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Solitons, Nonlinear Evolution Equations and Inverse Scattering

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Preface

An exciting and extremely active area of research investigation during the past twenty years has been the study of Solitons and the related issue of the construction of solutions to a wide class of nonlinear equations. Indeed there have been a few books written which serve to review aspect of this burgeoning field. A book coauthored by one of us (MJA) exactly ten years ago, discussed many of the relevant viewpoints as well as a variety of applications. Certain important and novel subareas of research such as the the application of the Inverse Scattering Transform (I.S.T.) to solve nonlinear wave equations on the infinite interval, in one spatial and one temporal dimension $(1+1)$, were described in detail. At that time the complete inverse scattering methodology had been carried out primarily for those nonlinear equations related to second order scattering problems. Although it was known that certain nonlinear evolution equations in one and two spatial dimensions were related to suitable (higher order and two dimensional) linear scattering problems, and special techniques were available, nevertheless it was not yet clear that a unified and effective procedure could be applied to all of these nonlinear equations. The main purpose of this book is the description of how the I.S.T. technique can be applied to these situations.

Our presentation begins with a "state of the art" introduction. Here we list as many integrable systems and relevant scattering problems that we are familiar with — though it is still possible that we have missed some interesting ones. In order to help the interested reader and to establish our point of view, in Chapter Two we have reviewed the I.S.T. technique as it applies to the famous Korteweg-de Vries (KdV) equation on the infinite line. In this case the KdV equation is related to the classical second order scalar time-independent Schrödinger scattering problem. Chapter Three discusses the inverse scattering associated with $N \times N$ systems and higher order systems of scattering problems; the 2×2 case is reviewed. The I.S.T. technique is then applied to various systems of partial differential equations. Chapter Three contains a brief discussion of discrete problems as well. Chapter Four details the inverse transform as it applies to a rather novel class of scattering problems which arise in the solution of singular integro-differential evolution equations such as the intermediate long wave and Benjamin-Ono equations. Chapter Five involves the inverse transform for $2+1$-dimensional nonlinear wave equations, and discusses the "DBAR" ($\bar{\partial}$) method of inverse scattering. Key solvable nonlinear wave equations include the Kadomtsev-Petviashvili, Davey-Stewartson and three wave equations in $2+1$-dimensions. Chapter Six analyzes certain multidimensional inverse problems (generalised Schrödinger and $N \times N$ systems) via the $\bar{\partial}$ method of inverse scattering. It is especially worth noting that Faddeev's inverse formulation for the time-independent Schrödinger problem in three(or more dimensions) follows naturally from the $\bar{\partial}$ method in a limiting case. There are few nonlinear wave equations in more than 2 + 1-dimensions which are now known to be solvable by I.S.T.. However the few that are known are discussed with special emphasis placed on the self-dual Yang-Mills (SDYM) system. Indeed the SDYM system plays a rather central role in integrable systems as the reader will understand. Chapter Seven details some related questions involving the properties and solution of certain nonlinear ordinary differential equations. Especially important amongst these equations are the classical equations of Painlevé and coworkers. Indeed the ideas of Painlevé lead naturally to the study of more general equations which possess the so called Painlevé property. Moreover there is a connection between nonlinear equations solvable by I.S.T. and such equations. Indeed recent work has shown that such properties can be exploited to provide useful *a priori* tests to determine when a given partial differential equation might be integrable. These ideas are also discussed in Chapter Seven. Finally we conclude with some remarks and brief discussion of some important open problems. A large and we hope reasonably complete bibliography is included. Given the rate at which this field has developed and the numerous applications it is natural that some references will be missed or forgotten. We apologise for such situations whenever they occur.

In this book we have not considered solutions to nonlinear wave equations with periodic boundary conditions, or the various methods of construction of special solutions (e.g., Hirota's direct method, pole expansions, dressing techniques etc.), or the in depth functional analysis which underlies a considerable portion of the theory. Nor have we considered the rather substantive issue of the underlying algebraic connections that serve to endow these nonlinear wave equations with considerable inherent structure. The main reason for this approach is that doing otherwise would have entailed a much larger project and hence a considerably expanded book. Since there is so much to cover and explain even with the reduced scope of this effort we felt that it would have been simply unwise and likely detrimental to the reader to have included much more than we did.

These notes review many years of work, partially supported by the mathematics divisions of the Air Force Office of Scientific Rèsearch, the National Science Foundation and the Office of Naval Research. Their support is gratefully appreciated. We sincerely acknowledge the collaboration of our many colleagues and note especially M.D. Kruskal, for innumerable valuable discussions and insights, A.S. Fokas who collaborated on many of the projects, A. Nachman who collaborated on the multi-dimensional inverse scattering, and J.B. McLeod for introducing one of us to this exciting field of research. We also thank A.P. Bassom for help in proof-reading the manuscript and A.C. Hicks and S. Hood for assistance in producing the figures. Of course we are deeply indebted to our wives Enid and Kim who endured many hours away from their husbands while this book was written.

Chapter One

Introduction.

1.1 Historical Remarks and Applications.

"Solitons" were first observed by J. Scott Russell in 1834 [1838, 1844] whilst riding on horseback beside the narrow Union canal near Edinburgh, Scotland. There are a number of discussions in the literature describing Russell's observations. Nevertheless we feel that his point of view is so insightful and relevant that we present it here as well. He described his observations as follows:

> "I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped — not so the mass of water in the channel which it had put in motion; it accumulates round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that rare and beautiful phenomenon which I have called the Wave of Translation ... ."

Subsequently, Russell did extensive experiments in a laboratory scale wave tank in order to study this phenomenon more carefully. Included amongst Russell's results are the following:

1. he observed solitary waves, which are long, shallow, water waves of permanent form, hence he deduced that they *exist*; this is his most significant result;
2. the speed of propagation, c, of a solitary wave in a channel of uniform depth h is given by $c^2 = g(h+\eta)$, where η is the amplitude of the wave and g the force due to gravity.

Further investigations were undertaken by Airy [1845], Stokes [1847], Boussinesq [1871, 1872] and Rayleigh [1876] in an attempt to understand this phenomenon. Boussinesq and Rayleigh independently obtained approximate descriptions of the solitary wave; Boussinesq derived a one-dimensional nonlinear evolution equation, which now bears his name, in order to obtain his result.

These investigations provoked much lively discussion and controversy as to whether the inviscid equations of water waves would possess such solitary wave solutions. The issue was finally resolved by Korteweg and de Vries [1895]. They derived a nonlinear evolution equation governing long one dimensional, small amplitude, surface gravity waves propagating in a shallow channel of water

$$\frac{\partial\eta}{\partial\tau} = \frac{3}{2}\sqrt{\frac{g}{h}}\,\frac{\partial}{\partial\xi}\left(\tfrac{1}{2}\eta^2 + \tfrac{2}{3}\alpha\eta + \tfrac{1}{3}\sigma\frac{\partial^2\eta}{\partial\xi^2}\right), \qquad \sigma = \tfrac{1}{3}h^3 - Th/(\rho g), \tag{1.1.1}$$

where η is the surface elevation of the wave above the equilibrium level h, α an small arbitrary constant related to the uniform motion of the liquid, g the gravitational constant, T the surface tension and ρ the density (the terms "long" and "small" are meant in comparison to the depth of the channel). The controversy was now resolved since equation (1.1.1), now known as the Korteweg-de Vries (KdV) equation, has permanent wave solutions, including *solitary wave solutions* (see §1.3 for details). Equation (1.1.1) may be brought into nondimensional form by making the transformation

$$t = \tfrac{1}{2}\sqrt{g/(h\sigma)}\,\tau, \qquad x = -\sigma^{-1/2}\xi, \qquad u = \tfrac{1}{2}\eta + \tfrac{1}{3}\alpha.$$

Hence, we obtain

$$u_t + 6uu_x + u_{xxx} = 0, \tag{1.1.2}$$

where subscripts denote partial differentiations. Henceforth, we shall consider the KdV equation in this form (1.1.2) (note that any constant coefficient may be placed in front of any of the three terms by a suitable scaling of the independent and dependent variables). (1.1.2) may be thought of as the simplest "nonclassical" partial differential equation since it has the minimum number of independent variables, two; the lowest order not considered classically, that is three; the fewest terms of that order, one; the simplest such term, an unmixed derivative; the fewest number of terms containing the other derivative, which is of first order; the simplest structure for these terms, linear; and the simplest additional term to make the equation nonlinear, quadratic. (It might be thought that a simpler nonlinear term would be u^2, however the KdV equation has an extra symmetry with uu_x (Galilean invariance) and if u is interpreted as a velocity, then the convective derivative, which arises in continuum mechanics and is familiar to physicists and engineers, is $\mathrm{d}u/\mathrm{d}t = u_t + uu_x$.)

As Miura [1976] points out, despite this early derivation of the KdV equation, it was not until 1960 that any new application of the equation was discovered. Gardner and Morikawa [1960] rediscovered the KdV equation in the study of collision-free hydromagnetic waves. Subsequently the KdV equation has arisen in a number of other physical contexts, stratified internal waves, ion-acoustic waves, plasma physics, lattice dynamics, ... (for details and further references see, for example, the articles by Jeffrey and Kakutani [1972]; Scott, Chu and McLaughlin [1973]; Miura [1976] and monographs by Ablowitz and Segur [1981]; Calogero and Degasperis [1982]; Dodd, Eilbeck, Gibbon and Morris [1982]; Lamb [1980]; Novikov, Manakov, Pitaevskii and Zakharov [1984] — for a discussion concentrating primarily on the pre-1965 history of the KdV equation, see Miles [1980b]).

It has been known for a long time that the KdV equation (1.1.2) possesses the solitary wave solution

$$u(x,t) = 2\kappa^2 \operatorname{sech}^2 \left\{\kappa(x - 4\kappa^2 t - x_0)\right\}, \tag{1.1.3}$$

where κ and x_0 are constants (in fact this solution was known to Korteweg and de Vries). Note that the velocity of this wave, $4\kappa^2$, is proportional to the amplitude, $2\kappa^2$; therefore taller waves travel faster than shorter ones. Zabusky and Kruskal [1965] discovered that these solitary wave solutions have the remarkable property that the interaction of two solitary wave solutions is elastic, and are called *solitons* (see §1.4 details).

The solitons observed by Russell were small amplitude surface waves. There have been several investigations examining the validity of the KdV equation (1.1.2) as a model of the evolution of small amplitude water waves as they propagate in one direction in shallow water. These studies have compared the solutions of (1.1.2) with experimental results (see, for example, Hammack and Segur [1974, 1978]). In physical terms the KdV equation arises if the water waves are strictly one-dimensional, that is one spatial dimension and time (for a derivation of the KdV equation see, for example, Chapter 4 of Ablowitz and Segur [1981]).

In many physical situations, internal waves can arise at the interface of two layers of fluid due the gravitational effects in a stably stratified fluid. Several theoretical models exist which govern the evolution of long internal waves with small amplitudes in a stably stratified fluid including the KdV equation (1.1.2), the intermediate long wave (ILW) equation

$$u_t + \delta^{-1} u_x + 2uu_x + \mathrm{T}u_{xx} = 0, \tag{1.1.4a}$$

where $\mathrm{T}u$ is the singular integral operator

$$(\mathrm{T}f)(x) = \frac{1}{2\delta} ⨍_{-\infty}^{\infty} \coth\left\{\frac{\pi}{2\delta}(y-x)\right\} f(y)\,\mathrm{d}y, \tag{1.1.4b}$$

with $⨍_{-\infty}^{\infty}$ the Cauchy principal value integral (Joseph [1977]; Kubota, Ko and Dobbs [1978]), and the Benjamin-Ono (BO) equation

$$u_t + 2uu_x + \mathrm{H}u_{xx} = 0, \tag{1.1.5a}$$

where $\mathrm{H}u$ is the Hilbert transform

$$(\mathrm{H}f)(x) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{f(y)}{y-x}\,\mathrm{d}y \tag{1.1.5b}$$

(Benjamin [1967]; Davies and Acrivos [1967]; Ono [1975]). In the shallow water limit, as $\delta \to 0$, (1.1.4) reduces to the KdV equation

$$u_t + 2uu_x + \tfrac{1}{3}\delta u_{xxx} = 0, \tag{1.1.6}$$

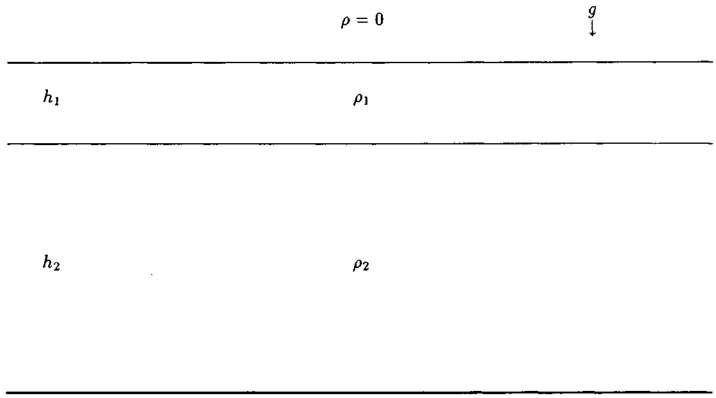


FIGURE 1.1.1 *The two-layer configuration.*

and in the deep-water limit as $\delta \to \infty$, to the BO equation. Therefore the ILW equation (1.1.4) may be thought of as being an equation intermediate between (1.1.5) and (1.1.6).

Consider two incompressible, immiscible fluids, with densities and depths h_1, h_2 ($h := h_1 + h_2$) with the lighter fluid, of height h_1, lying over a heavier fluid of height h_2, in a constant gravitational field (Figure 1.1.1). The lower fluid rests on a horizontal impermeable bed, and the upper fluid is bounded by a free surface.

Suppose the the characteristic wave amplitude is denoted by a and the characteristic wavelength by $\lambda = k^{-1}$. The basic assumptions for the derivation of the KdV equation (1.1.6), the ILW equation (1.1.4) and the BO equation (1.1.5), as models for internal waves are (see Chapter 4 of Ablowitz and Segur [1981] for details).

Korteweg de-Vries equation:

(A1) the waves are long waves in comparison with the total depth, $kh \ll 1$;
(A2) the amplitude of the waves is small, $\varepsilon = a/h \ll 1$;
(A3) the two effects (A1) and (A2) approximately balance, i.e., $kh = O(\varepsilon)$;
(A4) viscous effects may be neglected.

Intermediate-Long-Wave equation:

(B1) there is a thin (upper) layer, $\varepsilon = h_1/h_2 \ll 1$;
(B2) the amplitude of the waves is small, $a \ll h_1$;
(B3) the two effects (B1) and (B2) balance, i.e., $a/h_1 = O(\varepsilon)$;
(B4) the characteristic wavelength is comparable to the total depth of the fluid, $kh = O(1)$;
(B5) the waves are long waves in comparison with the thin layer, $kh_1 \ll 1$ [this is implied by (B1) and (B4)];
(B6) viscous effects may be neglected.

Note that in the ILW equation (1.1.4), the parameter δ is effectively kh. For derivations of the ILW equation (1.1.4) in which the fluid is confined between two rigid walls see Kubota, Ko and Dobbs [1978]; Segur and Hammack [1982] (these two derivations are slightly different).

Benjamin-Ono equation:

(C1) there is a thin (upper) layer, $h_1 \ll h_2$;

(C2) the waves are long waves in comparison with the thin layer, $kh_1 \ll 1$;

(C3) the waves are short in comparison with the total depth of the fluid, $kh \gg 1$;

(C4) the amplitude of the waves is small, $a \ll h_1$;

(C5) viscous effects may be neglected.

Under these assumptions one obtains the KdV, ILW and BO equations, after a suitable scaling. Segur and Hammack [1982] have examined the validity of the KdV and ILW equations as models for internal waves by comparing the theoretical solutions of the KdV and ILW equations with experimental results.

Recently there has been considerable interest in the observation of what scientists think might be solitons in the oceans. The availability of photographs taken from satellites and spacecraft orbiting the earth have greatly assisted in these observations. Peculiar striations, visible on satellite photographs of the surface of the Andaman and Sulu seas in the Far East, have been interpreted as secondary phenomena accompanying the passage of "internal solitons", solitary wavelike distortions of the boundary layer between the warm upper layer of sea water and cold lower depths. These internal solitons are travelling ridges of warm water, extending hundreds of meters down below the thermal boundary, and enormous energies which they carry are presumed to be the cause of unusually strong underwater currents experienced by deep-sea drilling rigs. In order to continue deep-sea drilling for oil in areas where these internal solitons occur, the drilling rigs will have to be built to withstand these forces.

Osborne and Burch [1980] (see also Osborne [1990]) investigated the underwater currents which were experienced by an oil rig in the Andaman sea, which was drilling at a depth of 3600ft (one drilling rig was apparently spun through ninety degrees and moved one hundred feet by the passage of a soliton below). Satellite photographs had shown that there were 100km long striations on the Andaman sea, separated by 6 to 15km and grouped in packets of typically 4 to 8. The average time between the arrival of these packets at the research vessel was 12 hours 26 minutes, suggesting that this was some kind of tidal phenomenon. Osborne and Burch spent four days measuring underwater currents and temperatures. The striations seen on satellite photographs turned out to be kilometer-wide bands of extremely choppy water, stretching from horizon to horizon, followed by about two kilometers of water "as smooth as a millpond". These bands of agitated water are called "tide rips", they arose in packets of 4 to 8, spaced about 5 to 10km apart (they reached the research vessel at approximately hourly intervals) and this pattern was repeated with the regularity of tidal phenomenon.

Osborne and Burch [1980] found that the amplitude of each succeeding soliton was less than the previous one, which, as we shall see in §1.7 below, is precisely what is expected for solitons (recall that the velocity of a solitary wave solution (1.1.3) of the KdV equation increases with amplitude). Thus if a number of solitons are generated together, then we expect them eventually to be arranged in an ordered sequence of decreasing amplitude. From the spacing between successive waves in a packet and the rate of separation calculated from the KdV equation, Osborne and Burch were able to estimate the distance the packet had travelled from its source and thus identify possible source regions. They concluded that the solitons are generated by tidal currents off northern Sumatra or between the islands of the Nicobar chain that extends beyond it.

Underwater measurements showed a rapid circulation of water associated with the solitary waves. The interaction of this internal circulation with ordinary surface waves produces the tide rip and appear to explain the origin of the forces experienced by the oil drilling rigs. Osborne and Burch [1980] conclude that, despite the irregular geometry of the Andaman sea, their observations have good general agreement with the predictions for internal solitons as given by the KdV equation.

Apel and Holbrook [1980] (see also Holbrook, Apel and Tsai [1980]) undertook a detailed study of internal waves in the Sulu sea. Satellite photographs had suggested that the source of these waves was near the southern end of the Sulu sea and their research ship followed one wave packet for more than 250 miles over a period of two days — an extraordinary coherent phenomenon. These internal solitons travel at speeds of about 8 kilometers per hour (5 miles per hour), with amplitude of about 100 meters and wavelength of about 1700 meters.

Further observations and studies of solitons in oceans include the Strait of Messina (Alpers and Salustri [1983]; Santoleri [1983]); the Strait of Gibraltar (Lacombe and Richez [1982]); off the western side of Baja California (Apel and Gonzalez [1983]); the Gulf of California (Fu and Holt [1984]); the Archipelago of La Maddalena (Manzella, Bohm and Salustri [1983]) and the Georgia Strait (Hughes and Gower [1983]).

Rossby waves (Rossby [1939]) are long waves between layers of the atmosphere, created by the rotation of the planet. As one might suspect, there is an analogy between internal waves and Rossby waves under suitable conditions. The KdV equation has been proposed as a model for the evolution of Rossby waves (Long [1964]; Benney [1966]; Redekopp [1977]; Redekopp and Weidman [1978] — see also Miles [1980b] and the references cited therein). There has been been a conjecture (Maxworthy and Redekopp [1976]) that Jupiter's Great Red Spot might be a solitary Rossby wave, though recent work seems to indicate that the red spot is a result of strong differential rotation (cf. Meyers, Sommeria and Swinney [1989]).

We have noted that the KdV equation (1.1.2) can describe the evolution of small amplitude water waves as they propagate in shallow water; the evolution described being weakly nonlinear and weakly dispersive, where these effects are of the same order. These solitary waves are usually regarded as propagating in a uniform rectangular channel. There have been several studies on the problem of quasi one-dimensional solitary waves in channels of varying width, depth or shape of the channel (for a review see Miles [1980b] and the references cited therein).

A further restriction in the application of the KdV equation as a practical model for water waves, is that the KdV equation is strictly only one-dimensional (that is one spatial-dimension plus time), whereas the surface is two-dimensional. A two-dimensional generalization of the KdV equation is the Kadomtsev-Petviashvili (KP) equation

$$(u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0, \tag{1.1.7}$$

where $\sigma^2 = \pm 1$ (Kadomtsev and Petviashvili [1970]). Whereas the KdV equation (1.1.2) describes the evolution of long water waves of small amplitude if they are strictly one-dimensional, the KP equation (1.1.7) describes their evolution if they are weakly two-dimensional (in §1.2 below, we give a derivation of the KP equation as a model for surface waves). The evolution described by the KP equation is weakly nonlinear, weakly dispersive and weakly two-dimensional with all three effects being of the same order; the choice of sign depends on the relevant magnitude of gravity and surface tension (cf. §1.2). The KP equation has also been proposed as a model for surface waves and internal waves in straits or channels of varying depth and width (see, for example, Santini [1981]; David, Levi and Winternitz [1987a, 1989]).

The KdV equation (1.1.2), the ILW equation (1.1.4), the BO equation (1.1.5) and the KP equation (1.1.7) have been extensively studied in recent years, primarily by mathematicians and physicists. We have already seen that they arise in the description of physically interesting phenomena, however much of the interest in these nonlinear evolution equations is due to the fact that they are thought of as being *completely integrable*, or *exactly solvable* equations. This terminology is a consequence of the fact that the initial value problem for each of these nonlinear equations can be solved exactly by a method employing *inverse scattering*, which we shall study in depth in these notes. The inverse scattering method was discovered by Gardner, Greene, Kruskal and Miura [1967] as a method of solving the initial value problem for the KdV equation (on the infinite line), for initial values that decay sufficiently rapidly at infinity. Subsequently numerous other physically interesting equations in one spatial dimension were solved by generalizations of this technique, which is now referred to as the *Inverse Scattering Transform* (I.S.T.), for example the nonlinear Schrödinger, Sine-Gordon, three-wave interaction, Modified KdV and Boussinesq equations. The I.S.T. scheme for the ILW (1.1.4) and the BO equation (1.1.5), which are integro-differential equations, was derived by Kodama, Ablowitz and Satsuma [1982] and

Fokas and Ablowitz [1983b], respectively. For the KP equation (1.1.7), the I.S.T. scheme is dependent upon the choice of sign for σ^2. The scheme for the KP equation with $\sigma^2 = -1$ (which is known as KPI) was derived by Manakov [1981] and Fokas and Ablowitz [1983d]; and the scheme in the case $\sigma^2 = 1$ (which is known as KPII), by Ablowitz, BarYaacov and Fokas [1983]. (As we show in Chapter 5, the I.S.T. schemes for KPI and KPII are fundamentally different.)

The organization of these notes is as follows. In Chapter 1 we give an introduction to several of the most important and significant aspects in the development of the I.S.T. schemes for nonlinear evolution equations. In Chapters 2 and 3 we discuss I.S.T. schemes for one-dimensional equations; discussing the KdV equation (1.1.2) in Chapter 2 and more general I.S.T. schemes in one dimension in Chapter 3 (including the I.S.T. schemes for differential-difference and partial-difference equations). In Chapter 4 we discuss the I.S.T. scheme for integro-differential equations using the ILW equation (1.1.4) and the BO equation (1.1.5) as the prototype examples. In Chapters 5 and 6 we discuss multidimensional I.S.T. schemes; in Chapter 5 discussing inverse scattering in two spatial dimensions respectively, using the KP equation (1.1.7) as prototype example and in Chapter 6 discussing inverse scattering in higher dimensions. In Chapter 7 we discuss several topics concerning integrability associated with the Painlevé equations, which are six second order, nonlinear ordinary differential equations; in particular, the so-called Painlevé tests and inverse scattering schemes for the Painlevé equations using the second Painlevé equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 2y^3 + xy + \alpha, \tag{1.1.8}$$

where α is a constant, as the prototype example. Finally, in Chapter 8, we make some concluding remarks and mention some difficult open problems.

1.2 Physical Derivation of the KP Equation.

In this section we present a physical derivation of the KP equation (1.1.7) as a model for surface waves.

The classical problem of water waves is to find the irrotational motion of an inviscid, incompressible, homogeneous fluid, subject to a constant gravitational force g. The fluid rests on a horizontal and impermeable bed of infinite extent at $z = -h$ and has a free surface at $z = \eta(x, y, t)$. Since the fluid is irrotational and incompressible, then it has a velocity potential ϕ satisfying

$$\nabla^2\phi = 0, \qquad -h < z < \eta(x, y, t). \tag{1.2.1a}$$

It is subject to the boundary conditions on the bottom $z = -h$

$$\phi_z = 0 \tag{1.2.1b}$$

(since the bed is impermeable), and on the free surface $z = \eta$

$$\frac{\mathrm{D}\eta}{\mathrm{D}t} \equiv \eta_t + \phi_x\eta_x + \phi_y\eta_y = \phi_z \tag{1.2.1c}$$

(kinematic condition) and

$$\phi_t + g\eta + \tfrac{1}{2}|\nabla\phi|^2 = \frac{T}{\rho}\,\frac{\eta_{xx}(1+\eta_y^2) + \eta_{yy}(1+\eta_x^2) - 2\eta_{xy}\eta_x\eta_y}{(1+\eta_x^2+\eta_y^2)^{3/2}} \tag{1.2.1d}$$

(dynamic condition), where T is the surface tension and ρ is the density of the fluid. Boundary conditions in (x, y) and initial conditions are also required. For isolated waves, then $\nabla\phi$ and η should vanish as $(x^2 + y^2) \to \infty$. (In other problems, periodic boundary conditions in x and y may be appropriate.)

This problem, first posed by Stokes [1847], is nonlocal, highly nonlinear and not surprisingly remains unsolved in its general form. To make further progress, we need to impose additional assumptions on the solutions to equations (1.2.1). The first such assumption is that the wave amplitudes should be small. If we interpret small to mean infinitesimal, then we may linearize equation (1.2.1) about $\nabla\phi = 0$, $\eta = 0$, and seek solutions of the linearized equations proportional to $\exp\{\mathrm{i}(kx + my - \omega t)\}$ (see Lamb [1932,§§228,266,267]). The result is the linearized dispersion relation

$$\omega^2 = \left(g\kappa + \kappa^3 T/\rho\right)\tanh(\kappa h), \tag{1.2.2}$$

where $\kappa^2 := k^2 + m^2$. From this one computes the group velocity and shows that the linearized problem is dispersive at most wave numbers, but not as $\kappa \to 0$ (i.e., long waves, or shallow water waves), where it is only weakly dispersive. The KdV and KP equations arise as models of the water wave problem in this weakly dispersive limit $\kappa h \ll 1$.

We orient the horizontal coordinates so that the x-direction is the principal direction of wave propagation. To derive the KP equation (1.1.7) we assume that:

1. wave amplitudes are small, $\varepsilon = |\eta|_{\max}/h \ll 1$;
2. the water is shallow relative to typical horizontal wavelengths, $(\kappa h)^2 \ll 1$;
3. the waves are nearly one-dimensional, $(m/k)^2 \ll 1$;
4. these three effects all have comparable influence (i.e., they balance), $(m/k)^2 = O((\kappa h)^2) = O(\varepsilon)$.

These assumptions imply a certain scaling of the original equation (see below); an evolution equation which follows may then be found using a multiple scales method.

At leading order, the equations are linear (from 1), nondispersive (from 2) and one-dimensional (from 3), the result being the linear wave equation

$$\eta_{tt} - gh\eta_{xx} = 0. \tag{1.2.3}$$

At this order every wave has permanent form, not because it is soliton but because we are solving the one-dimensional, linear wave equation.

When the perturbation expansion is continued to second order, the one-dimensional, linear wave equation has homogeneous (forcing) terms representing weak nonlinearity, weak dispersion and weak two-dimensionality. Each of these effects contributes to a secular term at second order. Define

$$r = \varepsilon^{1/2}\,\frac{x-\sqrt{gh}\,t}{h}, \quad s = \varepsilon^{1/2}\,\frac{x+\sqrt{gh}\,t}{h}, \quad \zeta = \varepsilon\,\frac{y}{h}, \quad \tau = \varepsilon^{3/2}\,\frac{\sqrt{gh}\,t}{6h}, \tag{1.2.4a}$$

and seek solutions of the form

$$\eta(x,y,t;\varepsilon) = \tfrac{2}{3}\varepsilon h\,[u(r,\zeta,\tau)+v(s,\zeta,\tau)] + O(\varepsilon^2 h). \tag{1.2.4b}$$

Since we are interested in problems where the initial disturbances are localized, then it is convenient to assume *a fortiori* that the physical quantities have compact support initially. In this case it is easily shown that u and v in (1.2.4b) have compact support as well. No secular terms arise at second order provided that $U(r,\zeta,\tau)$ and $U(s,\zeta,\tau)$ satisfy

$$[U_\tau + 6UU_r + (1-\hat{T})U_{rrr}]_r + 3U_{\zeta\zeta} = 0, \tag{1.2.5a}$$

$$[V_\tau - 6VV_s - (1-\hat{T})V_{sss}]_s - 3V_{\zeta\zeta} = 0, \tag{1.2.5b}$$

where $\hat{T} := T/(3\rho g h^2)$ is the dimensionless surface tension. Thus the left and right running waves each evolve according to their own KP equation, which describe how the two sets of waves each interact with themselves over a long time scale

$$\tau = \tfrac{1}{6}\varepsilon^{3/2}\,\sqrt{g/h}\,t = O(1).$$

To make the model quite explicit, we also write the KP equation for the right-going waves in its dimensional form for $\eta(x,y,t)$:

$$\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{gh}}\,\eta_t + \eta_x + \frac{3}{2h}\eta\eta_x + \left(\frac{3h^2\rho g - T}{18\rho g}\right)\eta_{xxx}\right] + \tfrac{1}{2}\eta_{yy} \sim 0. \tag{1.2.6}$$

For most cases of interest in water waves $1-\hat{T} > 0$ (in fact usually $\hat{T} \ll 1$ and so $\hat{T}$ may be neglected), which corresponds to $\sigma^2 = 1$ in equation (1.1.7) (i.e., KPII). Thus from equation (1.2.2), the linearized phase speed is a (local) maximum at $\kappa = 0$. In this case, equation (1.2.6) is equivalent to equation (1.1.7) with $\sigma = +1$, i.e.

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0, \tag{1.2.7}$$

which is usually called KPII.

Obviously every solution of the KdV equation (1.1.2) is also a solution of the KPII equation (1.2.7). More generally, Satsuma [1976] showed that the KPII equation (1.2.7) has N line-soliton solutions, with the N line-solitons travelling in N different directions and interacting obliquely. In particular, a one line-soliton solution is

$$u(x,y,t) = 2\kappa^2 \operatorname{sech}^2 \left\{\kappa[x + \lambda y - (4\kappa^2 + 3\lambda^2)t + \delta_0]\right\}. \tag{1.2.8}$$

Essentially these are one-dimensional solutions travelling at an angle to the y-axis. A two line-soliton (written in Hirota's notation — see §2.6.5 below) is

$$u(x,y,t) = 2\frac{\partial^2}{\partial x^2} \ln F(x,y,t), \tag{1.2.9a}$$

with

$$F(x,y,t) = 1 + \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_1 + \eta_2 + A_{12}), \tag{1.2.9b}$$

where

$$\eta_i = 2\kappa_i[x + \lambda_i y - (4\kappa_i^2 + 3\lambda_i^2)t] + \delta_i, \tag{1.2.9c}$$

$$\exp(A_{12}) = \frac{4(\kappa_1 - \kappa_2)^2 - (\lambda_1 - \lambda_2)^2}{4(\kappa_1 + \kappa_2)^2 - (\lambda_1 - \lambda_2)^2}. \tag{1.2.9d}$$

One and two line-soliton solutions of the KPII equation are illustrated in Figures 1.2.1 and 1.2.2, respectively. Note that in Figure 1.2.2, away from the interaction region, each wave is essentially a KdV-type soliton (as expected, there is a phase shift due to the interaction). We remark that solutions such as (1.2.8) and (1.2.9) do not decay as $(x^2 + y^2)^{1/2} \to \infty$ in all directions.

For very thin sheets of water $\hat{T} > 1$ (when surface tension dominates gravity), then equation (1.2.6) is equivalent to equation (1.1.7) with $\sigma^2 = -1$, i.e.

$$(u_t + 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{1.2.10}$$

which is usually called KPI. Here one-dimensional solitons are unstable (Kadomtsev and Petviashvili [1970]). (We remark that the KP equation with $\hat{T} > 1$ also applies if we neglect gravity, remove the horizontal bed (and the viscous boundary layer it creates) on which the thin sheet of water lies and restrict out attention to the symmetric modes so that $\phi_z = 0$ holds.) In this case, there exist "lump" solutions of the KPI equation,

$$u(x,y,t) = 4\,\frac{\left\{-\left[x + \lambda y + 3(\lambda^2 - \mu^2)t\right]^2 + \mu^2(y + 6\lambda t)^2 + 1/\mu^2\right\}}{\left\{\left[x + \lambda y + 3(\lambda^2 - \mu^2)t\right]^2 + \mu^2(y + 6\lambda t)^2 + 1/\mu^2\right\}^2}, \tag{1.2.11}$$

which decay algebraically as $(x^2 + y^2)^{1/2} \to \infty$ (cf. Manakov, Zakharov, Bordag, Its and Matveev [1977]; Satsuma and Ablowitz [1979]).

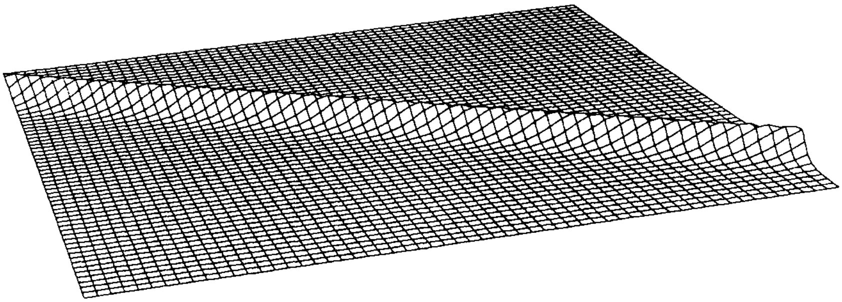


FIGURE 1.2.1 *One line-soliton solution of the KPII equation* (1.2.7).

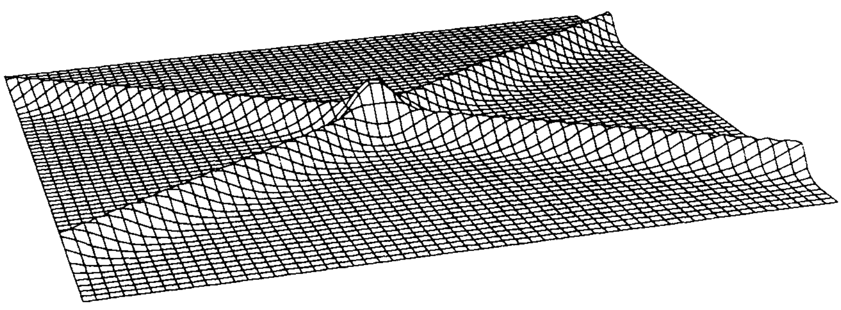


FIGURE 1.2.2 *Two line-soliton solution of the KPII equation* (1.2.7).

1.3 Travelling Wave Solutions of the KdV Equation.

The first interesting property of the KdV equation is the existence of permanent wave solutions, including solitary wave solutions.

DEFINITION 1.3.1

A solitary wave solution of a partial differential equation

$$\Delta(x,t,u) = 0,$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$ are temporal and spatial variables and $u \in \mathbb{R}$ the dependent variable, is a travelling wave solution of the form

$$u(x,t) = w(x - \gamma t) = w(z), \tag{1.3.1}$$

whose transition is from one constant asymptotic state as $z \to -\infty$ to (possibly) another constant asymptotic state as $z \to \infty$. (Note that some definitions of solitary waves require the constant asymptotic states to be equal — often to zero.)

To obtain travelling wave solutions of the KdV equation, we seek a solution in the form (1.3.1) which yields a third order ordinary differential equation for w

$$w''' + 6ww' - \gamma w' = 0, \tag{1.3.2}$$

where $' \equiv \mathrm{d}/\mathrm{d}z$. Integrating this twice gives

$$\tfrac{1}{2}(w')^2 = f(w) := -\tfrac{1}{2}(2w^3 - \gamma w^2 - Aw - B), \tag{1.3.3}$$

with A, B constants. Since we are interested in obtaining real, bounded solutions for the KdV equation, then we require that $f(w) \geq 0$ and so we study the zeros of $f(w)$. There are two cases to consider: (1) when $f(w)$ has only one real zero and (2) when $f(w)$ has three real zeros (for which there are three subcases).

Case 1. If $f(w)$ has only one real zero, a, then it is one of the two forms shown in Figure 1.3.1. If $w'(0) < 0$, then $f(w) > 0$ for all $z > 0$, and so w decreases monotonically to $-\infty$ as $z \to \infty$. If $w'(0) > 0$, then w increases until it reaches a, at say z_1 (that is $w(z_1) = a$), which is a simple maximum of w. Therefore $w'(z) < 0$, for $z > z_1$ and thereafter w decreases monotonically to $-\infty$ as $z \to \infty$. Hence there are no bounded solutions in this case.

Case 2. If $f(w)$ has three real zeros, a, b and c, then we may assume that $a \leq b \leq c$, and so we may write

$$f(w) = -(w-a)(w-b)(w-c), \tag{1.3.4}$$

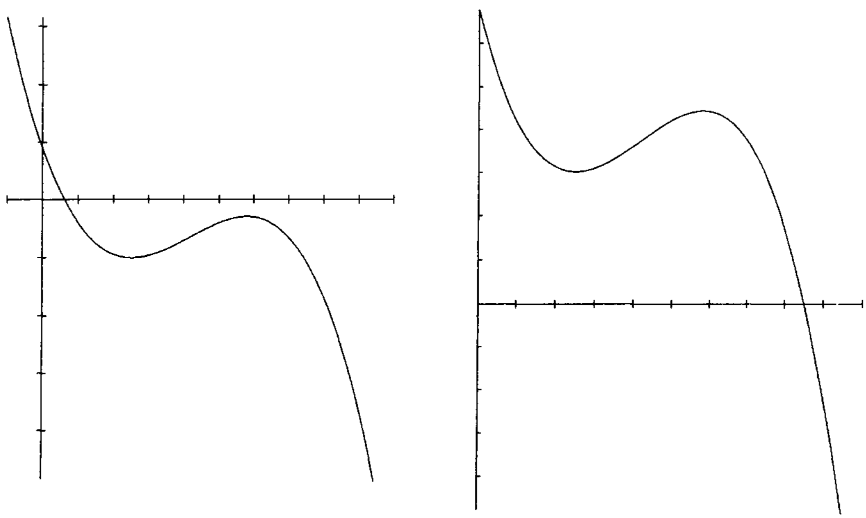


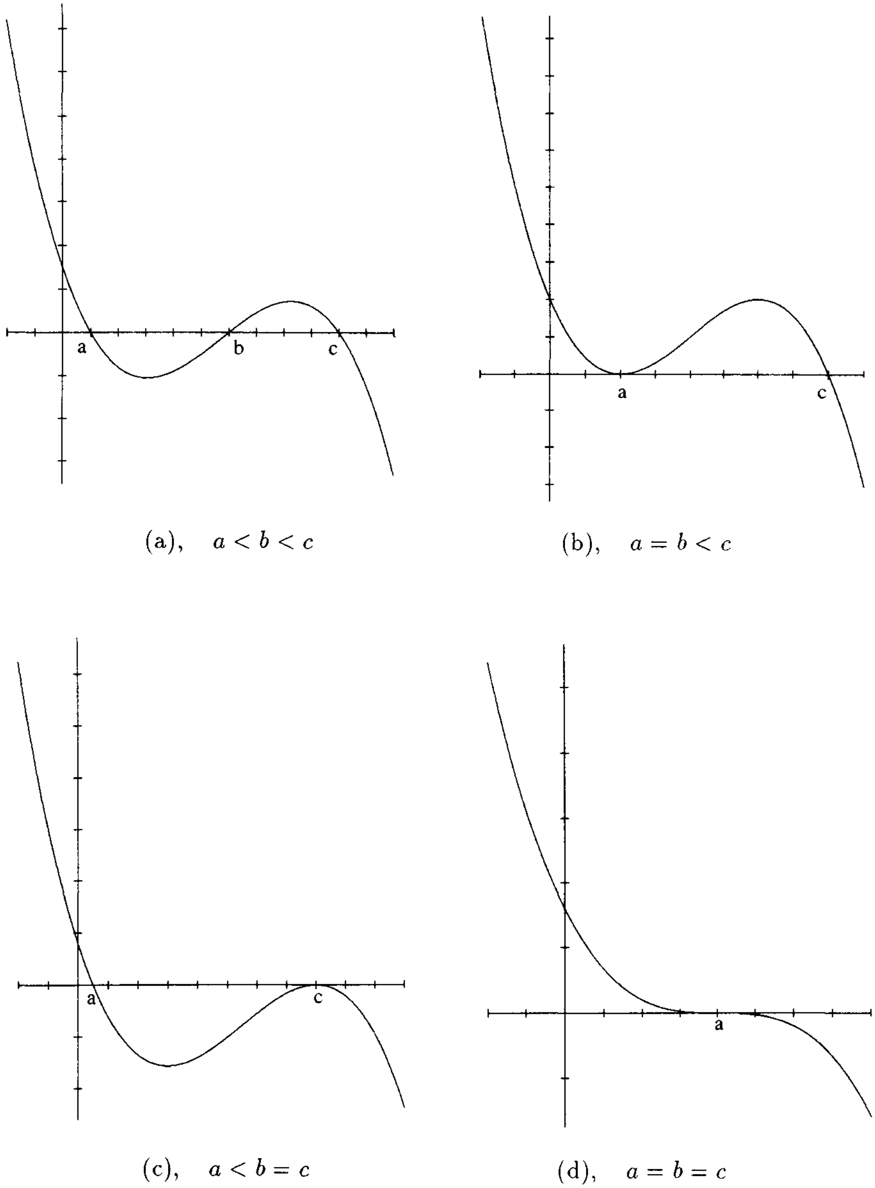
FIGURE 1.3.1 *Rough sketches of $f(w)$ when it has one real zero.*

where $\gamma = 2(a+b+c)$, $A = -2(ab+bc+ca)$, $B = 2abc$. In order that a, b and c are real it is necessary that

$$\Delta^2 := \gamma^2 + 6A \geq 0,$$
$$f\left(\tfrac{1}{3}(\gamma+\Delta)\right) \geq 0, \qquad \text{and} \qquad f\left(\tfrac{1}{3}(\gamma-\Delta)\right) \leq 0.$$

The possible behaviors of the function $f(w)$ against w when it has three real zeros is shown in Figure 1.3.2.

The general behavior of the function $f(w)$ against w is shown in Figure 1.3.2a. In this case all the roots are distinct and the real solution of (1.3.3) represents a nonlinear oscillation between b and c. There are two special cases; one when $b = a$ (Figure 1.3.2b), which corresponds to the solitary wave solution, and the other is when $b = c$ (Figure 1.3.2c), which gives rise to a constant solution related to linear sinusoidal waves. If all three roots coalesce (Figure 1.3.2d), then again we obtain a constant solution as the only bounded solution (see below). We shall now consider the four cases corresponding to Figures 1.3.2a, 1.3.2b, 1.3.2c and 1.3.2d.



(a), $a < b < c$

(b), $a = b < c$

(c), $a < b = c$

(d), $a = b = c$

FIGURE 1.3.2 *Rough sketches of $f(w)$ when it has three real zeros.*

Case 2a. If a, b, c are distinct, then the solution of the KdV equation is of the form of ***cnoidal waves***, which may be expressed in terms of the Jacobian elliptic function $\mathrm{cn}(z;m)$

$$u(x,t) = b + (c-b)\,\mathrm{cn}^2\left\{\left[\tfrac{1}{2}(c-a)\right]^{1/2}[x-(a+b+c)t+x_0]\,;m\right\}, \tag{1.3.5a}$$

with

$$m = \frac{c-b}{c-a}, \qquad \text{that is} \qquad 0 < m < 1, \tag{1.3.5b}$$

where x_0 is a constant and m denotes the modulus of the Jacobian elliptic function (cf. Whittaker and Watson [1927]). These cnoidal wave solutions of the KdV equation were originally found by Korteweg and de Vries themselves and appear their celebrated paper [1895] (in fact they coined the term cnoidal wave).

Case 2b. $a = b \neq c$. If in (1.3.5) we take the limit $b \to a$ (i.e. $m \to 1$), then, provided that $b \neq c$, it reduces to the solitary wave solution

$$u(x,t) = a + (c-a)\,\mathrm{sech}^2\left\{\left[\tfrac{1}{2}(c-a)\right]^{1/2}[x-(2a+c)t+x_0]\right\}, \tag{1.3.6}$$

since $\mathrm{cn}(z;1) = \mathrm{sech}(z)$ (Abramowitz and Stegun [1972, p.571]). If in (1.3.6) we now set $u_0 = a$ and $\kappa^2 = \frac{1}{2}(c-a)$ then we obtain the solitary wave solution

$$u(x,t) = u_0 + 2\kappa^2\,\mathrm{sech}^2\left\{\kappa[x-(4\kappa^2+3u_0)t+x_0]\right\}. \tag{1.3.7}$$

From this we can easily make the following observations.

1. The speed of the wave relative to the uniform state u_0 is proportional (in this case by a factor of two) to the amplitude (that is, if $u_0 = 0$, then the wave speed is twice the amplitude).
2. The width of the solitary wave $2\pi/\kappa$ is inversely proportional to the square root of the wave amplitude and therefore taller solitary waves are narrower in width and travel faster than shorter ones.
3. The amplitude is independent of the uniform state u_0, the limit of $u(x,t)$ as $|t| \to \infty$.

[We could have derived the solitary wave solution (1.3.7) directly from (1.3.3) by imposing the boundary conditions $w \to u_0$, $w' \to 0$, $w'' \to 0$, as $|z| \to \infty$; note that these compare favorably with the observations of Russell mentioned above.]

Case 2c. If $a \neq b = c$, then the general solution of (1.3.4) is

$$w(z) = b - (b-a)\sec^2\left\{\left[\tfrac{1}{2}(b-a)\right]^{1/2}(z+z_0)\right\},$$

where z_0 is a constant. Therefore the only bounded real solution of the KdV equation is the constant solution $u(x,t) = b$, which may be obtained by taking the limit $c \to b$, that is $m \to 0$, in (1.3.5). Suppose m is very small, and define $\kappa^2 := \frac{1}{2}(c-a)$, then (1.3.5) becomes

$$u(x,t) = b + 2\kappa^2 m \,\mathrm{cn}^2\left\{\kappa[x-(a+b+c)t+x_0];m\right\}.$$

Since $\mathrm{cn}(z;m) = \cos(z) + O(m)$ for small m (Abramowitz and Stegun [1972, p.573]), it follows that m sufficiently small

$$u(x,t) \sim \tfrac{1}{2}(b+c) + \kappa^2 m \cos\left\{2\kappa[x-(a+b+c)t+x_0]\right\},$$

so that the case $c \to b$ may be thought of as the limiting case of a sinusoidal wave with finite period.

Case 2d. If $a=b=c$, then the general solution of (1.3.4) is

$$w(z) = a - \tfrac{1}{2}(z+z_0)^{-2},$$

where z_0 is a constant. Therefore, as for Case 2c, the only bounded real solution of the KdV equation is the constant solution $u(x,t)=a$ [which may have been obtained by taking the limit $c \to a$ in (1.3.6)].

For further details on travelling wave solutions of the KdV equation, see Drazin and Johnson [1989].

1.4 The Discovery of the Soliton.

The physical model which motivated the recent discoveries associated with the KdV equation was the Fermi-Pasta-Ulum (FPU) problem of a one-dimensional anharmonic lattice of equal masses coupled by nonlinear strings which was studied numerically by Fermi, Pasta and Ulum [1955]. They felt that any smooth initial state would eventually relax into equilibrium due to the nonlinear couplings.

The model Fermi, Pasta and Ulum [1955] considered consisted of identical masses connected by nonlinear springs with the force law $F(\Delta) = -K(\Delta + \alpha\Delta^2)$. The equations of motion governing this lattice are

$$m\frac{\partial^2 y_n}{\partial t^2} = K(y_{n+1} - 2y_n + y_{n-1})[1 + \alpha(y_{n+1} - y_{n-1})], \tag{1.4.1}$$

for $n = 1,2,\ldots,N-1$, with $y_0 = 0 = y_N$ and initial condition $y_n(0) = \sin(n\pi/N)$, $y_{n,t}(0) = 0$ (typically N was taken to be 32). Here y_n measures the displacement of the nth mass from equilibrium.

A great surprise was encountered. Rather than equilibrating, the energy eventually recurred. That is, after flowing back and forth amongst the low order modes, the energy recollected into the lowest mode to within an accuracy of one or two percent, and from there on the process approximately repeated.

In order to understand this phenomenon, Zabusky and Kruskal [1965] considered a continuum model. Calling the length between springs h, $t' = \omega t$ ($\omega = \sqrt{K/m}$), $x' = x/h$ with $x = nh$ and expanding $y_{n\pm 1}$ in a Taylor series, equation (1.4.1) reduces to (dropping the primes)

$$y_{tt} = y_{xx} + \varepsilon y_x y_{xx} + \tfrac{1}{12}h^2 y_{xxxx} + O(\varepsilon h^2, h^4), \tag{1.4.2}$$

where $\varepsilon = 2\alpha h$. A further reduction is possible by looking for an asymptotic solution of the form (unidirectional waves)

$$y \sim \phi(\xi, \tau), \qquad \xi = x - t, \qquad \tau = \tfrac{1}{2}\varepsilon t,$$

whereupon (1.4.2) gives

$$\phi_{\xi\tau} + \phi_\xi \phi_{\xi\xi} + \delta^2 \phi_{\xi\xi\xi\xi} + O(h^2, h^4\varepsilon^{-1}) = 0, \tag{1.4.3}$$

where $\delta^2 = \frac{1}{12}h^2\varepsilon^{-1}$ and the $O(h^2, h^4\varepsilon^{-1})$ terms are small. By setting $u = \phi_\xi$ and ignoring small terms, (1.4.3) reduces to the KdV equation

$$u_\tau + u u_\xi + \delta^2 u_{\xi\xi\xi} = 0 \tag{1.4.4}$$

(for further details of this derivation see, for example, Kruskal [1975]).

We saw in §1.3 that the KdV equation possesses a solitary solution

$$u(x,t) = 2\kappa^2 \operatorname{sech}^2 \left\{\kappa(x - 4\kappa^2 t - x_0)\right\}, \tag{1.4.5}$$

where κ and x_0 are constants [set $u_0 = 0$ in (1.3.6)]. This special solution of the KdV equation had been well known for a long time, though it was not until Zabusky and Kruskal [1965] did extensive numerical studies for the KdV equation (1.4.4) that the remarkable properties of these solitary waves were discovered. Zabusky and Kruskal considered the initial-value problem for the KdV equation (1.4.4) with the initial condition

$$u(\xi, 0) = \cos(\pi\xi), \qquad 0 \le \xi \le 2, \tag{1.4.6}$$

and u, u_ξ, $u_{\xi\xi}$ periodic on $[0,2]$ for all t; they took $\delta = 0.022$ (the periodic problem is more complicated than the one on the infinite line, but is better suited to numerical computations). Zabusky and Kruskal discovered that after a short time the wave steepens and almost produces a shock, but the dispersive term ($\delta^2 u_{xxx}$) then becomes significant and a balance between the nonlinear and dispersion terms ensues. Later the solutions develops a train of eight well-defined waves, each like sech^2 functions, with the faster (taller) waves continually catching up and overtaking the slower (smaller) waves. When two solitary waves of the form (1.4.5) (with different wave speeds), are initially well separated, with the larger one to the left (the waves are travelling from left to right), then the faster, taller wave overlaps the slower, smaller one, and the waves interact nonlinearly. After the interaction, the waves separate, with the larger one on the right, having regained their initial amplitude and velocity and the only effect of the interaction are phase shifts, that is the centers of the waves are at different positions than where they would have been had there been no interaction. Finally, after a long time the initial profile recurs.

At the center of these observations is the discovery that these nonlinear waves can interact elastically and continue afterwards almost as if there had been no interaction at all. Because of the analogy with particles, Zabusky and Kruskal named these special waves as *solitons*.

DEFINITION 1.4.1

A <u>soliton</u> is a solitary wave which asymptotically preserves its shape and velocity upon nonlinear interaction with other solitary waves, or more generally, with another (arbitrary) localized disturbance.

Kruskal and Zabusky's remarkable numerical discovery demanded an analytical explanation and detailed mathematical study of the KdV equation. However the KdV equation is nonlinear and at that time no general method of solution for nonlinear equations existed. (We remark that Lax and Levermore [1983a,b,c] and Venakides [1985a,b] have reinvestigated (1.4.4) with δ small.)

1.5 Fourier Transforms.

Before discussing the inverse scattering method of solution for the KdV equation we shall discuss a general method for solving *linear* partial differential equations.

Consider the partial differential equation

$$u_t = \Delta(x, t, u(x,t)), \tag{1.5.1}$$

where $\Delta(x,t,u)$ is a function of $u(x,t)$ and its spatial derivatives, with $t \in \mathbb{R}$, and $x \in \mathbb{R}$, the temporal and spatial variables, respectively. A typical problem associated with equation (1.5.1) is to solve it subject to the given initial condition

$$u(x,0) = f(x), \tag{1.5.2}$$

provided that the problem is well posed for f in some Banach space $\mathcal{B}$ (usually either a space of functions vanishing sufficiently rapidly as $|x| \to \infty$ or a space of periodic functions). If (1.5.1) is a linear partial differential equation, then it is usually solvable by Fourier transform methods (on the infinite line, provided that $f(x)$ decays sufficiently rapidly as $|x| \to \infty$, for example $f \in L^1(\mathbb{R})$). Given the linear partial differential equation in the form

$$u_t = -\mathrm{i}\omega(-\mathrm{i}\partial_x)u, \tag{1.5.3}$$

where $\partial_x \equiv \partial/\partial x$ (usually ω is a polynomial function with constant coefficients, referred to as the dispersion relation); its solution, given the initial condition (1.5.2), is obtained by use of the Fourier transform pair

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,t)\,\mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}k, \tag{1.5.4}$$

$$\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x. \tag{1.5.5}$$

Assuming the validity of the interchange of derivative and integral, by taking the Fourier transform of (1.5.3) we obtain a linear ordinary differential equation for $\hat{u}(k,t)$

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}t} = -\mathrm{i}\omega(k)\hat{u}, \tag{1.5.6}$$

which has general solution

$$\hat{u}(k,t) = \hat{u}(k,0)\exp\{-\mathrm{i}\omega(k)t\}, \tag{1.5.7a}$$

where

$$\hat{u}(k,0) = \hat{f}(k) = \int_{-\infty}^{\infty} f(x)\,\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x. \tag{1.5.7b}$$

Therefore from (1.5.4)

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\exp\{\mathrm{i}[kx - \omega(k)t]\}\,\mathrm{d}k. \tag{1.5.8}$$

Conceptually the Fourier transform method may be viewed as follows.

1. The initial data is transformed into Fourier space by means of (1.5.7b).
2. The time evolution in Fourier space is particularly simple since it satisfies (1.5.6).
3. $u(x,t)$ is recovered from the inverse transform (1.5.4) and is given by (1.5.8).

Schematically this may be written as:

$$\begin{tikzcd}
u(x,0) \arrow[r, "\text{Fourier Transform}"] & \hat{u}(k,0) \arrow[d, "\omega(k)\text{: dispersion relation}"] \\
u(x,t) & \hat{u}(k,t) \arrow[l, "\text{Inverse Fourier Transform}"']
\end{tikzcd}$$

Although (1.5.8) is a solution in quadrature form, useful information may be obtained from it in the asymptotic limit $t \to \infty$. If the system is conservative ($\omega(k)$ is real for real k) and dispersive ($\omega''(k) \neq 0$) then the initial data decays into wave packets which move with their group velocity $\omega'(k)$ and decay algebraically as $t \to \infty$ (in general, asymptotic formulae can be given to describe this see also below). Hence the asymptotic behavior of these linear problems is relatively simple.

EXAMPLE 1.5.1 *The linearized Korteweg-de Vries equation*

A simple example of the method of Fourier transforms is given by the linearized KdV equation

$$u_t + u_{xxx} = 0, \tag{1.5.9a}$$

with

$$u \to 0, \qquad \text{as} \quad |x| \to \infty, \qquad u(x,0) = f(x) \in L^1(\mathbb{R}). \tag{1.5.9b}$$

The dispersion relation is given by $\omega(k) = -k^3$ and so the Fourier transform solution of (1.5.9) is

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)\exp\left\{\mathrm{i}(kx + k^3 t)\right\} \mathrm{d}k.$$

Asymptotically the solution has the representation as $|t| \to \infty$

$$\begin{aligned} u(x,t) \sim (3t)^{-1/3}\,\mathrm{Ai}(z)\left(\frac{\hat{f}(y) + \hat{f}(-y)}{2}\right) \\ + (3t)^{-2/3}\,\mathrm{Ai}'(z)\left(\hat{f}(y) - \frac{\hat{f}(-y)}{2\mathrm{i}k}\right), \end{aligned}$$

where $z = x/(3t)^{1/3}$ and $y = (-x/(3t))^{1/2}$, and $\mathrm{Ai}(z)$ is the Airy function given by

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left\{\mathrm{i}\left(sz + \tfrac{1}{3}s^3\right)\right\} \mathrm{d}s$$

(see Ablowitz and Segur [1981, p.363] for further details).

1.6 An Infinite Number of Conserved Quantities.

An important stage in the development of the general method of solution for the KdV equation was the discovery that KdV had an infinite number of independent conservation laws.

DEFINITION 1.6.1

For the partial differential equation

$$\Delta(x,t,u(x,t)) = 0, \tag{1.6.1}$$

where $t \in \mathbb{R}$, $x \in \mathbb{R}$ are temporal and spatial variables and $u(x,t) \in \mathbb{R}$ the dependent variable, a <u>conservation law</u> is an equation of the form

$$\mathrm{D}_t T_i + \mathrm{D}_x X_i = 0, \tag{1.6.2}$$

which is satisfied for all solutions of the equation (1.6.1)*, where $T_i(x,t)$, the <u>conserved density</u>, and $X_i(x,t)$, the <u>associated flux</u>, are, in general, functions of x, t, u and the partial derivatives of u; D_t denotes the total derivative with respect to t; and D_x the total derivative with respect to x.*

If additionally, $u \to 0$ as $|x| \to \infty$ sufficiently rapidly, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} T_i(x,t)\,\mathrm{d}x = 0.$$

Therefore

$$\int_{-\infty}^{\infty} T_i(x,t)\,\mathrm{d}x = c_i,$$

where c_i, constant, is the *conserved density*. For the KdV equation, the first three conservation laws are

$$\begin{aligned}
&(u)_t + (3u^2 + u_{xx})_x = 0,\\
&(u^2)_t + (4u^3 + 2uu_{xx} - u_x^2)_x = 0,\\
&\left(u^3 - \tfrac{1}{2}u_x^2\right)_t + \left(\tfrac{9}{2}u^4 + 3u^2u_{xx} - 6uu_x^2 - u_xu_{xxx} + \tfrac{1}{2}u_{xx}^2\right)_x = 0.
\end{aligned}$$

The first two conservation laws correspond to conservation of momentum and energy, respectively. The third (and less obvious one) was discovered by Whitham [1965].

The fourth and fifth conservation laws for the KdV equation were found by Kruskal and Zabusky [1963]. Subsequently four additional ones were discovered and it was conjectured that there was an infinite number. Despite rumors that only nine existed, Miura worked out the tenth. After studying the conservation laws of the KdV equation, and those associated with another partial differential equation, the Modified Korteweg-de Vries (mKdV) equation

$$v_t - 6v^2v_x + v_{xxx} = 0, \tag{1.6.3}$$

Miura [1968] discovered the following transformation, now known as *Miura's transformation*. If v is a solution of the mKdV equation (1.6.3), then

$$u = -(v^2 + v_x), \tag{1.6.4}$$

is a solution of the KdV equation (1.1.2). This is readily seen from the relation

$$u_t + 6uu_x + u_{xxx} = -(2v + \partial_x)(v_t - 6v^2 v_x + v_{xxx}), \tag{1.6.5}$$

where $\partial_x \equiv \partial/\partial_x$. Note that every solution of the mKdV equation (1.6.3) leads, via Miura's transformation (1.6.4), to a solution of the KdV equation, but the converse is not true (that is not every solution of the KdV equation can be obtained from a solution of the mKdV equation — see, for example, Ablowitz, Kruskal and Segur [1979]). Miura's transformation leads to many other important results related to the KdV equation. Initially it formed the basis of a proof that the KdV and mKdV equations have an infinite number of conserved densities (Miura, Gardner and Kruskal [1968]), which we outline below. However, even more significant was the motivation given by Miura's transformation (1.6.4) in the development of the *inverse scattering method* for solving the initial value problem for the KdV equation (1.1.2), which we shall discuss in §1.7.

Early on there was an ingenious proof of the existence of an infinite number of conservation laws (Miura, Gardner and Kruskal [1968]). Define w by the relation

$$u = w - \varepsilon w_x - \varepsilon^2 w^2, \tag{1.6.6}$$

which may be thought of as generalization of Miura's transformation (1.6.4). Then the equivalent relation to (1.6.5) is

$$u_t + 6uu_x + u_{xxx} = \left(1 - \varepsilon\partial_x - 2\varepsilon^2 w\right)\left(w_t + 6(w - \varepsilon^2 w^2)w_x + w_{xxx}\right). \tag{1.6.7}$$

Therefore u, as defined by (1.6.6), is a solution of the KdV equation provided that w is a solution of

$$w_t + 6(w - \varepsilon^2 w^2)w_x + w_{xxx} = 0 \tag{1.6.8}$$

(as with Miura's transformation, the transformation (1.6.6) is only one way). Since the KdV equation does not contain ε, then it's solution u depends only upon x and t, however w, a solution of equation (1.6.8), depends on x, t and ε. Then seek a formal power series solution of (1.6.6), in the form

$$w(x,t;\varepsilon) = \sum_{n=0}^{\infty} w_n(x,t)\varepsilon^n. \tag{1.6.9}$$

Since the equation (1.6.8) is in conservation form, then

$$\int_{-\infty}^{\infty} w(x,t;\varepsilon)\,\mathrm{d}x = \text{constant},$$

and so

$$\int_{-\infty}^{\infty} w_n(x,t)\,\mathrm{d}x = \text{constant},$$

for each $n = 0, 1, 2, \ldots$. Substituting (1.6.9) into (1.6.6) and equating coefficients of powers of ε and solving recursively gives

$$w_0 = u, \tag{1.6.10a}$$

$$w_1 = w_{0,x} = u_x, \tag{1.6.10b}$$

$$w_2 = w_{1,x} + w_0^2 = u_{xx} + u^2, \tag{1.6.10c}$$

$$w_3 = w_{2,x} + 2w_0w_1 = u_{xxx} + 4uu_x, \tag{1.6.10d}$$

$$w_4 = w_{3,x} + 2w_0w_2 + w_1^2 = u_{xxxx} + 6uu_{xx} + 5u_x^2 + 2u^3, \tag{1.6.10e}$$

etc.. Continuing to all powers of ε gives an infinite number of conserved densities. The corresponding conservation laws may be found by substituting (1.6.9-10) into equation (1.6.8) and equating coefficients of powers of ε (odd powers of ε actually give no useful information since they are the derivative of the previous even power, however the even powers give independent conservation laws for the KdV equation).

1.7 The Associated Linear Scattering Problem and Inverse Scattering.

1.7.1 The Inverse Scattering Method. Many physical problems are modelled by nonlinear partial differential equations for which, unfortunately, the Fourier transform method fails to solve the problem. There was no unified method by which classes of nonlinear partial differential equations could be solved, and the solutions were often obtained by rather *ad hoc* methods. Therefore a most significant result was the development by Gardner, Greene, Kruskal and Miura [1967, 1974] of a method for the exact solution of the initial-value problem for the KdV equation (1.1.2) for initial values which decay sufficiently rapidly, through a series of *linear* equations. Prior to this work, the only known exact solutions of the KdV equation were the solitary wave and cnoidal wave solutions described in §1.3. The aim is to solve the KdV equation (1.1.2) for $(x,t) : x \in \mathbb{R}$, $t > 0$, subject to the initial condition

$$u(x,0) = f(x), \tag{1.7.1}$$

where $f(x)$ decays sufficiently rapidly as $|x| \to \infty$.

The basic idea is to relate the KdV equation to the time-independent Schrödinger scattering problem

$$\mathrm{L}v := v_{xx} + u(x,t)v = \lambda v, \tag{1.7.2}$$

which has been extensively studied by mathematicians and physicists. The motivation for this equation came from studying the Miura transformation relating solutions of the KdV and mKdV equations. Recall that if $U(x,t)$ is a solution of the mKdV equation

$$U_t - 6U^2U_x + U_{xxx} = 0,$$

then

$$u = -(U^2 + U_x), \tag{1.7.3}$$

is a solution of the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.7.4}$$

(1.7.3) may be viewed as a Riccati equation for U in terms of u, and it is well known that it may be linearized by the transformation $U = v_x/v$, which yields

$$v_{xx} + uv = 0. \tag{1.7.5}$$

Since the KdV equation is Galilean invariant, that is invariant under the transformation

$$(x, t, u(x,t)) \to \left(x - ct, t, u(x,t) + \tfrac{1}{6}c\right),$$

where c is some constant, then it is natural to consider (1.7.2) rather than (1.7.5), in which t plays the role of a parameter and $u(x,t)$ the potential. For (1.7.2), the eigenvalues and the behavior of the eigenfunctions as x determine the *scattering data*, $S(\lambda, t)$, which depends upon the potential $u(x,t)$. The *direct scattering problem* is to map the potential into the scattering data. The *inverse scattering problem* is to reconstruct the potential from the scattering data.

The time dependence of the eigenfunctions of (1.7.2) is given by

$$v_t = (\gamma + u_x)v - (4\lambda + 2u)v_x, \tag{1.7.6}$$

where γ is an arbitrary constant. Assuming that $\lambda_t = 0$, then from (1.7.2) and (1.7.6) we obtain

$$v_{txx} = \left[(\gamma + u_x)(\lambda - u) + u_{xxx} + 6uu_x\right]v - (4\lambda + 2u)(\lambda - u)v_x, \tag{1.7.7a}$$

$$v_{xxt} = \left[(\lambda - u)(\gamma + u_x) - u_t\right]v - (\lambda - u)(4\lambda + 2u)v_x. \tag{1.7.7b}$$

Therefore (1.7.2) and (1.7.6) are compatible (i.e., $v_{xxt} = v_{txx}$), if and only if u satisfies the KdV equation (1.7.4). Similarly, if (1.7.4) is satisfied, then necessarily the eigenvalues must be time independent (i.e., $\partial\lambda/\partial t = 0$).

The solution of (1.7.4), corresponding to $u \to 0$ as $|x| \to \infty$, proceeds as follows:

1. Direct problem. At time $t = 0$, given $u(x,0)$ we solve the direct scattering problem. The spectrum of the Schrödinger equation (1.7.2) consists of a finite number of discrete eigenvalues, $\lambda = \kappa_n^2$, $n = 1, 2, \ldots, N$, for $\lambda > 0$ and a

continuum, $\lambda = -k^2$, for $\lambda < 0$. The eigenfunctions corresponding to these eigenvalues may be computed and their asymptotic behavior written as: for $0 < \lambda = \kappa_n^2$,

$$v_n(x,t) \sim c_n(t) \exp(-\kappa_n x), \qquad \text{as} \quad x \to \infty, \tag{1.7.8a}$$

with

$$\int_{-\infty}^{\infty} v_n^2 \, \mathrm{d}x = 1, \tag{1.7.8b}$$

and for $0 > \lambda = -k^2$,

$$v(x,t) \sim \mathrm{e}^{-\mathrm{i}kx} + r(k,t)\, \mathrm{e}^{\mathrm{i}kx}, \qquad \text{as} \quad x \to \infty, \tag{1.7.9a}$$

$$v(x,t) \sim a(k,t)\, \mathrm{e}^{-\mathrm{i}kx}, \qquad \text{as} \quad x \to -\infty, \tag{1.7.9b}$$

where $r(k,t)$ is the *reflection coefficient* and $a(k,t)$ the *transmission coefficient*. Therefore we have the scattering data at time $t = 0$;

$$S(\lambda, 0) = \left(\{\kappa_n, c_n(0)\}_{n=1}^{N}, r(k,0), a(k,0) \right).$$

2. Time evolution. From equation (1.7.6) we can determine the time evolution of the scattering data. It may be shown that

$$\kappa_n = \text{constant}, \qquad n = 1, 2, \ldots, N \tag{1.7.10a}$$

$$c_n(t) = c_n(0) \exp(4\kappa_n^3 t), \qquad n = 1, 2, \ldots, N \tag{1.7.10b}$$

$$a(k,t) = a(k,0), \tag{1.7.10c}$$

$$r(k,t) = r(k,0) \exp(8\mathrm{i}k^3 t), \tag{1.7.10d}$$

so we have the scattering data at time t;

$$S(\lambda, t) = \left(\{\kappa_n, c_n(t)\}_{n=1}^{N}, r(k,t), a(k,t) \right).$$

3. Inverse Problem. Given the scattering data at the initial time, it is possible to determine its time evolution. The inverse scattering problem is to reconstruct from knowledge of the scattering $S(\lambda, t)$ data, the potential $u(x,t)$ which is the required solution of the KdV equation. This problem was considered by Gel'fand and Levitan [1955] (see also Faddeev [1963] — for a recent discussion see Deift and Trubowitz [1979]; later, we shall view this techniques in a different way — as a Riemann-Hilbert boundary value problem). The results may be summarized as follows. First, using the scattering data (1.7.10), define the function

$$F(x;t) = \sum_{n=1}^{N} c_n^2(t) \exp(-\kappa_n x) + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k,t)\, \mathrm{e}^{\mathrm{i}kx} \, \mathrm{d}k. \tag{1.7.11}$$

Then solve the linear integral equation

$$K(x,y;t) + F(x+y;t) + \int_x^\infty K(x,z;t)F(z+y;t)\,\mathrm{d}z = 0, \tag{1.7.12}$$

called the *Gel'fand-Levitan-Marchenko equation.* Finally the potential is reconstructed by the relation

$$u(x,t) = 2\frac{\partial}{\partial x}\left[K(x,x;t)\right]. \tag{1.7.13}$$

This method is therefore conceptually analogous in many ways to the Fourier transform method for solving linear equations, except however that the final step of solving the inverse scattering problem is highly nontrivial. Schematically this may be written as

$$\begin{tikzcd}
u(x,0) \arrow[r, "\text{direct scattering}"] & S(\lambda,0) = \left(\{\kappa_n, c_n(0)\}_{n=1}^N, r(k,0), a(k,0)\right) \arrow[d, "\omega(k)\text{: dispersion relation}"] \\
u(x,t) & S(\lambda,t) = \left(\{\kappa_n, c_n(t)\}_{n=1}^N, r(k,t), a(k,t)\right) \arrow[l, "\text{inverse scattering}"']
\end{tikzcd}$$

The scattering data plays the role of the Fourier transform and the inverse scattering problem the inverse Fourier transform.

1.7.2 Reflectionless Potentials. If the reflection coefficient is zero (i.e., $r(k,0) = 0 = r(k,t)$), then we obtain the special soliton solutions. If $r(k,t) = 0$ in (1.7.11), then the kernel and inhomogeneous terms in the Gel'fand-Levitan-Marchenko integral equation are reduced to finite sums over the discrete spectrum. This is solvable by separation of variables (cf., Kay and Moses [1956] and Gardner, Greene, Kruskal and Miura [1974]).

If $r(k,t) = 0$ in (1.7.11) then

$$F(x;t) = \sum_{n=1}^N c_n^2(t)\exp(-\kappa_n x), \tag{1.7.14}$$

with $c_m(t) = c_m(0)\exp(4\kappa_j^3 t) > 0$ and distinct $\kappa_m > 0$, $m = 1,2,\ldots,N$ and so the Gel'fand-Levitan-Marchenko equation becomes

$$\begin{aligned} K(x,y;t) &+ \sum_{n=1}^N c_n^2(t)\exp\{-\kappa_n(x+y)\} \\ &+ \int_x^\infty K(x,z;t)\sum_{n=1}^N c_n^2(t)\exp\{-\kappa_n(z+y)\}\,\mathrm{d}z = 0. \end{aligned} \tag{1.7.15}$$

The solution of this equation now takes the form

$$K(x,y;t) = \sum_{n=1}^{N} c_n(t) v_n(x) \exp(-\kappa_n y),$$

then (1.7.15) becomes

$$v_m(x) + \sum_{n=1}^{N} \frac{c_m(t)\, c_n(t)}{\kappa_m + \kappa_n} \exp\{-(\kappa_m + \kappa_n)x\} v_n(x) = c_m(t) \exp(-\kappa_m x), \qquad (1.7.16)$$

for $m = 1, 2, \ldots, N$. This is a system of algebraic equations which can be written in the form

$$(\mathbf{I} + \mathbf{C})\boldsymbol{v} = \boldsymbol{f}, \qquad (1.7.17)$$

where $\boldsymbol{v} = (v_1, v_2, \ldots, v_N)$, $\boldsymbol{f} = (f_1, f_2, \ldots, f_N)$ with $f_m := c_m(t)\exp(-\kappa_m x)$, $m = 1, 2, \ldots, N$, $\mathbf{I}$ is the $N \times N$ identity matrix and $\mathbf{C}$ is a symmetric, $N \times N$ matrix with entries

$$C_{mn} = \frac{c_m(t)\, c_n(t)}{\kappa_m + \kappa_n} \exp\{-(\kappa_m + \kappa_n)x\}, \qquad m, n = 1, 2, \ldots, N.$$

A sufficient condition for (1.7.17) to have a unique solution is that $\mathbf{C}$ is positive definite. Consider the quadratic form

$$\begin{aligned}\boldsymbol{\xi}^{\mathrm{T}}\mathbf{C}\boldsymbol{\xi} &= \sum_{m=1}^{N} \sum_{n=1}^{N} \frac{c_m c_n \xi_m \xi_n}{\kappa_m + \kappa_n} \exp\{-(\kappa_m + \kappa_n)x\} \\ &= \int_x^\infty \left[\sum_{n=1}^{N} c_n \xi_n \exp(-\kappa_n y)\right]^2 \mathrm{d}y,\end{aligned}$$

where $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_N)$, which clearly positive and equal to zero only if $\boldsymbol{\xi} = \mathbf{0}$, and so $\mathbf{C}$ is positive definite. The unique solution to the KdV equation in this case is

$$u(x,t) = 2\frac{\partial^2}{\partial x^2} \{\ln \det(\mathbf{I} + \mathbf{C})\}, \qquad (1.7.18)$$

This is the N-soliton solution for the KdV equation, which corresponds to N waves which asymptotically as $t \to \pm\infty$ have the form

$$u_n(x,t) \sim 2\kappa_n^2 \operatorname{sech}^2 \left\{\kappa_n(x - 4\kappa_n^2 t + x_n)\right\}, \qquad n = 1, 2, \ldots, N,$$

where $x_n = \ln c_n^2(0)$ is a constant. There is a one-to-one relationship between the number of discrete eigenvalues and the number of solitons which emerge asymptotically (see below). As mentioned previously above, these waves interact in such a way as to preserve their identities in the limit asymptotically, however by using the inverse scattering method to solve the KdV equation, we are able to mathematically confirm the numerical observations of Zabusky and Kruskal.

If in (1.7.18) we set $N = 1$, then we obtain the one-soliton solution

$$u(x,t) = 2\kappa_1^2 \operatorname{sech}^2 \left\{ \kappa_1(x - 4\kappa_1^2 t + x_1) \right\},$$

as derived in equation (1.3.7) above.

For $N = 2$, we write

$$\begin{aligned} \Delta :&= \det(\mathbf{I} + \mathbf{C}) \\ &= 1 + \exp(2\eta_1) + \exp(2\eta_2) + \exp(2\eta_1 + 2\eta_2 + A_{12}), \end{aligned} \tag{1.7.19a}$$

with

$$\eta_n = -\kappa_n(x - 4\kappa_n^2 t + \ln c_n^2(0)), \qquad A_{mn} = 2\ln\left(\frac{\kappa_m - \kappa_n}{\kappa_m + \kappa_n}\right) \tag{1.7.19b}$$

(below we explicitly give the two-soliton solution of the KdV equation). From equation (1.7.19) it is easily shown that the only effect of the interaction of two solitary waves is a phase shift (see Figure 1.7.1). Consider the trajectory $\eta_1 =$ constant, and assume that $\kappa_1 > \kappa_2 > 0$. Then

$$\Delta \sim 1 + \exp(2\eta_1), \qquad \text{as} \quad t \to -\infty, \tag{1.7.20a}$$

$$\Delta \sim \exp(2\eta_2) + \exp(2\eta_1 + 2\eta_2 + A_{12}), \qquad \text{as} \quad t \to \infty. \tag{1.7.20b}$$

Therefore from (1.7.18) it follows that for fixed η_1

$$\begin{aligned} u(x,t) &= 2\frac{\partial^2}{\partial x^2}\{\ln\Delta\}, \\ &\sim 2\kappa_1^2 \operatorname{sech}^2(\eta_1 + \delta_1^{\pm}), \qquad \text{as} \quad t \to \pm\infty, \end{aligned} \tag{1.7.21a}$$

with

$$\delta_1^+ = \tfrac{1}{2}A_{12}, \qquad \delta_1^- = 0. \tag{1.7.21b}$$

Similarly, for fixed η_2

$$u(x,t) \sim 2\kappa_2^2 \operatorname{sech}^2(\eta_2 + \delta_2^{\pm}), \qquad \text{as} \quad t \to \pm\infty, \tag{1.7.22a}$$

with

$$\delta_2^+ = 0, \qquad \delta_2^- = \tfrac{1}{2}A_{12}. \tag{1.7.22b}$$

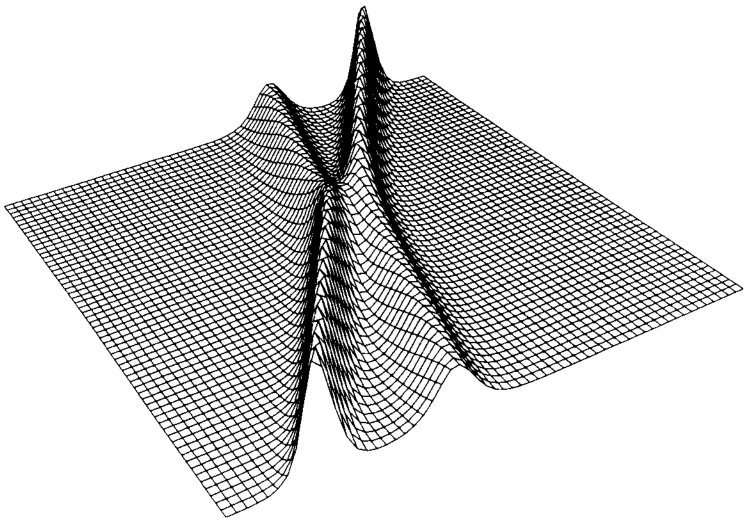


FIGURE 1.7.1 *Two-soliton solution of the KdV equation.*

For large negative time, the larger soliton lies to the left of the smaller soliton, and vice-versa for large positive time (the solitons are travelling from right to left). From (1.7.21,22) it follows that as a consequence of the interaction, the larger and smaller solitons undergo phase shifts given by

$$\delta_1 = \delta_1^+ - \delta_1^- = \tfrac{1}{2}A_{12} = \ln\left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}\right) < 0, \tag{1.7.23a}$$

$$\delta_2 = \delta_2^+ - \delta_2^- = -\tfrac{1}{2}A_{12} = -\ln\left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}\right) > 0, \tag{1.7.23b}$$

respectively. Therefore, after the interaction, the larger soliton has been shifted to the right of where it would have been had there been no interaction, and the smaller shifted to the left by the same amount.

A similar calculation can be done in the N-soliton case (see Gardner, Greene, Kruskal and Miura [1974]; Zakharov [1971]; Wadati and Toda [1972]; Tanaka [1972]). Assume that $\kappa_1 > \kappa_2 > \ldots > \kappa_N > 0$, then for fixed η_n

$$u(x,t) \sim 2\kappa_n^2 \operatorname{sech}^2(\eta_n + \delta_n^\pm), \qquad \text{as} \quad |t| \to \infty. \tag{1.7.24}$$

Therefore the nth soliton undergoes a phase shift given by

$$\delta_n = \delta_n^+ - \delta_n^- = \sum_{m=n+1}^{N} \ln\left(\frac{\kappa_n - \kappa_m}{\kappa_n + \kappa_m}\right) - \sum_{m=1}^{n-1} \ln\left(\frac{\kappa_m - \kappa_n}{\kappa_m + \kappa_n}\right). \tag{1.7.25}$$

It is clear from this expression that the total phase shift experienced by the nth soliton is equivalent to the sum of phase shifts that arise from pairwise interactions with every other soliton. The general question of the interaction of N-soliton and continuous spectra has been examined by Ablowitz and Kodama [1980]; Alonso [1985a,b,c]; Schuur [1986].

To demonstrate the relationship between the number of discrete eigenvalues and the number of solitons, suppose that the initial condition is given by

$$u(x,0) = N(N+1)\,\mathrm{sech}^2\, x.$$

In this case the scattering problem, with $\lambda = \kappa^2$, is

$$v_{xx} + \left\{N(N+1)\,\mathrm{sech}^2\, x - \kappa^2\right\} v = 0. \tag{1.7.26}$$

If we make the transformation $\mu = \tanh x$, then (1.7.26) becomes

$$(1-\mu^2)\frac{\mathrm{d}^2 v}{\mathrm{d}\mu^2} - 2\mu\frac{\mathrm{d}v}{\mathrm{d}\mu} + \left(N(N+1) - \frac{\kappa^2}{1-\mu^2}\right) v = 0, \tag{1.7.27}$$

which is the *Associated Legendre equation* (cf. Abramowitz and Stegun [1972, Chapter 8]). Equation (1.7.27) has N distinct eigenvalues $\kappa_n = n$, $n = 1, 2, \ldots, N$ and bounded associated eigenfunctions $v_1(x), v_2(x), \ldots, v_N(x)$, given by

$$v_n(x) = \gamma_n P_N^n(\tanh x) \sim c_n \exp(-nx), \qquad \text{as} \quad x \to \infty,$$

where $P_N^n(\mu)$ is the Associated Legendre polynomial defined by

$$P_N^n(\mu) = (\mu^2-1)^{n/2}\frac{\mathrm{d}^n}{\mathrm{d}\mu^n}\left(\frac{1}{2^N N!}\frac{\mathrm{d}^N}{\mathrm{d}\mu^N}(\mu^2-1)^N\right),$$

and $c_n(0)$ is determined from the normalization condition (1.7.8b), and so from equation (1.7.10b)

$$c_n(t) = c_n(0)\exp(4n^3 t).$$

Therefore, from equation (1.7.11), the functions $F(x;t)$ in the Gel'fand-Levitan-Marchenko equation is given by

$$F(x;t) = \sum_{n=1}^{N} c_n^2(t)\exp(-nx),$$

and so, from we equation (1.7.18) we obtain the N-solution of the KdV equation

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\left\{\ln\det(\mathbf{I}+\mathbf{C})\right\}, \tag{1.7.28a}$$

where

$$C_{mn} = \frac{c_m(t)\,c_n(t)}{m+n}\exp\{-(m+n)x\}. \tag{1.7.28b}$$

In particular, the two-soliton solution of the KdV equation is [see also (1.7.19)]

$$u(x,t) = 12\left\{\frac{3+4\cosh(2x-8t)+\cosh(4x-64t)}{[3\cosh(x-28t)+\cosh(3x-36t)]^2}\right\}. \tag{1.7.29}$$

If we introduce $\xi = x - 16t$ and $\eta = x - 4t$, then the two-soliton solution can be expressed as

$$u(x,t) = 12\left\{\frac{3+4\cosh(2\xi+24t)+\cosh(4\xi)}{[3\cosh(\xi-12t)+\cosh(3\xi+12t)]^2}\right\}, \tag{1.7.30}$$

and

$$u(x,t) = 12\left\{\frac{3+4\cosh(2\eta)+\cosh(4\eta-48t)}{[3\cosh(x-24t)+\cosh(3\eta-24t)]^2}\right\}. \tag{1.7.31}$$

Expanding (1.7.30,31) as $t \to \pm\infty$, keeping ξ and η fixed respectively, shows that as $t \to \pm\infty$

$$u(x,t) \sim 2\,\mathrm{sech}^2\left(\eta \pm \tfrac{1}{2}\ln 3\right) + 8\,\mathrm{sech}^2\left(2\xi \mp \tfrac{1}{2}\ln 3\right).$$

(The structure of the two-soliton solution of the KdV equation during the interaction has been discussed analytically by a number of authors, e.g., Lax [1968]; Ablowitz and Kodama [1980]; Moloney and Hodnett [1986]; LeVeque [1987]; Hodnett and Moloney [1989].)

For a discussion of the case when the reflection coefficient is nonzero see for example Ablowitz and Kodama [1980]; Ablowitz and Segur [1981, §1.7]; Alonso [1985a,b,c]; Schuur [1986] and the references therein.

1.8 Lax's Generalization.

Lax [1968] put the inverse scattering method for solving the KdV equation into a more general framework which subsequently paved the way to generalizations of the technique as a method for solving other partial differential equations. Consider two operators L and M, where L is the operator of the spectral problem and M is the operator governing the associated time evolution of the eigenfunctions

$$\mathrm{L}v = \lambda v, \tag{1.8.1a}$$

$$v_t = \mathrm{M}v. \tag{1.8.1b}$$

Now take $\partial/\partial t$ of (1.8.1a), giving

$$\mathrm{L}_t v + \mathrm{L}v_t = \lambda_t v + \lambda v_t,$$

hence using (1.8.1b)

$$\begin{aligned} \mathrm{L}_t v + \mathrm{LM}v &= \lambda_t v + \lambda \mathrm{M}v, \\ &= \lambda_t v + \mathrm{M}\lambda v, \\ &= \lambda_t v + \mathrm{ML}v, \end{aligned}$$

Therefore we obtain

$$[\mathrm{L}_t + (\mathrm{LM} - \mathrm{ML})]\, v = \lambda_t v,$$

and hence in order to solve for nontrivial eigenfunctions $v(x,t)$

$$\mathrm{L}_t + [\mathrm{L}, \mathrm{M}] = 0, \tag{1.8.2}$$

where

$$[\mathrm{L}, \mathrm{M}] := \mathrm{LM} - \mathrm{ML},$$

if and only if $\lambda_t = 0$. Equation (1.8.2) is called *Lax's equation*, and contains a nonlinear evolution equation for suitably chosen L and M. For example if we take

$$\mathrm{L} := \frac{\partial^2}{\partial x^2} + u, \tag{1.8.3a}$$

$$\mathrm{M} := (\gamma + u_x) - (4\lambda + 2u)\frac{\partial}{\partial x}, \tag{1.8.3b}$$

then L and M satisfy (1.8.2) provided that u satisfies the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.8.4}$$

Therefore, the KdV equation may be thought of as the compatibility condition of the two linear operators given by (1.8.3). As we shall see below, there is a general class of equations which are associated with the Schrödinger operator (1.8.3a).

If a nonlinear partial differential equation arises as the compatibility condition of two such operators L and M, then (1.8.2) is called the *Lax representation* of the partial differential equation and L and M are the *Lax pair*. Lax [1968] indicates how given L, an associated M may be constructed so that (1.8.2) is nontrivial. A major difficulty is that for a given partial differential equations there is to date no completely systematic method of determining whether or not it has a Lax representation and, if so, how to determine the associated operators L and M. Work has usually relied on inspired guesswork, either fixing the form of the operators L and M and seeing what partial differential equations result or looking for operators of a certain form given the partial differential equation (the so called prolongation structure method due to Wahlquist and Estabrook [1975, 1976], which we discuss in §2.6.8 below — see also Kaup [1980c]; Dodd and Fordy [1983]).

1.9 Linear Scattering Problems and Associated Nonlinear Evolution Equations.

Following the development of the method of inverse scattering to solve the initial value problem for the KdV equation by Gardner, Greene, Kruskal and Miura [1967], it was then of considerable interest to determine whether the method would be applicable to other physically important nonlinear evolution equations. The method of inverse scattering is highly nontrivial and was thought by some to be a fluke, a clever transformation analogous to the Cole-Hopf transformation (Cole [1951]; Hopf [1950]) which linearizes Burgers' equation

$$u_t + 2uu_x - u_{xx} = 0.$$

If we make the transformation $u = -\phi_x/\phi$, then $\phi(x,t)$ satisfies the linear heat equation $\phi_t - \phi_{xx} = 0$. (We remark that Forsyth [1906, p101] first pointed out the relationship between Burgers' equation and the linear heat equation.)

However, Zakharov and Shabat [1972] proved that the method indeed was no fluke by extending Lax's ideas in order to relate the nonlinear Schrödinger equation

$$\mathrm{i}u_t + u_{xx} + \kappa u^2 u^* = 0, \tag{1.9.1}$$

where $*$ denotes the complex conjugate and κ is a constant to a certain linear scattering problem. They showed that if

$$\mathrm{L} = \mathrm{i}\begin{pmatrix} 1+k & 0 \\ 0 & 1-k \end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix} 0 & u^* \\ u & 0 \end{pmatrix}, \tag{1.9.2a}$$

$$\mathrm{M} = \mathrm{i}k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{\partial^2}{\partial x^2} + \begin{pmatrix} \dfrac{-\mathrm{i}uu^*}{1+k} & u_x^* \\ -u_x & \dfrac{\mathrm{i}uu^*}{1-k} \end{pmatrix}, \tag{1.9.2b}$$

with $\kappa = 2/(1-k^2)$, then L and M satisfy Lax's equation

$$\mathrm{L}_t + [\mathrm{L}, \mathrm{M}] = 0, \tag{1.9.3}$$

if and only if $u(x,t)$ satisfies the nonlinear Schrödinger equation (1.9.1). Using the operators (1.9.2), Zakharov and Shabat were able to solve (1.9.1), given initial data $u(x,0) = f(x)$ (provided that $f(x)$ decays sufficiently rapidly as $|x| \to \infty$). Shortly thereafter, Wadati [1972] gave the method of solution for the Modified KdV equation

$$u_t - 6u^2 u_x + u_{xxx} = 0, \tag{1.9.4}$$

and Ablowitz, Kaup, Newell and Segur [1973a], motivated by several important observations by Kruskal, solved the Sine-Gordon equation

$$u_{xt} = \sin u. \tag{1.9.5}$$

Ablowitz, Kaup, Newell and Segur [1973b, 1974] developed a procedure, which showed that the initial value problem for a remarkably large class of physically interesting nonlinear evolution equations could be solved by this method. Because of the analogy between the Fourier transform method for solving the initial value problem for linear evolution equations and the inverse scattering method for solving the initial value problem for nonlinear evolution equations, they termed the inverse scattering technique the *Inverse Scattering Transform* (I.S.T.). (Referred to briefly in §1.1, by the I.S.T. scheme for a nonlinear evolution equation, we shall mean the inverse scattering method for solving the initial value problem of the equation.)

Consider two linear equations

$$\boldsymbol{v}_x = \mathbf{X}\boldsymbol{v}, \tag{1.9.6a}$$

$$\boldsymbol{v}_t = \mathbf{T}\boldsymbol{v}, \tag{1.9.6b}$$

where $\boldsymbol{v}$ is an n-dimensional vector and $\mathbf{X}$ and $\mathbf{T}$ are $n \times n$ matrices. If we require that (1.9.6) are compatible, that is requiring that $\boldsymbol{v}_{xt} = \boldsymbol{v}_{tx}$, then $\mathbf{X}$ and $\mathbf{T}$ must satisfy

$$\mathbf{X}_t - \mathbf{T}_x + [\mathbf{X}, \mathbf{T}] = 0. \tag{1.9.7}$$

This equation (1.9.7) and Lax's equation (1.9.3) are similar; (1.9.6) is somewhat more general as it allows eigenvalue dependence other than $Lv = \lambda v$.

As an example, consider the 2×2 scattering problem [a generalization of (1.9.2a)] given by

$$v_{1,x} = -\mathrm{i}kv_1 + q(x,t)v_2, \tag{1.9.8a}$$

$$v_{2,x} = \mathrm{i}kv_2 + r(x,t)v_1, \tag{1.9.8b}$$

and the most general linear time dependence is given by

$$v_{1,t} = Av_1 + Bv_2, \tag{1.9.9a}$$

$$v_{2,t} = Cv_1 + Dv_2, \tag{1.9.9b}$$

where A, B, C and D are scalar functions of $q(x,t)$, $r(x,t)$ and k, independent of (v_1, v_2). Essentially, we just specify that

$$\mathbf{X} = \begin{pmatrix} -\mathrm{i}k & q \\ r & \mathrm{i}k \end{pmatrix}, \qquad \mathbf{T} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{1.9.10}$$

in (1.9.6). Note that if there were any x-derivatives on the right-hand side of (1.9.9) then they can be eliminated by use of (1.9.8). Furthermore, when $r = -1$, then (1.9.8) reduces to the Schrödinger scattering problem

$$v_{2,xx} + (k^2 + q)v_2 = 0, \tag{1.9.11}$$

which is equivalent to (1.7.2), with $k^2 = -\lambda$. It is interesting to note that the most physically interesting nonlinear evolution equations arise from this procedure when either $r = -1$ or $r = q^*$ (or $r = q$ if q is real).

This procedure provides a simple technique which allows us the find nonlinear evolution equations expressible in the form (1.9.7). The compatibility of equations (1.9.8-9), that is requiring that $v_{j,xt} = v_{j,tx}$, for $j = 1, 2$, and assuming that the eigenvalue k is time-independent, that is $\mathrm{d}k/\mathrm{d}t = 0$, imposes a set of conditions which A, B, C and D must satisfy. Therefore

$$\begin{aligned} v_{1,xt} &= -\mathrm{i}kv_{1,t} + q_t v_2 + qv_{2,t}, \\ &= -\mathrm{i}k(Av_1 + Bv_2) + q_t v_2 + q(Cv_1 + Dv_2), \\ v_{1,tx} &= A_x v_1 + Av_{1,x} + B_x v_2 + Bv_{2,x}, \\ &= A_x v_1 + A(-\mathrm{i}kv_1 + qv_2) + B_x v_2 + B(\mathrm{i}kv_2 + rv_1). \end{aligned}$$

Hence by equating the coefficients of v_1 and v_2, we obtain

$$A_x = qC - rB, \tag{1.9.12a}$$
$$B_x + 2\mathrm{i}kB = q_t - (A - D)q, \tag{1.9.12b}$$

respectively. Similarly

$$\begin{aligned} v_{2,xt} &= \mathrm{i}kv_{2,t} + r_t v_1 + rv_{1,t}, \\ &= \mathrm{i}k(Cv_1 + Dv_2) + r_t v_1 + a(A_1 + Bv_2), \\ v_{2,tx} &= C_x v_1 + Cv_{1,x} + D_x v_2 + Dv_{2,x}, \\ &= C_x v_1 + C(-\mathrm{i}kv_1 + qv_2) + D_x v_2 + D(\mathrm{i}kv_2 + rv_1), \end{aligned}$$

and equating the coefficients of v_1 and v_2 we obtain

$$C_x - 2\mathrm{i}kC = r_t + (A - D)r, \tag{1.9.13a}$$
$$(-D)_x = qC - rB. \tag{1.9.13b}$$

Therefore, from (1.9.12a) and (1.9.13b), without loss of generality we may assume $D = -A$, and hence it is seen that A, B and C necessarily satisfy the compatibility conditions

$$A_x = qC - rB, \tag{1.9.14a}$$
$$B_x + 2\mathrm{i}kB = q_t - 2Aq, \tag{1.9.14b}$$
$$C_x - 2\mathrm{i}kC = r_t + 2Ar. \tag{1.9.14c}$$

We now solve (1.9.14) for A, B and C (thus ensuring that (1.9.8) and (1.9.9) are compatible). In general, this can only be done if another condition (on r and q) is satisfied, this condition being the evolution equation. Since k, the eigenvalue, is a free parameter, we may find solvable evolution equations by seeking finite power series expansions for A, B and C:

$$A = \sum_{j=0}^{n} A_j k^j, \qquad B = \sum_{j=0}^{n} B_j k^j, \qquad C = \sum_{j=0}^{n} C_j k^j. \tag{1.9.15}$$

Substituting (1.9.15) into (1.9.14) and equating coefficients of powers of k, we obtain $3n+5$ equations. There are $3n+3$ unknowns, A_j, B_j, C_j, $j = 0, 1, \ldots, n$, and so we also obtain two nonlinear evolution equations for r and q. Now let us consider some examples.

EXAMPLE 1.9.1 $n = 2$

Suppose that A, B and C are quadratic in k, that is

$$A = A_2 k^2 + A_1 k + A_0, \tag{1.9.16a}$$

$$B = B_2 k^2 + B_1 k + B_0, \tag{1.9.16b}$$

$$C = C_2 k^2 + C_1 k + C_0. \tag{1.9.16c}$$

Substitute (1.9.16) into (1.9.14) and equate powers of k. The coefficients of k^3 immediately give $B_2 = C_2 = 0$. At order k^2, we obtain $A_2 = a$, constant, $B_1 = \mathrm{i}aq$, $C_1 = \mathrm{i}ar$. At order k^1, we obtain $A_1 = b$, constant, for simplicity we set $b = 0$ (if $b \neq 0$ then a more general evolution equation is obtained), then $B_0 = -\frac{1}{2}aq_x$ and $C_0 = \frac{1}{2}ar_x$. Finally, at order k^0, we obtain $A_0 = \frac{1}{2}arq + c$, with c a constant (again for simplicity we set $c = 0$). Therefore we obtain the following evolution equations

$$-\tfrac{1}{2}aq_{xx} = q_t - aq^2 r, \tag{1.9.17a}$$

$$\tfrac{1}{2}ar_{xx} = r_t + aqr^2. \tag{1.9.17b}$$

If in (1.9.17) we set $r = \mp q^*$ and $a = 2\mathrm{i}$, then we obtain the nonlinear Schrödinger equation

$$\mathrm{i}q_t = q_{xx} \pm 2q^2 q^*. \tag{1.9.18}$$

In summary, given the scattering problem (1.9.8) and the associated time dependence (1.9.9), then these are compatible provided (1.9.14) holds. In this example, setting $r = -q^*$, then we find that

$$A = 2\mathrm{i}k^2 \pm \mathrm{i}qq^*, \tag{1.9.19a}$$

$$B = 2qk + \mathrm{i}q_x, \tag{1.9.19b}$$

$$C = \mp 2q^* k \pm \mathrm{i}q_x^*, \tag{1.9.19c}$$

satisfy (1.9.14) provided that $q(x,t)$ satisfies the nonlinear Schrödinger equation (1.9.18).

EXAMPLE 1.9.2 $n = 3$

If we substitute the third order polynomials in k

$$A = a_3k^3 + a_2k^2 + \tfrac{1}{2}(a_3qr + a_1)k + \tfrac{1}{2}a_2qr - \tfrac{1}{4}\mathrm{i}a_3(qr_x - rq_x) + a_0, \tag{1.9.20a}$$
$$B = \mathrm{i}a_3qk^2 + \left(\mathrm{i}a_2q - \tfrac{1}{2}a_3q_x\right)k + \left[\mathrm{i}a_1q - \tfrac{1}{2}a_2q_x + \tfrac{1}{4}\mathrm{i}a_3(2q^2r - q_{xx})\right], \tag{1.9.20b}$$
$$C = \mathrm{i}a_3rk^2 + \left(\mathrm{i}a_2r + \tfrac{1}{2}a_3r_x\right)k + \left[\mathrm{i}a_1r + \tfrac{1}{2}a_2r_x + \tfrac{1}{4}\mathrm{i}a_3(2r^2q - r_{xx})\right], \tag{1.9.20c}$$

in (1.9.14), with a_3, a_2, a_1 and a_0 constants, then we find that $q(x,t)$ and $r(x,t)$ satisfy the evolution equations

$$q_t + \tfrac{1}{4}\mathrm{i}a_3(q_{xxx} - 6qrq_x) + \tfrac{1}{2}a_2(q_{xx} - 2q^2r) - \mathrm{i}a_1q_x - 2a_0q = 0, \tag{1.9.21a}$$
$$r_t + \tfrac{1}{4}\mathrm{i}a_3(r_{xxx} - 6qrr_x) - \tfrac{1}{2}a_2(r_{xx} - 2qr^2) - \mathrm{i}a_1r_x + 2a_0r = 0. \tag{1.9.21b}$$

For special choices of the constants a_3, a_2, a_1 and a_0 in (1.9.21) we find physically interesting evolution equations. If $a_0 = a_1 = a_2 = 0$, $a_3 = -4\mathrm{i}$ and $r = -1$, then we obtain the KdV equation

$$q_t + 6qq_x + q_{xxx} = 0.$$

If $a_0 = a_1 = a_2 = 0$, $a_3 = -4\mathrm{i}$ and $r = q$, then we obtain the mKdV equation

$$q_t - 6q^2q_x + q_{xxx} = 0.$$

[Note that if $a_0 = a_1 = a_3 = 0$, $a_2 = -2\mathrm{i}$ and $r = -q^*$, then we obtain the nonlinear Schrödinger equation (1.9.18).]

We can also consider expansions of A, B and C in inverse powers of k.

EXAMPLE 1.9.3 $n = -1$

Suppose that

$$A = \frac{a(x,t)}{k}, \qquad B = \frac{b(x,t)}{k}, \qquad C = \frac{c(x,t)}{k}, \tag{1.9.22}$$

then the compatibility conditions (1.9.14) are satisfied if

$$a_x = \tfrac{1}{2}(qr)_t, \qquad q_{xt} = -4\mathrm{i}aq, \qquad r_{xt} = -4\mathrm{i}ar. \tag{1.9.23}$$

Special cases of these are (i),

$$a = \tfrac{1}{4}\mathrm{i}\cos u, \qquad b = c = \tfrac{1}{4}\mathrm{i}\sin u, \qquad q = -r = -\tfrac{1}{2}u_x, \tag{1.9.24}$$

then u satisfies the Sine-Gordon equation

$$u_{xt} = \sin u.$$

and (ii),

$$a = \tfrac{1}{4}\mathrm{i}\cosh u, \qquad b = c = \tfrac{1}{4}\mathrm{i}\sinh u, \qquad q = r = \tfrac{1}{2}u_x, \tag{1.9.25}$$

where u satisfies the Sinh-Gordon equation

$$u_{xt} = \sinh u.$$

These three examples only show a few of the many nonlinear evolution equations which may be obtained by this procedure. We saw above that when $r = -1$, the scattering problem (1.9.8) reduced to the Schrödinger equation (1.9.11). In this case we take an alternative associated time dependence

$$v_t = Av + Bv_x. \tag{1.9.26}$$

By requiring that this and

$$v_{xx} + (k^2 + q)v = 0,$$

are compatible and assuming that $\mathrm{d}k/\mathrm{d}t = 0$, yields equations for A and B analogous to equations (1.9.14), then by expanding in powers of k^2, we obtain a general class of equations.

There have been numerous applications and generalizations of this method. Early on Ablowitz and Haberman [1975b] generalized the 2×2 case to the $n \times n$ system (see §1.10 below). Other generalizations includes that of Wadati, Konno and Ichikawa [1979a,b] (see also Shimizu and Wadati [1980]; Ishimori [1981, 1982]; Wadati and Sogo [1983]; Konno and Jeffrey [1984]). For example, instead of (1.9.8), consider the scattering problem

$$v_{1,x} = -f(k)v_1 + g(k)q(x,t)v_2, \tag{1.9.27a}$$

$$v_{2,x} = f(k)v_2 + g(k)r(x,t)v_1, \tag{1.9.27b}$$

where $f(k)$ and $g(k)$ are functions of the eigenvalue k, and the time dependence is given (as previously) by

$$v_{1,t} = Av_1 + Bv_2, \tag{1.9.28a}$$

$$v_{2,t} = Cv_1 - Av_2, \tag{1.9.28b}$$

The compatibility of (1.9.27) and (1.9.28) requires that A, B and C satisfy

$$A_x = g(qC - rB), \tag{1.9.29a}$$

$$B_x + 2fB = gq_t - 2Agq, \tag{1.9.29b}$$

$$C_x - 2fC = gr_t + 2Agr. \tag{1.9.29c}$$

As earlier, postulating that A, B, C, f and g have finite power series expansions in k (where the expansions for f and g have constant coefficients), then one obtains a variety of physically interesting evolution equations.