$;
(b) $[G, A, A] = [G, A]$;
(c) if $[G, A, A] = 1$, then A acts trivially on G;
(d) if G is abelian, then $G = C_G(A) \times [G, A]$; and
(e) if G is abelian and $C_G(A)$ contains every element of prime order in G, then A acts trivially on G.

Proof. For (a), let $H = [G, A]$ in Proposition 1.5(d). For (b) and (c), see the proof of **G**, Theorem 5.3.6, p. 181. For (d), see the proof of **G**, Theorem 5.2.3, p. 177. Finally, note that (e) follows from (d). $\square$

In the following lemma we list some of the basic properties of the Frattini subgroup of a finite group.

Lemma 1.7. Suppose that G is a group and R is a p-group for some prime p. Then:

(a) if H is a subgroup of G and $G = H\Phi(G)$, then $G = H$;
(b) $R/\Phi(R)$ is elementary abelian;
(c) $\Phi(R) = 1$ if and only if R is elementary abelian; and
(d) $\Phi(R) = \langle R', x^p \mid x \in R \rangle$.

Proof. (a) **G**, Theorem 5.1.1, p. 173. (b) **G**, Theorem 5.1.3, p. 174. (c) **G**, Theorem 5.1.3, p. 174. (d) Let $S = \langle R', x^p \mid x \in R \rangle$. By (b), $S \subseteq \Phi(R)$. Since R/S is elementary abelian and $\Phi(R/S) = \Phi(R)/S$, (c) yields (d). $\square$

Theorem 1.8 (Burnside). Suppose that A is an operator group on a p-group R and $(|A|, |R|) = 1$. Assume that A centralizes $R/\Phi(R)$. Then A centralizes R.

Proof. By Proposition 1.5(d), $R = C_R(A)\Phi(R)$. By Lemma 1.7(a), $R = C_R(A)$. (This is **G**, Theorem 5.1.4, p. 174.) $\square$

Lemma 1.9. Suppose that π is a set of primes, G is a finite solvable π-group, and A is an operator group on G that stabilizes a normal series of G. Then $A/C_A(G)$ is a π-group.

Proof. It suffices to show that A acts trivially on G if A is a π'-group. This follows from Proposition 1.5(d) by induction on the length of the normal series. $\square$

Proposition 1.10. Suppose that A is an operator group on a nilpotent group G and $(|A|, |G|) = 1$. Let $C = C_G(A)$. If $C_G(C) \subseteq C$, then A acts trivially on G.