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The Technique of Pseudodifferential Operators

H. O. Cordes

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To my 6 children, Stefan and Susan

Sabine and Art, Eva and Sam

TABLE OF CONTENTS

P R E F A C E

It is generally well known that the Fourier-Laplace transform converts a linear constant coefficient PDE $P(D)u=f$ on $\mathbb{R}^n$ to an equation $P(\xi)u^{\sim}(\xi)=f^{\sim}(\xi)$, for the transforms $u^{\sim}$, $f^{\sim}$ of u and f, so that solving $P(D)u=f$ just amounts to division by the polynomial $P(\xi)$. The practical application was suspect, and ill understood, however, until theory of distributions provided a basis for a logically consistent theory. Thereafter it became the Fourier-Laplace method for solving initial-boundary problems for standard PDE. We recall these facts in some detail in sec's 1-4 of ch.0.

The technique of pseudodifferential operator extends the Fourier-Laplace method to cover PDE with variable coefficients, and to apply to more general compact and noncompact domains or manifolds with boundary. Concepts remain simple, but, as a rule, integrals are divergent and infinite sums do not converge, forcing lengthy, often endlessly repetitive, discussions of 'finite parts' (a type of divergent oscillatory integral existing as distribution integral) and asymptotic sums (modulo order $-\infty$).

Of course, pseudodifferential operators (abbreviated ψdo's) are (generate) abstract linear operators between Hilbert or Banach spaces, and our results amount to 'well-posedness' (or normal solvability) of certain such abstract linear operators. Accordingly both, the Fourier-Laplace method and theory of ψdo's, must be seen in the context of modern operator theory.

To this author it always was most fascinating that the same type of results (as offered by elliptic theory of ψdo;'s) may be obtained by studying certain examples of Banach algebras of linear operators. The symbol of a ψdo has its abstract meaning as Gelfand function of the coset modulo compact operators of the abstract operator in the algebra.

On the other hand, hyperbolic theory, generally dealing with a group exp(Kt) (or an evolution operator U(t)) also has its manifestation with respect to such operator algebras: conjugation with exp(Kt) amounts to an automorphism of the operator algebra, and of the quotient algebra. It generates a flow in the symbol space essentially the characteristic flow of singularities. In [C_1], [C_2] we were going into details discussing this abstract approach.

We believe to have demonstrated that ψdo's are not necessary to understand these fact. But the technique of ψdo's, in spite of its endless formalisms (as a rule integrals are always 'distribution integrals', and infinite series are asymptotically convergent, not convergent), still provides a strongly simplifying principle, once the technique is mastered. Thus our present discussion of this technique may be justified.

On the other hand, our hyperbolic discussions focus on invariance of ψdo-algebras under conjugation with evolution operators, and do not touch the type of oscillatory integral and further discussions needed to reveal the structure of such evolution operators as Fourier integral operators. In terms of Quantum mechanics we prefer the Heisenberg representation, not the Schroedinger representation.

In particular this leads us into a discussion of the Dirac equation and its invariant algebra, in chapter X. We propose it as algebra of observables.

The basis for this volume is (i) a set of notes of lectures given at Berkeley in 1974-80 (chapters I-IV) published as preprint at U. of Bonn, and (ii) a set of notes on a seminar given in 1984 also at Berkeley (chapters VI-IX). The first covers elliptic (and parabolic) theory, the second hyperbolic theory. One might say that we have tried an old fashiened PDE lecture in modern style.

In our experience a newcomer will have to reinvent the theory before he can feel at home with it. Accordingly, we did not try to push generality to its limits. Rather, we tend to focus on the simplest nontrivial case, leaving generalizations to the reader. In that respect, perhaps we should mention the problems (partly of research level) in chapters I-IV, pointing to manifolds with conical tips or cylindrical ends, where the 'Fredholm-significant symbol' becomes operator-valued.

The material has been with the author for a long time, and was subject of many discussions with students and collaborators. Especially we are indebted to R. McOwen, A.Erkip, H. Sohrab, E. Schrohe, in chronclogical order. We are grateful to Cambridge University Press for its patience, waiting for the manuscript.

Berkeley, November 1993 Heinz O. Cordes

Chapter 0. INTRODUCTORY DISCUSSIONS.

In the present introductory chapter we give comprehensive discussions of a variety of nonrelated topics. All of these bear on the concept of pseudo-differential operator, at least in the author's mind. Some are only there to make studying ψdo's appear a natural thing, reflecting the author's inhibitions to think along these lines.

In sec.1 we discuss the elementary facts of the Fourier transform, in sec.'s 2 and 3 we develop Fourier-Laplace transforms of temperate and nontemperate distributions. In sec.4 we discuss the Fourier-Laplace method of solving initial-value problems and free space problems of constant coefficient partial differential equations. Sec.5 discusses another problem in PDE, showing how the solving of an abstract operator equation together with results on hypo-ellipticity and "boundary-hypo-ellipticity" can lead to existence proofs for classical solutions of initial-boundary problems. Sec.6 is concerned with the operator e^{Lt} , for a first order differential expression L . Sec.'s 7 and 8 deal with the concept of characteristics of a linear differential expression and learning how to solve a first order PDE. Sec.9 gives a mini-introduction to Lie groups, focusing on the mutual relationship between Lie groups and Lie algebras. (Note the relation to ψdo's discussed in ch.8).

We should expect the reader to glance over ch.0 and use it to have certain prerequisites handy, or to get oriented in the serious reading of later chapters.

0. Some special notations.

The following notations, abbreviations, and conventions will be used throughout this book.

(a) $\kappa_n = (2\pi)^{-n/2}$, $đx = \kappa_n dx_1 dx_2 \ldots dx_n = \kappa_n dx$.

(b) $\langle x\rangle = (1+|x|^2)^{1/2}$, $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$, etc.

(c) Derivatives are written in various ways, at convenience: For $u=u(x)=u(x_1,\dots,x_n)$ we write $u^{(\alpha)}=\partial_x^\alpha u=\partial_{x^1}^{\alpha^1}\partial_{x^2}^{\alpha^2}\dots u =$ $=\partial^{\alpha_1}/\partial x^{\alpha_1}\dots\partial^{\alpha_n}/\partial x^{\alpha_n}u$. Or, $u_{|x_j}=\partial_{x_j}u$, $u_{|x}$ to denote the n-vector with components $u_{|x_j}$, $\nabla_x^k u$ for the k-dimensional array with components $u_{|x_{i_1}x_{i_2}\dots}$. For a function of $(x,\xi)=(x_1,\dots,x_n,\xi_1,\dots,\xi_n)$ it is often convenient to write $u_{(\beta)}^{(\alpha)}=\partial_\xi^\alpha\partial_x^\beta u$.

(d) A multi-index is an n-tuple of integers $\alpha=(\alpha_1,\dots,\alpha_n)$. We write $|\alpha|=|\alpha_1|+\dots+|\alpha_n|$, $\alpha!=\alpha_1!\dots\alpha_n!$, $\binom{\alpha}{\beta}=\binom{\alpha}{\beta}\dots\binom{\alpha}{\beta}$, $x^\alpha=x_1^{\alpha_1}\dots x_n^{\alpha_n}$, etc., $\mathbb{N}^n$={all multi-indices}.

(e) Some standard spaces: $\mathbb{R}^n$ = n-dimensional Euclidean space $\mathbb{B}^n$=directional compactification of $\mathbb{R}^n$ (one infinite point ∞x added in every direction (of a unit vector x).

(f) Spaces of continuous or differentiable complex-valued functions over a domain or differentiable manifold X (or sometimes only $X=\mathbb{R}^n$): C(X) = continuous functions on X ; CB(X)= bounded continuous functions on X; CO(X)= continuous functions on X vanishing at ∞ ; CS(X) = continuous functions with directional limits; $C_0(X)$ = continuous functions with compact support; $C^k(X)$= functions with derivatives in C, to order k, (incl. $k=\infty$). $CB^\infty(X)$="all derivatives exist and are bounded". The Laurent-Schwartz notations $\mathcal{D}(X)=C_0^\infty(X)$, $\mathcal{E}(X)=C^\infty(X)$ are used. Also $\mathcal{S}=\mathcal{S}(\mathbb{R}^n)$= "rapidly decreasing functions" (All derivatives decay stronger as any power of x). Also, distribution spaces $\mathcal{D}'$, $\mathcal{E}'$, $\mathcal{S}'$.

(g) L^p-spaces: For a measure space X with measure $d\mu$ we write $L^p(X)=L^p(X,d\mu)$={measurable functions u(x) with $|u|^p$ integrable} for $1\le p<\infty$; $L^\infty(X)$={essentially bounded functions}.

(h) Maps between general spaces: C(X,Y) denotes the continuous maps $X\to Y$. Similar for the other symbols under (f), i.e., CB(X,Y) ,.... .

(i) Classes of linear operators ($\mathcal{X}$= Banach space) : $\mathcal{L}(\mathcal{X})$ ($\mathcal{K}(\mathcal{X})$)= continuous (compact) operators; $\mathcal{GL}(\mathcal{X})$ ($\mathcal{U}(\mathcal{H})$) = invertible (unitary) operators of $\mathcal{L}(\mathcal{X})$ (of $\mathcal{L}(\mathcal{H})$, $\mathcal{H}$=Hilbert space); $\mathcal{U}_n=\mathcal{U}(\mathbb{C}_n)$. For operators $\mathcal{X}\to\mathcal{Y}$, again, $\mathcal{L}(\mathcal{X},\mathcal{Y})$, etc.

(j) The convolution product: For $u,v\in L^1(\mathbb{R}^n)$ we write $w(x)$ $=(u*v)(x)=\kappa_n\int dyu(x-y)v(y)$ (Note the factor $\kappa_n=(2\pi)^{-n/2}$).

(k) Special notation: " $X\subset\subset Y$ " means that X is contained in a compact subset of Y .

(l) For technical reason we may write $\lim_{\varepsilon\to 0} a(\varepsilon) = a|_{\varepsilon\to 0}$.

(m) Abbreviations used: ODE (PDE) = ordinary (partial) differential equation (or "expression"). FOLPDE (or folpde)= first order linear partial differential equation (or "expression"); ψdo= pseudodifferential operator.

(n) Integrals need not be existing (proper or improper) Riemann or Lebesgue integrals, unless explicitly stated, but may be distribution integrals By this term we mean that either (i) the integral may be interpreted as value of a distribution at a testing function-the integrand may be a distribution, or (ii) the limit of Riemann sums exists in the sense of weak convergence of a sequence of (temperate) distributions, or (iii) the limit defining an improper Riemann integral exists in the sense of weak convergence, as above, or (iv) the integral may be a 'finite part' (cf. I,4).

(o) Adjoints: For a linear operator A we use 'distribution adjoint' $A^{\sim}$ and 'Hilbert space adjoint' A^*, corresponding to transpose A^T and adjoint $\bar{A}^T = A^*$, in case of a matrix $A=((a_{jk}))$, respectively. For a symbols $a(x,\xi)$, a^* (or a^+) may denote the symbol of the adjoint ψdo of $a(x,D)$, as specified in each section.

(p) supp u (sing supp u (or s.s.u)) denotes the (singular) support of the distribution u.

1. The Fourier transform; elementary facts.

Let $u \in L^1(\mathbb{R}^n)$ be a complex-valued integrable function. Then we define the Fourier transform $u^\wedge = Fu$ of u by the integral

$$(1.1) \qquad u^\wedge(x) = \int_{\mathbb{R}^n} d\!\!\!/\xi\, u(\xi) e^{-ix\xi} \quad , \; x \in \mathbb{R}^n ,$$

with $x\xi = x.\xi = \sum_{j=1}^n x_j\xi_j$, an existing Lebesgue integral. Clearly,

$$(1.2) \qquad |u^\wedge(x)| \le \|u\|_{L_1} = \int_{\mathbb{R}^n} d\!\!\!/x\, |u(x)| \; .$$

Note that $u^\wedge$ is uniformly continuous over $\mathbb{R}^n$: We get

$$(1.3) \qquad \begin{aligned} |u^\wedge(x) - u^\wedge(y)| &\le 2\int d\!\!\!/\xi\, |\sin(x-y)\xi/2|\, |u(\xi)| \\ &\le N|x-y|\,\|u\|_{L^1} + 2\int_{|\xi|\ge N} d\!\!\!/\xi\, |u(\xi)| \; , \end{aligned}$$

where the right hand side is $<\varepsilon$ if N is chosen for $\int_{|\xi|\ge N} <\varepsilon/4$, and then $|x-y|< \varepsilon/(2N\|u\|_{L^1})$. Moreover, we get $u^\wedge \in CO(\mathbb{R}^n)$, i.e., $\lim_{|x|\to\infty} u^\wedge(x)=0$, a fact, known as the <u>Riemann-Lebesgue lemma</u>.

To prove the latter, we reduce it to the case of $u \in C_0^\infty(\mathbb{R}^n)$: The space C_0^∞ is known to be dense in L^1. By (1.1) we get

$$(1.4)\qquad |u^\wedge(x)-v^\wedge(x)| \le \|u-v\|_{L^1} < \varepsilon/2 \ , \text{ as } v \in C_0^\infty \ , \ \|u-v\|_{L^1}<\varepsilon/2.$$

Hence $\lim_{|x|\to\infty} v^\wedge(x)=0$ implies $|u^\wedge| \le |u^\wedge - v^\wedge| + |v^\wedge| < \varepsilon$ whenever x is chosen according to $|v^\wedge| < \varepsilon/2$.

But for $v \in C_0^\infty$ the Fourier integral extends over a ball $|\xi| \le N$ only, since $v=0$ outside. We may integrate by parts for

$$(1.5)\qquad |x|^2 u^\wedge(x) = -\int \Delta_\xi(e^{-ix\xi})v(\xi)d\!\!\!/\xi = -\int d\!\!\!/\xi e^{-ix\xi}(\Delta v)(\xi) = -(\Delta v)^\wedge(x) \ ,$$

with the Laplace differential operator $\Delta_\xi = \sum_{j=1}^n \partial_{\xi_j}^{\ 2}$. Clearly we have $\Delta v \in C_0^\infty \subset L^1$ as well, whence (1.1) applies to Δv, for

$$(1.6)\qquad |v^\wedge(x)| \le \|\Delta v\|_{L^1}/|x|^2 \to 0 \ , \text{ as } |x| \to \infty \ ,$$

completing the proof.

The above partial integration describes a general method to be applied frequently in the sequel. (1.6) may be derived under the weaker assumptions that $v\in C^2$, and that all derivatives $v^{(\alpha)}$, $|\alpha|\le 2$, are in L^1 (cf. pbm.5). On the other hand, there are simple examples of $u\notin L^1$ such that $u^\wedge$ does not decay as rapidly as (1.6) indicates. In particular, $u\in L^1$ exists with $u^\wedge \notin L^1$ (cf.pbm.4).

This matter becomes important if we think of inverting the linear operator $F:L^1 \to CO$ defined by (1.1), because formally an inverse seems to be given by almost the same integral. Indeed, define the (complex) <u>conjugate Fourier transform</u> $\bar{F}:L^1 \to CO$ by $\bar{F}u = \overline{(F\bar{u})}$, or, $u^\vee = \bar{F}u$, where

$$(1.7)\qquad u^\vee(x) = \int d\!\!\!/\xi e^{ix\xi}u(\xi) \quad , \ u \in L^1(\mathbb{R}^n) \ .$$

Then, in essence, it will be seen that $\bar{F}$ is the inverse of the operator F. More precisely we will have to restrict F to a (dense) subspace of L^1, for this result. Or else, the definition of the operator $\bar{F}$ must be extended to certain non-integrable functions, for which existence of the Lebesgue integral (1.7) cannot be expected. Both things will be done, eventually.

It turns out that F induces a unitary operator of the Hilbert space $L^2(\mathbb{R}^n)$: We have <u>Parseval's</u> <u>relation</u>:

$$(1.8)\qquad \int_{\mathbb{R}^n} đx|u^\wedge(x)|^2 = \int_{\mathbb{R}^n} đx|u(x)|^2 \ , \text{ for all } u \in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n) \ .$$

Formula (1.8) is easier to prove as the <u>Fourier</u> <u>inversion</u> <u>formula</u>, asserting $u^{\wedge\vee}=u^{\vee\wedge}=u$ for certain u: We may write

$$(1.9)\qquad \int_{Q_N} đx\bar{u}^\wedge(x)v^\wedge(x) = \int đξđη\bar{u}(\xi)v(\eta)\Pi_{j=1}^n \int_{-N}^{N} e^{ix_j(\xi_j-\eta_j)}đx_j \ ,$$

assuming that $u,v\in L^1(\mathbb{R}^n)$, with the 'cube' $Q_N=\{|x_j|\le N, j=1,..,N\}$, some integer N>0. Indeed, the interchange of integrals leading to (1.9) is legal, since the integrand is $L^1(Q_N\times\mathbb{R}^n\times\mathbb{R}^n)$.

Note that $\int_{-N}^{N} e^{ist}dt = 2\frac{\sin sN}{s}$, $s\neq 0$, $= 2N$, $s=0$, allowing evaluation of the inner integrals at right of (1.9). With $\int đξđη = \int đξ\int đη$, and $\eta = \xi - \zeta/N$, $đη = N^{-n}đζ$, (1.9) assumes the form

$$(1.10)\qquad \int_{Q_N} đx\bar{u}^\wedge(x)v^\wedge(x) = \int đξ\bar{u}(\xi)\int đζ v(\xi-\zeta/N)\Pi_{j=1}^n\varphi(\zeta_j) \ ,$$

where $\varphi(t) = (2/\pi)^{1/2}\frac{\sin t}{t}$, $t\neq 0$, continuously extended into t=0.

For $v\in C(R^n)$, as $N\to\infty$, the function $v(\xi-\zeta/N)$ will converge to $v(\xi)$, independent of ζ . Thus one expects the inner integral at right of (1.10) to converge to $v(\xi)\int\Pi_{j=1}^n\varphi(\zeta_j)đζ_j = v(\xi)$, since

$$(1.11)\qquad \int_0^\infty \sin t\ dt/t = \pi/2 \ .$$

Legalization of this argument will confirm Parseval's relation, since the right hand converges to the right hand side of (1.8), as $N\to\infty$. With $u\in L^1$ and $v\in C_0^\infty$ (setting $\varphi_n(\zeta)=\Pi\varphi(\zeta_j)$) write

$$(1.12)\qquad \int đξ\bar{u}(\xi)\int đζ\varphi_n(\zeta)(v(\xi-\zeta/N)-v(\xi)) = \int_{Q_N} đx\bar{u}^\wedge v^\wedge - \int_{R^n} đx\bar{u}v \ .$$

To show that the inner integral at left goes to 0 as $N\to\infty$ it is more skilful to use the integration variable $\theta=\zeta/N$, $d\zeta=N^n d\theta$. For $n=1$, $\int \sin N\theta\ (v(\xi-\theta)-v(\xi))d\theta/\theta = \int_{|\theta|\le\delta} + \int_{|\theta|\ge\delta} = I_0 + I_\infty$.

Here we get (with $w(\theta) = (v(\xi-\theta)-v(\xi))/\theta$)

$$|I_0| \le\delta\|v'\|_{L^\infty},\ I_\infty=\tfrac{1}{N}((w(\theta)\cos(N\theta)|_{\theta=-\delta}^{\theta=\delta} + \int_{|\theta|\ge\delta}\cos(N\theta)w_{|\theta}(\theta)d\theta).$$

The latter gives $I_\infty \le \frac{c}{N\delta}(\|v\|_{L^\infty}+\|v'\|_{L^\infty})$, with a constant c, only depending on the volume of supp v, i.e., it is fixed after fixing v . The estimates imply the inner integral to go to 0, uniformly as $x \in \mathbb{R}^n$. For $u \in L^1$ the Lebesgue theorem then implies the left hand side of (1.12) to tend to 0, as $N \to \infty$, for each fixed $v \in C_0^\infty$.

For general n the proof is a bit less transparent, but remains the same: Split the inner integral into a sum of integrals over a small neighbourhood of 0 and its complement. In the first term use differentiability of v; in the second an integration by parts.

We now have a 'polarized' Parseval relation, in the form

$$(1.13) \qquad \int_{R^n} \not{d}x \bar{u}^\wedge v^\wedge = \int_{R^n} \not{d}x \bar{u} v \text{ , for } u \in L^1 \text{ , } v \in C_0^\infty .$$

For $u \in L^1 \cap L^2$ pick a sequence $u_j \in C_0^\infty$ with $\|u-u_j\|_{L^1} \to 0$, $\|u-u_j\|_{L^2} \to 0$, as is possible. Then, since $u_j - u_l \in C_0^\infty \subset L^2$, (1.13) with $u=v=u_j-v_j$ implies $\|u_j^\wedge - u_l^\wedge\|_{L^2} = \|u_j - u_l\|_{L^2} \to 0$, $j,l \to \infty$. In other words, u_j and $u_j^\wedge$ both converge in L^2. Clearly, $u_j^\wedge \to u^\wedge$. Indeed, initially we showed uniform convergence over $\mathbb{R}^n$, while the L^2-limit $z = \lim u_j^\wedge$ satisfies $(u^\wedge, \varphi) = \int \bar{z} \varphi dx$ for all $\varphi \in C_0^\infty$. This yields $\int (u^\wedge - z)\varphi dx = 0$ for all such φ, hence $\hat{u} = z$ (almost everywhere), since C_0^∞ is dense in L^2. Substituting $u=v=u_j$ in (1.13), letting $j \to \infty$, it follows that (1.8) is valid for all $u \in L^1 \cap L^2$, confirming Parseval's relation.

Clearly (1.13) also holds for all $u,v \in L^1 \cap L^2$. We use it to prove the Fourier inversion Let n=1. For $v \in L^1 \cap L^2$, $u = \chi_{[0,x_0]}$, some $x_0 > 0$ apply (1.13). Confirm by calculation of the integral that

$$(1.14) \qquad (2\pi)^{1/2} u^\wedge(x) = (e^{-ixx_0} - 1)/(-ix) = h_{x_0}(x) \text{ , } \quad x \neq 0 \text{ ,}$$

hence

$$(1.15) \qquad \int_0^{x_0} v(x)dx = \int \not{d}x v^\wedge(x) \bar{h}_{x_0}(x) dx \quad .$$

The Fourier inversion formula is a matter of differentiating (1.15) for x_0 under the integral sign, assuming that this is legal Consider the difference quotient:

$$(1.16) \qquad (2\delta)^{-1} \int_{x_0 - \delta}^{x_0 + \delta} v(x)dx = \int \not{d}x v^\wedge(x) e^{ixx_0} \frac{\sin \delta x}{\delta x} .$$

Assuming only that v, $v^\wedge$ both are in L^1, it follows indeed that

$$(1.17) \ \lim_{\delta \to 0} (2\delta)^{-n} \int_{Q_{x_0,\delta}} v(x)dx = \int \not{d}x v^\wedge(x) e^{ixx_0} = (v^\wedge)^\vee(x_0), \ x_0 \in \mathbb{R}^n.$$

(Actually, our proof works for n=1, $x_0 > 0$ only, but can easily be extended to all x_0, and general n. One must replace the derivative d/dx_0 by a mixed derivative $\partial^n/(\partial x_{0_1} \ldots \partial x_{0_n})$.) Indeed, letting $\delta\to 0$ in (1.17) we obtain (1.15), using that $\sin(\delta x)/(\delta x) \to 1$ uniformly on compact sets, and boundedly on $\mathbb{R}$, as $\delta\to 0$.

If v is continuous at x_0 then clearly the left hand side of (1.17) equals $v(x_0)$, giving the Fourier inversion formula, as it is well known. For $n=1$, if v has a jump at x_0 then the left hand side of (1.17) equals the mean value $(v(x_0+0)+v(x_0-0))/2$.

Again for $n=1$ a limit of (1.16), as $\delta\to 0$ exists, if only

$$\lim_{\alpha\to\infty}\int_{-\alpha}^{+\alpha} v^\wedge(x)đx \quad , \tag{1.18}$$

the principal value, exists (cf. pbm.6), without requiring $v^\wedge\in L^1$.

We summarize our results thus far:

<u>Proposition 1.1</u>. The Fourier transform $u^\wedge$ of (1.1) and its complex conjugate $u^\vee = (\bar{u}^\wedge)^-$ are defined for all $u\in L^1(\mathbb{R}^n)$, and we have $u^\wedge$, $u^\vee \in CO(\mathbb{R}^n)$. For $u\in L^1(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ we have Parseval's relation (1.8). If both $u\in L^1(\mathbb{R}^n)$ and $u^\wedge\in L^1(\mathbb{R}^n)$ hold, then we have $u^{\wedge\vee}(x) = u^{\vee\wedge}(x) = u(x)$ for almost all $x\in\mathbb{R}^n$.

It is known that the Banach space $L^1(\mathbb{R}^n)$ is a commutative Banach algebra under the convolution product

$$u*v = w \ , \ w(x) = \int đy\, u(x-y)w(y) = \int đy\, v(x-y)u(y) \ . \tag{1.19}$$

Indeed,

$$\|w\|_{L^1} = \int |w(x)|dx \le \kappa_n \int dx\int dy|u(x-y)||v(y)| = \kappa_n\|u\|_{L^1}\|v\|_{L^1} \ . \tag{1.20}$$

Prop.1.2, below, clarifies the role of the Fourier transform F for this Banach-algebra: F provides the Gelfand homomorphism.

<u>Proposition 1.2</u>. For $u,v\in L^1$ let $w = u*v$. Then we have

$$w^\wedge(\xi) = u^\wedge(\xi)v^\wedge(\xi) \ , \ \xi\in\mathbb{R}^n \ . \tag{1.21}$$

<u>Proof</u>. We have

$$w^\wedge(\xi) = \int đx e^{-ix\xi}\int đy\, u(x-y)v(y) = \int đy e^{-iy\xi}\int dx\, u(x-y)e^{-i(x-y)\xi} \ .$$

The substitution $x-y=z$, $dy=dz$ thus confirms (1.21), q.e.d.

The importance of the Fourier transform for PDE's hinges on

<u>Proposition 1.3</u>. If $u^{(\beta)}\in L^1$ for all $\beta\le\alpha$ then

$$u^{(\alpha)\wedge}(\xi) = i^{|\alpha|}\xi^\alpha u^\wedge(\xi) \ , \ \xi\in\mathbb{R}^n \ . \tag{1.22}$$

Proof. Partial integration gives $\int dx e^{-ix\xi} u^{(\alpha)}(\xi) = (-1)^{|\alpha|} \int dx \partial_x^\alpha (e^{ix\xi})$ (with vanishing boundary integrals), implying (1.21), q.e.d.

Given a linear partial differential equation

$$(1.23) \qquad P(D)u = f \quad , \; P(D) = \sum_{|\alpha| \le N} a_\alpha D_x^\alpha \; ,$$

where $f \in L^1(\mathbb{R}^n)$, $D_{x_j} = -i\partial_{x_j}$, one might attempt to find solutions by taking the Fourier transform. With proper assumptions (1.21) gives

$$(1.24) \qquad P(\xi) u^\wedge(\xi) = f^\wedge(\xi) \; .$$

Assuming that $e = (\frac{1}{P(x)})^\vee$ exists, (1.24) will assume the form

$$(1.25) \qquad u^\wedge(\xi) = e^\wedge(\xi) f^\wedge(\xi) \; ,$$

which by prop.1.2 (and Fourier inversion) is equivalent to

$$(1.26) \qquad u(x) = \int đy\, e(x-y) f(y) \; .$$

Presently, (1.26) can only have a formal meaning, since normally $(1/P) \notin L^1$, or $f \notin L^1$, or $u \notin L^1$, in practical applications.

However, as to be discussed in the sections below, the Fourier transform may be extended to more general classes of functions and to generalized functions. Then (1.26) yields a powerful tool for solving problems in constant coefficient PDE's (cf. sec.4).

Problems. 1) For n=1 obtain the Fourier transforms of the functions a) $(a^2+x^2)^{-1}$, $a>0$; b) $(\sin^2 ax)/x^2$, $a>0$; c) $1/\cosh x$.

2) For general n obtain the Fourier transform of $e^{-\alpha x^2}$, $\alpha>0$.
3) Obtain the Fourier transform of $f(x) = (1+|x|^2)^{-\nu}$, where $\nu>n/2$ (Hint: A knowledge of Bessel functions is required for this problem). 4) Construct a function $f(x) \in L^1(\mathbb{R}^n)$ such that $f^\wedge \notin L^1$.
5) The Riemann-Lebesgue lemma states that $f^\wedge \in CO$ whenever $f \in L^1$. Is it true that even $f^\wedge(x) = O(\langle x \rangle^{-\varepsilon})$ for each $f \in L^1$ with some $\varepsilon>0$? 6) Combining some facts, derived above, show that, for n=1, every piecewise smooth function $f(x) \in L^1(\mathbb{R})$ has a Fourier transform satisfying $f(x) = O(1/x)$, as $|x|$ is large, and satisfying

$$(1.27) \qquad (f(x+0)+f(x-0))/2 = \lim_{\alpha \to \infty} \int_{-\alpha}^{\alpha} đy\, e^{ixy} f^\wedge(y) \; , \; x \in \mathbb{R} \; .$$

Here 'piecewise smooth' means, that $\mathbb{R}$ may be divided into finitely many closed subintervals in each of which f is C^1 , possibly after changing its value at boundary points.

2. Fourier analysis for temperate distributions on $\mathbb{R}^n$.

We assume that the reader is familiar with the concept of distribution, as a continuous linear functional on the space $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$. A linear functional $f:\mathcal{D} \to \mathbb{C}$ is said to be continuous if $\langle f,\varphi_j\rangle \to 0$ whenever $\varphi_j \to 0$ in $\mathcal{D}$. The latter means that (i) $f_j \in \mathcal{D}$, $j=1,2,\ldots$, (ii) supp $\varphi_j \in K \subset\subset \mathbb{R}^n$, K independent of j, (iii) $\sup\{|\varphi^{(\alpha)}(x)|: x\in \mathbb{R}^n\} \to 0$, as $j\to\infty$, for every α. The space of distributions on $\mathbb{R}^n$ is called $\mathcal{D}'=\mathcal{D}'(\mathbb{R}^n)$. The space $L^1_{loc}(\mathbb{R}^n)$ of locally integrable functions is naturally imbedded in $\mathcal{D}'$ by defining

$$(2.1) \qquad \langle f,\varphi\rangle = \int f(x)\varphi(x)dx \ , \text{ for } f\in L^1_{loc} \ .$$

The derivatives $f^{(\alpha)}=\partial_x^\alpha f$ of a distribution $f \in \mathcal{D}'$ are defined by

$$(2.2) \qquad \langle f^{(\alpha)},\varphi\rangle = (-1)^{|\alpha|}\langle f,\varphi^{(\alpha)}\rangle \ , \ \varphi \in \mathcal{D} \ ,$$

the product of a distribution $f \in \mathcal{D}'$ and a $C^\infty(\mathbb{R}^n)$ function g by

$$(2.3) \qquad \langle gf,\varphi\rangle = \langle f,g\varphi\rangle \ , \ \varphi \in \mathcal{D} \ .$$

Thus Lf is defined for any distribution $f\in \mathcal{D}'(R^n)$ and linear differential operator $L=\sum a_\alpha \partial_x^\alpha$ with coefficients $a_\alpha(x) \in C^\infty(\mathbb{R}^n)$.

While the value f(x) of a distribution at a point x is a meaningless concept, one may talk about the restriction $f|\Omega$ of $f\in \mathcal{D}'(\mathbb{R}^n)$ to an open subset Ω , and its properties: First of all, the space $\mathcal{D}'(\Omega)$ of distributions over Ω consists of the continuous linear functionals on $\mathcal{D}(\Omega)=C_0^\infty(\Omega)$, with continuity defined as for $\mathbb{R}^n$. For $f\in \mathcal{D}'(\mathbb{R}^n)$, the restriction $f|\mathcal{D}(\Omega)$ defines a distribution of $\mathcal{D}'(\Omega)$, denoted by $f|\Omega$. Thus, for example, it is meaningful to say that $f\in \mathcal{D}'(\mathbb{R}^n)$ is a function (a $C^k(\Omega)$-function, etc.) in an open set $\Omega\subset \mathbb{R}^n$ - it means that $f|\Omega$ has this property. For a distribution $f\in \mathcal{D}'(\Omega)$ on an open set the derivatives and product with $g\in C^\infty(\Omega)$ is defined as in (2.2) , (2.3) . The support supp f (singular support sing supp f) of $f\in \mathcal{D}'$ is defined as the smallest closed set E (intersection of all closed sets E) such that f=0 (such that f is C^∞) in the complement of E .

The concept of Fourier transform can be generalized to distributions on $\mathbb{R}^n$, with multiple benefit: Some non-L^1-functions will get distributions as Fourier transforms. Certain distributions will get functions as Fourier transforms. The Fourier inversion formula and many assumptions (limit interchanges) will simplify.

We used the Fourier integral of (1.1) only for $u \in L^1(\mathbb{R}^n)$. It is practical to introduce a growth restriction for $u \in \mathcal{D}'(\mathbb{R}^n)$ if we want $u^\wedge$ to be a distribution again. Later on (sec.3) we also define $u^\wedge$ for general $u \in \mathcal{D}'(\mathbb{R}^n)$, but it no longer will be a distribution in $\mathcal{D}'(\mathbb{R}^n)$. We follow [$\mathrm{Schw}_1$] here, but [GS] in sec.3.

The growth restriction is imposed by requesting that $u \in \mathcal{D}'$ allows an extension to a larger space of testing functions called $\mathcal{S}$. Here $\mathcal{S}$ - the space of rapidly decreasing functions- consists of all $\varphi \in C^\infty(\mathbb{R}^n)$ such that for all multi-indices α and $k=1,2,\ldots$,

$$\varphi^{(\alpha)}(x) = O(\langle x\rangle^{-k}) \ . \tag{2.4}$$

- the derivatives of φ decay faster than every power $\langle x\rangle^{-k}$.

Note that, equivalently, we could have prescribed that for every α one (and the same) of the following conditions be satisfied:

$$\langle x\rangle^k u^{(\alpha)}(x) \text{ (for every } k=1,2,..\text{), or } x^\beta u^{(\alpha)}(x) \text{ (for every } \beta\text{)},$$

$$\text{or } (x^\beta u(x))^{(\alpha)} \text{ (for every } \beta\text{), is } O(1) \text{ , or is } o(1) \text{ , or is} \tag{2.5}$$

$$CB \text{ , or } CO \text{ , or } L^2 \text{ , or } L^p \text{ (for some } 1\le p\le\infty) \ .$$

Indeed, for a given α one of these conditions may be weaker or stronger than the other. However for <u>all</u> α simultaneously all conditions are equally strong. One must use Leibniz' formula to handle interchanges of ∂_x^α and multiplications (cf. lemma 2.8).

The above at once gives the following:

<u>Theorem 2.1</u>. We have $\mathcal{S} \subset L^1(\mathbb{R}^n)$, so that $u^\wedge$ of (1.1) (and $u^\vee$) are defined on $\mathcal{S}$. Moreover, for $u \in \mathcal{S}$, we have $u^\wedge, u^\vee \in \mathcal{S}$, and

$$(u^\wedge)^\vee(x) = (u^\vee)^\wedge(x) = u(x) \ , \ x \in \mathbb{R}^n \ . \tag{2.6}$$

The Fourier transform and its conjugate therefore define bijective linear maps $\mathcal{S} \leftrightarrow \mathcal{S}$, inverting each other.

<u>Proof</u>. Using repeated partial integration and $x^\alpha e^{-ix\xi} = i^{|\alpha|}\partial_\xi^\alpha e^{-ix\xi}$, $\partial_x^\beta e^{-ix\xi} = i^{-|\beta|}\xi^\beta e^{-ix\xi}$ get $\int dx e^{-ix\xi} x^\alpha u^{(\beta)}(x) = i^{|\alpha|}\partial_\xi^\alpha \int dx e^{-ix\xi} u^{(\beta)}(x)$

$= i^{|\alpha|+|\beta|}\partial_\xi^\alpha \xi^\beta \int dx e^{-ix\xi} u(x)$, hence

$$(x^\beta u^{(\alpha)})^\wedge(\xi) = i^{|\alpha|+|\beta|}(\xi^\alpha u^\wedge(\xi))^{(\beta)} \ . \tag{2.7}$$

In fact, we get $x^\beta u^{(\alpha)} \in L^1$, for every α,β , by the equivalence

(2.5) , for $u \in \mathcal{S}$. Therefore the right hand side is in CO , for every α,β , so that $u^\wedge \in \mathcal{S}$, again by the equivalence (2.5). Thus we get $u^\wedge \in \mathcal{S}$ for all $u \in \mathcal{S}$. Similarly for "$^\vee$" . Also, the Fourier inversion formula holds for $u \in \mathcal{S}$, and the left hand side of (1.17) equals v(x). This implies (2.5), also by taking complex conjugates. The bijectivity then follows at once, q.e.d.

Following Schwartz we introduce distributions with controlled growth at infinity - so called <u>temperate distributions</u> - over $\Omega=\mathbb{R}^n$ as <u>continuous linear functionals over</u> $\mathcal{S}$. The space of all temperate distributions is denoted by $\mathcal{S}'$. Clearly, $\mathcal{S} \supset \mathcal{D}$, so that a functional u over $\mathcal{S}$ induces a functional over $\mathcal{D}$ - its restriction $u|\mathcal{D}$.

<u>Definition 2.2</u>. A sequence of functions $\varphi_j \in \mathcal{S}$ is said to converge to 0 (in $\mathcal{S}$) if for every multi-index α and $k = 0,1,2,...$ the sequence $\langle x\rangle^k \varphi_j^{(\alpha)}(x)$ converges to zero uniformly for all $x \in \mathbb{R}^n$.

<u>Definition 2.3</u>. A linear functional u over $\mathcal{S}$ is said to be continuous if $\varphi_j \in \mathcal{S}$, $\varphi_j \to 0$ in $\mathcal{S}$ implies $\langle u,\varphi_j\rangle \to 0$.

<u>Temperate distributions are distributions</u>. More precisely speaking: For $u\in \mathcal{S}'$ the restriction $u|\mathcal{D}$ determines u uniquely, and $u|\mathcal{D} \in \mathcal{D}'(\mathbb{R}^n)$. To confirm this we must prove:

<u>Lemma 2.4</u>. a) If $\varphi_j \in \mathcal{D}$, $\varphi_j \to 0$ in $\mathcal{D}$, then we also have $\varphi_j \to 0$ in $\mathcal{S}$.
b) For $\varphi \in \mathcal{S}$ there exists a sequence $\varphi_j \in \mathcal{D}$ such that $\varphi-\varphi_j \to 0$ in $\mathcal{S}$.

From lemma 2.4 it follows that for $u\in \mathcal{S}'$ the restriction $v=u|\mathcal{D}$ is continuous over $\mathcal{D}$: If $\varphi_j \to 0$ in $\mathcal{D}$, then $\varphi_j \to 0$ in $\mathcal{S}$ (by (a)), hence $\langle v,\varphi_j\rangle = \langle u,\varphi_j\rangle \to 0$. Hence $v\in \mathcal{D}'$. Furthermore, if u, $w\in \mathcal{S}'$ have $u|\mathcal{D}=w|\mathcal{D}=v\in \mathcal{D}'$, then for $\varphi\in \mathcal{S}$ let φ_j be a sequence of (b) above. Get $u-w\in \mathcal{S}'$, $\langle u-w,\varphi-\varphi_j\rangle \to 0$. Hence $0=\langle u-w,\varphi_j\rangle = \langle v-v,\varphi_j\rangle \to \langle u-w,\varphi\rangle$, implying that $\langle u,\varphi\rangle = \langle v,\varphi\rangle$ for all $\varphi\in \mathcal{S}$, or u=v, so that indeed $u\in \mathcal{S}'$ is uniquely determined by its restriction $v=u|\mathcal{D}\in \mathcal{D}'$.

<u>Proof of lemma 2.4</u>. (a): If $\varphi_j \in \mathcal{D}$, $\varphi_j \to 0$ in $\mathcal{D}$ then $\operatorname{supp} \varphi_j^{(\alpha)} \subset K \subset\subset \mathbb{R}^n$, while the functions $\langle x\rangle^k$ are bounded in K. Thus the uniform convergence $\langle x\rangle^k \varphi_j^{(\alpha)}(x) \to 0$ in $\mathbb{R}^n$ follows from the uniform convergence $\varphi_j^{(\alpha)}(x) \to 0$ in $\mathbb{R}^n$, and we have $\varphi_j \to 0$ in $\mathcal{S}$, proving (a).

To prove (b), let $\chi(x)\in C_0^\infty(\mathbb{R}^n)$ satisfy $\chi(x)=1$ near 0. For a $\varphi\in \mathcal{S}$ define $\varphi_j(x)=\varphi(x)\chi(x/j)$, $j=1,2,...$, so that $\varphi_j\in \mathcal{D}$. Setting $\omega_j(x)=1-\chi(x/j)$, get $\psi_j=\varphi-\varphi_j=\varphi\omega_j=0$ in $|x|\le 1$ for large j. Note, $\langle x\rangle^k \psi_j^{(\alpha)}$ is a linear combination of $\theta_{\beta\gamma,j} = \langle x\rangle^k \varphi^{(\beta)} \omega_j^{(\gamma)}$, $\beta+\gamma=\alpha$,

where $\sup\{|\theta_{\beta\gamma}{}^{,j}(x)| : x \in \mathbb{R}^n\} \le \sup\{|\omega_j^{(\gamma)}|\} \sup\{\langle x\rangle^k \varphi^{(\beta)} : |x|\ge l\}$. Since $\varphi\in \mathcal{S}$ the second sup at right goes to 0 as $l\to\infty$ (i.e.,as $j\to\infty$). Also, $\sup\{|\omega_j^{(\gamma)}|\}=j^{-|\gamma|}\sup\{\omega(x):x\in \mathbb{R}^n\}\le c$. Thus $\psi_j \to 0$ in $\mathcal{S}$, q.e.d.

Note that polynomials, and delta functions $\delta^{(\alpha)}(x-a)$ are examples of temperate distribution. However, $e^x \notin \mathcal{S}(\mathbb{R})$ (pbms.2,3).

To generalize F we still require the following.

Corollary 2.5. The transforms F and $\bar{F}$ both have the property that

$\varphi_j \in \mathcal{S}$, $\varphi_j \to 0$ in $\mathcal{S}$ implies $F\varphi_j \to 0$ $\bar{F}\varphi_j \to 0$ in $\mathcal{S}$.

It is sufficient to prove this for F. Again we need an equivalence like (2.2), now for the property '$\varphi_j \to 0$ in $\mathcal{S}$' :

Proposition 2.6. Let $\varphi_j \in \mathcal{S}$, $j=1,\ldots$. Then '$\varphi_j \to 0$ in $\mathcal{S}$' is equivalent to each of the following conditions:

$$\langle x\rangle^k \varphi_j^{(\alpha)} \to 0 \ , \text{ or } x^\beta \varphi_j^{(\alpha)} \to 0 \ , \text{ or } (x^\beta \varphi_j)^{(\alpha)} \to 0$$

(2.8) for all multi-indices α , β , or $k=0,1,2,\ldots$, in one (and the same) of the norms of $CB(\mathbb{R}^n)$ or $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$.

For the proof cf. lemma 2.8.

Using prop.2.6, lemma 2.5 is a matter of (1.2), and (2.7). Indeed, if $\varphi_j \to 0$ in $\mathcal{S}$, we have $\|x^\beta \varphi_j^{(\alpha)}\|_{L^1} \to 0$, $j \to \infty$, hence $\|(x^\alpha \varphi_j^\wedge)^{(\beta)}\|_{CB} \to 0$, implying $\varphi_j^\wedge \to 0$, q.e.d.

For a given $u \in \mathcal{S}'$, observe that $u^\wedge$, defined by

$$\langle u^\wedge, \varphi\rangle = \langle u, \varphi^\wedge\rangle \ , \ \varphi \in \mathcal{S} \ , \tag{2.9}$$

defines a functional in $\mathcal{S}'$, since $\varphi_j \to 0$ in $\mathcal{S}$ implies $\varphi_j^\wedge \to 0$ in $\mathcal{S}$ (by cor.2.5) , hence $\langle u, \varphi^\wedge\rangle \to 0$. If $u \in L^1(\mathbb{R}^n)$ then it follows that $u \in \mathcal{S}'$ (cf. pbm.3). In that case we have

$$\langle u, \varphi^\wedge\rangle = \int đx\, u(x) \int đ\xi\, \varphi(\xi) e^{-ix\xi} = \langle \int đ\xi\, e^{-ix\xi} u(\xi), \varphi\rangle \ , \ \varphi \in \mathcal{S} \ , \tag{2.10}$$

by Fubini's theorem, since the integrand is $L^1(\mathbb{R}^{2n})$. Thus, for $u \in L^1$, (2.10) implies that the functional (2.9) coincides with that of the Fourier transform $u^\wedge$ of (1.1). Accordingly, <u>for a general</u> $u \in \mathcal{S}'$ <u>we define the Fourier transform</u> $u^\wedge$ as the functional of (2.9) and the conjugate Fourier transform $u^\vee$ by

$$\langle u^\vee, \varphi\rangle = \langle u, \varphi^\vee\rangle \ , \ \varphi \in \mathcal{S} \ . \tag{2.11}$$

It is clear at once that we have

Theorem 2.7. The (conjugate) Fourier transform coincides with the (conjugate) Fourier transform previously defined for L^1-functions (cf.(1.1), and (1.7)). We have the Fourier inversion formula

$$(2.12)\qquad (u^\wedge)^\vee = (u^\vee)^\wedge = u \text{ , for all } u \in \mathcal{S}' .$$

Also, for $u \in \mathcal{S}'$ we have $x^\alpha u^{(\beta)} \in \mathcal{S}'$, and (2.7) holds as well.

Prop.2.6 and (2.2) follow from the (evident) lemma, below.

<u>Lemma 2.8</u>. a) We have (using Leibniz' formula and its adjoint)

$$(2.13)\quad (x^\alpha u)^{(\beta)} = \sum c_{\alpha\beta\gamma} x^{\alpha-\gamma} u^{(\beta-\gamma)} \text{ , } x^\alpha u^{(\beta)} = \sum d_{\alpha\beta\gamma} (x^{\alpha-\gamma} u)^{(\beta-\gamma)} \text{ ,}$$

with finite sums and constants $c_{\alpha\beta\gamma}$, $d_{\alpha\beta\gamma}$.

b) We have

$$(2.14)\qquad |x^\alpha| \le \langle x\rangle^{|\alpha|} \text{ , and } \langle x\rangle^k \le c_k \sum_{|\alpha|\le k} |x^\alpha| \text{ with a constant } c_k .$$

c) We have

$$(2.15)\qquad \|u\|_{L^p} \le \|\langle x\rangle^{-k}\|_{L^p} \|\langle x\rangle^k u\|_{L^\infty} \text{ , } 1\le p<\infty \text{ , } k>n/p .$$

d) We have

$$(2.16)\qquad \|u\|_{L^\infty} \le c\|u^\wedge\|_{L^1} \le c\|(1+|x|)^{n+1} u^\wedge\|_{L^\infty} \le c \sum_{|\alpha|\le n+1} \|x^\alpha u^\wedge\|_{L^\infty}$$

$$= c \sum_{|\alpha|\le n+1} \|(u^{(\alpha)})^\wedge\|_{L^\infty} \le c \sum_{|\alpha|\le n+1} \|u^{(\alpha)}\|_{L^1} .$$

<u>Problems</u>: 1) Show that the following functionals define distributions in $D'(\mathbb{R}^n)$: a) $\langle f,\varphi\rangle = \varphi^{(\alpha)}(x^0)$, for given multiindex α and $x^0 \in \mathbb{R}^n$; b) $\langle f,\varphi\rangle = \int_{|x|=1} \varphi(x)dS$, dS=surface measure ; c) $\langle \text{p.v.}\frac{1}{x},\varphi\rangle = \lim_{\eta\to 0} \int_{|x|\ge\eta} \varphi(x)\frac{dx}{x}$ (for n=1). 2) Obtain the first partials of the distributions of pbm.1. 3) Show that distributions $f_\pm \in \mathcal{D}'(\mathbb{R})$ are defined by $\langle f_\pm,\varphi\rangle = \lim_{\varepsilon\to 0,\varepsilon>0} \int_{-\infty}^{\infty} \varphi(x)\frac{dx}{x\pm i\varepsilon}$. Relate $f_\pm$ with p.v.$\frac{1}{x}$ of pbm.1. 4) The distribution derivative satisfies Leibniz' formula and its adjoint (cf. [C₁],I,(1.23)). 5) Show that a distribution $f \in \mathcal{D}'(\Omega)$ with $f^{(\alpha)} \in C(\Omega)$, $|\alpha|\le k$ is a function in $C^k(\Omega)$. 6) Let L^1_{pol} be the class of all $u \in L^1_{loc}(\mathbb{R}^n)$ with $\langle x\rangle^{-k} u \in L^1(\mathbb{R}^n)$ for some $k = k(u)$. Show that $L^1_{pol} \subset \mathcal{S}'$. 7) Show that $p(x) = \sum_{|\alpha|\le m} a_\alpha x^\alpha \in L^1_{pol} \subset \mathcal{S}'$. Also that $CB(\mathbb{R}^n) \subset L^1_{pol}$, and $L^p(\mathbb{R}^n) \subset L^1_{pol}$, $1\le p\le\infty$. 8) Show that e^{ax}

$\in \mathcal{D}'(\mathbb{R})$, but $e^{ax} \notin \mathcal{S}'$, as Re a $\neq 0$. 9) Let $\mathcal{T}_{pol}$ be the class of all $a \in C^\infty(\mathbb{R}^n)$ with $a^{(\alpha)}(x) = O(\langle x \rangle^{k_\alpha})$, for some $k_\alpha \in \mathbb{Z}$, for every α. Show that differentiation and multiplication by $a \in \mathcal{T}_{pol}$ leaves $\mathcal{S}'$ invariant. That is, for $u \in \mathcal{S}'$, $a \in \mathcal{T}_{pol}$, $\alpha \in \mathbb{Z}^n_+$ we have $au \in S'$, $u^{(\alpha)} \in \mathcal{S}'$. 10) Obtain the Fourier transform of the following distributions (If necessary, show, they are in $\mathcal{S}'$): a) x^α, $\alpha \in \mathbb{Z}^n_+$; b) $\delta_{x_0}^{(\beta)}$, $\beta \in \mathbb{Z}^n_+$; c) e^{iax}, $a \in \mathbb{R}^n$. 11) Obtain $(\text{p.v.}\frac{1}{x})^\wedge$, for the distribution of pbm.1. 12) Define a distribution $\text{p.v.}\frac{1}{\sinh x} \in \mathcal{S}'$, using the same kind of 'principal-value integral' as in pbm 1. Calculate $(\text{p.v.}\frac{1}{\sinh x})^\wedge$. 13) Obtain the Fourier transform of a 2π-periodic $C^\infty(\mathbb{R})$- function a(x). Hint: Use that a(x) has a uniformly convergent <u>Fourier</u> <u>series</u> $a(x) = \sum_{-\infty}^{\infty} a_m e^{imx}$, $a_m = \frac{1}{2\pi}\int_0^{2\pi} a e^{-imx} dx$

14) Let $f(x) = |\sin x|$. Show that $f \in \mathcal{S}'$ and evaluate $f^\wedge$.

<u>3. The Paley-Wiener theorem; Fourier transform of a general</u> $u \in \mathcal{D}'$.

The support of a distribution $u \in D'$ was defined as smallest closed set Q with u=0 in $\Omega \setminus Q$. We now consider u with supp $u \subset\subset \Omega$.

A simple but important remark is that a compactly supported distribution $u \in D'(\Omega)$, as linear functional over $\mathcal{D}(\Omega)$, admits a natural extension to the larger space $\mathcal{E} = C^\infty(\Omega)$. (The notation was introduced by Schwartz again.) Indeed, for a given $\chi(x) \in C_0^\infty(\Omega)$ with $\chi(x) = 1$ near supp u, define the extension of $\langle u,.\rangle$ to $\mathcal{E}$ by

$$(3.1) \qquad \langle u,\varphi\rangle = \langle u,\chi\varphi\rangle \ , \text{ for all } \varphi \in \mathcal{E}(\Omega) = C^\infty(\Omega) \ .$$

This defines an extension: if $\varphi \in \mathcal{D}(\Omega)$, then $(1-\chi)\varphi \in \mathcal{D}(\Omega)$, and supp $(1-\chi)\varphi \subset$ supp $(1-\chi)$ is disjoint from supp u, hence $\langle u,(1-\chi)\varphi\rangle = 0$, or, $\langle u,\varphi\rangle = \langle u,\chi\varphi\rangle$. The extension is independent of the choice of χ. If $\theta \in \mathcal{D}(\Omega)$ has the property of χ then $\tau - \chi = 0$ near supp u, $\Rightarrow$

$$(3.2) \qquad \langle u,\theta\varphi\rangle = \langle u,\chi\varphi\rangle \ , \text{ for all } \varphi \in \mathcal{E}(\Omega) \ .$$

The class of all distributions $u \in \mathcal{D}'(\Omega)$ with compact support is commonly denoted by $\mathcal{E}'(\Omega)$. We have seen that $\mathcal{E}'(\Omega)$ is naturally identified with a class of linear functionals on the space $\mathcal{E}(\Omega)$.

<u>Proposition 3.1</u>. The set $\mathcal{E}'(\Omega)$ of all (above extensions of) compactly supported $u \in \mathcal{D}'(\Omega)$ coincides with the set of continuous lin-

ear functionals over $E(\Omega)$ (i.e., the functionals u over $E(\Omega)$ such that $\varphi_j \in E$, $\varphi_j \to 0$ in E implies $\langle u,\varphi_j\rangle \to 0$). Here $\varphi_j \to 0$ in E means that $\varphi_j^{(\alpha)}(x) \to 0$ uniformly on compact sets of Ω, for all α .

Clearly the extension (3.1) to E of $u \in D'$ with supp $u \subset\subset \Omega$ is a continuous linear functional over E, in the above sense: If $\varphi_j \in E$, $\varphi_j \to 0$ in E , then $\chi\varphi_j \to 0$ in D, as a consequence of Leibniz' formula. Vice versa, for a continuous linear functional u over E the restriction $v=u|D$ is a distribution in D', since $\varphi_j \in D$, $\varphi_j \to 0$ in D trivially implies $\varphi_j \to 0$ in E. Prop.3.1 follows if we can show that supp $v \subset\subset \Omega$. Suppose not, then a sequence of balls B_j may be constructed such that $u \neq 0$ in B_j, while every set $K \subset\subset \Omega$ is disjoint from all but finitely many of the B^j. Construct $\varphi_j \in D$, supp $\varphi_j \subset B_j$ with $\langle u,\varphi_j\rangle = 1$. Observe that $\varphi_j \to 0$ in E while $\langle u,\varphi_j\rangle$ $=1$ does not tend to zero, a contradiction. Q.E.D.

For a compactly supported distribution on $\mathbb{R}^n$ we always have a Fourier transform in the sense of sec.2, i.e.,we get $E'(\mathbb{R}^n) \subset S'$:

<u>Theorem 3.2</u>. All compactly supported distributions over $\mathbb{R}^n$ are temperate. Moreover, for $u \in E' \subset S'$, $u^\wedge$ is a C^∞-function given by

$$(3.3) \qquad u^\wedge(x) = \int d\!\!{}^-\xi e^{-ix\xi} u(\xi) = \langle u, e_x\rangle \ , \ e_x(\xi) = e^{-ix\xi} \ ,$$

with a distribution integral, given by the third expression (3.3).

In fact, the function $u^\wedge(x)$ is entire analytic, in the n complex variables x_j, in the sense that $v(z)=\langle u,e_z\rangle$, $e_z(x)=e^{-izx}$, is meaningful for all $z \in \mathbb{C}^n$, (not only $\mathbb{R}^n$), and defines an extension of $u^\wedge$ of (3.3) to $\mathbb{C}^n$ having continuous partial derivatives in the complex sense with respect to each of the variables $z_1,\dots,z_n$.

Note that formula (3.3) is meaningful only by virtue of our extension (3.1) of $u \in E'$ to all of E .

<u>Proof</u>. For $u \in D'(\mathbb{R}^n)$, supp $u \subset\subset \mathbb{R}^n$, the natural extension to E may be restricted to S again to provide a continuous linear functional on S, since "$\varphi_j \to 0$ in S " implies "$\varphi_j \to 0$ in E". Hence $u \in S'$. The function $v(z)$ indeed is meaningful for all $z \in \mathbb{C}^n$. Existence of $\partial v/\partial z_j$ is a matter of the continuity of the functional u over E: For a fixed z , $h \in \mathbb{C}^n$, form the difference quotient

$$(3.4) \qquad w_\varepsilon = (v(z+\varepsilon h)-v(z))/\varepsilon = \langle u,(e_{z+\varepsilon h}-e_z)/\varepsilon\rangle \ , \ \varepsilon > 0 \ .$$

For the directional derivative $\nabla_h e_z$ of e_z at z , we get

$$(3.5) \qquad \psi_\varepsilon = (e_{z+\varepsilon h} - e_z)/\varepsilon - \nabla_h e_z \to 0 \text{ in } E \ ,$$

Indeed, this only means that $\partial_x^\alpha \psi_\varepsilon \to 0$ uniformly on $K \subset\subset \mathbb{R}^n$, as readily verified for $e_z(\xi)$. Continuity of u then implies

$$(3.6) \qquad \lim_{\varepsilon\to 0,\varepsilon\neq 0} w_\varepsilon = \langle u, \nabla_h e_z\rangle \;,$$

confirming that v(z) is analytic for all z. Formally we then get

$$(3.7) \qquad \langle u^\wedge, \varphi\rangle = \langle u, \int d\!\!\!/\xi e_\xi \varphi(\xi)\rangle = \int d\!\!\!/\xi \langle u, e_\xi\rangle \varphi(\xi) = \int d\!\!\!/\xi v(\xi)\varphi(\xi) \;,$$

with v(x) as defined, where the interchange of limit leading to the second equality remains to be confirmed. Clearly (3.7) implies $u^\wedge = v$, i.e., (3.3) and thm.3.2 follows. For the interchange of limit show existence of the improper Riemann integral $\int d\!\!\!/\xi e_\xi \varphi(\xi)$ in the sense of convergence in $\mathcal{E}$: For $K \subset\subset \mathbb{R}^n$ we must show that $\int_K d\!\!\!/\xi e_\xi \varphi(\xi) - S_k \to 0$ in $\mathcal{E}$, as $k\to\infty$. Here S_k is any sequence of Riemann sums, with maximum partition diameter tending to 0 as $k\to\infty$. Also, that $\int_{\mathbb{R}^n\setminus K} d\!\!\!/\xi e_\xi \varphi(\xi) \to 0$, as K runs through a sequence K_j with $\cup K_j = \mathbb{R}^n$, again, with convergence in $\mathcal{E}$. Again, convergence in $\mathcal{E}$ just means local uniform convergence with all derivatives. One confirms easily the local uniform convergence in the parameter x , since the function $e_\xi(x) = e^{-ix\xi}$ is continuous. Similarly for the x-derivatives, again continuous in x and ξ . This, and the fact that the x-derivatives of the Riemann sums are Riemann sums again, indeed allows to confirm the desired convergences. Q.E.D.

As examples for Fourier transforms of compactly supported distributions we mention those of the delta-function and its derivatives. As seen in 2,pbm.5 we get $\delta_0^{(\alpha)\wedge} = i^{|\alpha|}\kappa_n x^\alpha$. In fact, this is an immediate consequence of (3.3), above.

We observe that the entire analytic function $u^\wedge(z)$ of (3.3), as a function of complex arguments z , has a growth property which characterizes the Fourier transforms of compactly supported distributions. The result is called the Paley-Wiener theorem.

Theorem 3.3. An entire analytic function v(z) over $\mathbb{C}^n$ is the Fourier transform of a compactly supported distribution $u \in \mathcal{D}'(\mathbb{R}^n)$ if and only if there exists an integer $k > 0$ and a real $\eta > 0$ such that

$$(3.8) \qquad v(z) = O(\langle z\rangle^k e^{\eta|\mathrm{Im}\, z|}) \text{ for all } z\in\mathbb{C}^n \;,\; \langle z\rangle = (1+\sum|z_j|^2)^{1/2}.$$

Moreover, the constant η may be chosen as the radius of the smallest ball $|x|\le r$ containing supp u . Furthermore, $u \in \mathcal{D}(\mathbb{R}^n)$ if and only if (3.8) holds for all k with $\eta = \max\{|x| : x\in \text{supp } u\}$.

Proof. For $u \in \mathcal{E}'$ we must have

$$(3.9)\qquad |\langle u,\varphi\rangle| \le c\,\sup\{|\varphi^{(\alpha)}(x)|: x \in K\ ,\ |\alpha|\le k\}\ .$$

for some c, k, and some compact $K \supset \operatorname{supp} u$ and all $\varphi\in \mathcal{E}$. Otherwise for every $c=k=j$ and $|x|\le j$ there exists $\varphi=\varphi_j\in \mathcal{E}$ with $\langle u,\varphi_j\rangle=1$, and ">" holds in (3.9). Or, $|\varphi_j^{(\alpha)}(x)|\le \frac{1}{j}$ for all $|\alpha|\le j$, $|x|\le j$, $j=1, 2,\dots$, implying uniform convergence $\varphi_j^{(\alpha)}(x)\to 0$, $j\to\infty$, a contradiction, since $1=\langle u,\varphi_j\rangle$ does not tend to 0.

We get $u^\wedge(z)=\langle u,\chi_z e_z\rangle$, $\chi_z=\chi(|z|(|x|-\eta))$ where $\chi\in C^\infty(\mathbb{R})$, $\chi(t)=1$, $t<\frac{1}{2}$, $=0$, $t>1$, χ decreasing. It follows that $\operatorname{supp}\chi_z \subset \{|x|\le\eta+\frac{1}{|z|}\}$ so that $(\chi_z e_z)^{(\alpha)}(x)= O(e^{\eta|\mathrm{Im} z|+1}\langle z\rangle^k)$. Combining this with (3.9) we get (3.8) with the proper constant η .

Next assume $u\in C_0^\infty(\Omega)$. We trivially get (3.9) with k=0 and $K=\operatorname{supp} u \subset\subset \Omega$, since $u\in L^1$. Similarly for $u^{(\alpha)}$. Accordingly, for all α we get $|z^\alpha u^\wedge(z)|=|\langle u,e_z^{(\alpha)}\rangle|=O(e^{\eta|\mathrm{Im}\ z|})$, hence (3.8) for all k.

Vice versa, (3.8) for all k implies $x^\alpha v|\mathbb{R}^n\in L^1 \subset \mathcal{S}'$. Then $v^\vee$ is given by the conjugate Fourier integral. We have $u=(v|\mathbb{R}^n)^\vee \in CO$, and even $u^{(\alpha)}\in CO$, i.e., $u\in C^\infty(\mathbb{R}^n)$. To show that $\operatorname{supp} u\subset\subset \mathbb{R}^n$ write

$$(3.10)\qquad u(x) = \int đ\xi e^{ix\xi}v(\xi)\ .$$

If $\theta \in \mathbb{R}^n$ is given arbitrary then we also may write (3.10) as

$$(3.11)\qquad u(x) = \int đ\xi e^{ix(\xi+i\theta)}v(\xi+i\theta)\ .$$

Indeed, this is a matter of Cauchy's integral theorem, applied for a rectangle in the complex ζ_j-plane with sides $\mathrm{Re}\ \zeta_j=\pm A$, $\mathrm{Im}\ \zeta_j=0$ or θ_j. In such a rectangle the integrand $e^{ix\zeta}v(\zeta)$ is holomorphic as a function of ζ_j for constant other variables, so that the complex integral over the boundary vanishes. For $A\to\infty$ the integrals over $\mathrm{Re}\ \zeta_j=\pm A$, $0<\mathrm{Im}\ \zeta_j<\theta_j$, tend to zero, in view of (3.8) for k=-2 for example. The integration pathes have length θ_j and the integrand is $O(\langle A\rangle^{-2}e^{(\eta-x)|\theta|})$. The integral (3.10) may be written as n-fold iterated integral over $\mathbb{R}$. The above proceedure allows the transfer of the integration from $\mathbb{R}$ to the line $\{\xi_j+i\theta_j: x_j\in \mathbb{R}\}$.

Next let us estimate (3.11):

$$(3.12)\qquad u(x) = O(\int đ\xi e^{\eta|\theta|-x\theta}\langle\xi+i\theta\rangle^{-k}) = O(e^{\eta|\theta|-x\theta})\ ,$$

setting k=n+1 (it holds for every k), and using that $\langle\xi+i\theta\rangle \ge \langle\xi\rangle$. The 'O(.)- constant' is independent of θ. Hence we can set $\theta=tx$, $t>0$, for $u(x)=O(e^{t|x|(\eta-|x|)})$. The exponent is <0 as $|x|>\eta$, and

$u(x)=0$ follows. Thus supp $u \subset \{|x| \le \eta\}$, $u \in \mathcal{D}$, if (3.8) for all k.

Finally, if (3.8) holds for some k, let $\chi(x) \in \mathcal{D}$, supp $\chi \subset \{|x| \le 1\}$, $\chi(x) \ge 0$, $\int \chi(x) dx = 1$. For $\varepsilon > 0$ let $\chi_\varepsilon(x) = \varepsilon^{-n}\chi(\frac{x}{\varepsilon})$. Note that $\chi_\varepsilon{}^\wedge(\xi) = \chi^\wedge(\varepsilon\xi) \to \chi^\wedge(0) = 1$, as $\varepsilon \to 0$. Moreover, for any $\varphi \in \mathcal{S}$ get $\varphi\chi_\varepsilon{}^\wedge - \varphi \to 0$ in $\mathcal{S}$. Since supp $\chi_\varepsilon \subset \{|x| \le \varepsilon\}$ we have (3.8) for $\chi_\varepsilon{}^\wedge(\zeta)$ with η replaced by ε for all k. Hence the product $v(z)\chi_\varepsilon{}^\wedge(z)$ satisfies (3.8) with η replaced by $\eta+\varepsilon$ again for all k. It follows that $v\chi_\varepsilon{}^\wedge = u_\varepsilon{}^\wedge$ with $u_\varepsilon \in \mathcal{D}$, supp $u_\varepsilon \subset \{|x| \le \eta+\varepsilon\}$. Also $\langle u_\varepsilon, \varphi\rangle = \langle u_\varepsilon{}^\wedge, \varphi^\vee\rangle = \langle v, \chi_\varepsilon{}^\wedge \varphi^\vee\rangle \to \langle v, \varphi^\vee\rangle = \langle u, \varphi\rangle$, for all $\varphi \in \mathcal{S}$. The latter implies that supp $u \subset \{|x| \le \eta+\varepsilon\}$, all $\varepsilon > 0$. It follows that $u \in \mathcal{E}$, supp $u \subset \{|x| \le \eta\}$, q.e.d.

Let $\mathcal{Z}$ denote the space of all entire analytic functions $v(z) = v(z_1, \ldots, z_n)$ in n complex variables such that for $k=0,1,2,\ldots$, and some $\eta \ge 0$ we have (3.8) satisfied. We shall say that a sequence $v_j \in \mathcal{Z}$ tends to 0 in $\mathcal{Z}$ if (i) estimates (3.8) hold with constants independent of j, and (ii) $m_j = \mathrm{Max}\{|v_j(x)| : x \in \mathbb{R}^n\} \to 0$, as $j \to \infty$.

Corollary 3.4. The Fourier transform $F: u \to u^\wedge$ establishes a linear bijection $\mathcal{D} \leftrightarrow \mathcal{Z}$ which is continuous in either direction, in the sense that $u_j \to 0$ in $\mathcal{D}$ holds if and only if $u_j{}^\wedge \to 0$ in $\mathcal{Z}$.

Proof. After thm.3.3 we focus on continuity only. If $u_j \to 0$ in $\mathcal{D}$ then supp $u_j \subset \{|x| \le a\}$ for a independent of j. This yields (3.8) with $\eta = a$ independent of j, by thm.3.3. Inspecting the first part of thm.3.3's proof we also find the O(.) constant independent of j.

Vice versa, if $v_j \to 0$ in $\mathcal{Z}$, then (3.8) with η independent of j implies supp $u_j{}^\vee \subset \{|x| \le \eta\}$. But (3.8), for real $z=x$, implies $v_j = O(\langle x\rangle^{-k})$, uniformly in x and j, for every k. Thus conclude from cdn.(ii) that $\|x^\alpha v_j\|_{L^1} \to 0$, as $j \to \infty$. For the inverse Fourier transform $u_j = v_j{}^\wedge$ we get $\|u_j^{(\alpha)}\|_{L^\infty} \to 0$, so that indeed $u_j \to 0$ in $\mathcal{D}$. Q.E.D.

Following [GS] we now define a Fourier transform of a general distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ regardless of growth at infinity, as a continuous linear functional $f^\wedge : \mathcal{Z} \to \mathbb{C}$. Here of course "$f^\wedge$ continuous" means that "$\langle f^\wedge, \varphi_j\rangle \to 0$, whenever $\varphi_j \to 0$ in $\mathcal{Z}$". We define $f^\wedge$ by

$$(3.13) \qquad \langle f^\wedge, \varphi\rangle = \langle f, \varphi^\wedge\rangle, \text{ for all } \varphi \in \mathcal{Z},$$

taking into account that $\varphi^\wedge = (\bar{\varphi}^\vee)^- \in \mathcal{D}$ for $\varphi \in \mathcal{Z}$.

This definition is compatible with the earlier ones. Indeed, we have $\mathcal{Z} \subset \mathcal{S}$, in the sense that for $u \in \mathcal{Z}$ the restriction $u|\mathbb{R}^n$ determines u uniquely and is contained in $\mathcal{S}$. Moreover, $\mathcal{Z}$ is dense in $\mathcal{S}$, since $\mathcal{Z} = \mathcal{D}^\wedge$, and $\mathcal{D}$ is dense in $\mathcal{S}$ while F and F^- are continuous maps $S \to S$. Also $\varphi_j \to 0$ in Z implies $\varphi_j \to 0$ in S. For $u \in S'$ the restriction $u|Z$ determines u and we have $u|Z \in Z'$. Hence get a natural imbedding $S' \to Z'$. For $u \in S' \subset Z'$ we earlier defined $\langle u^\wedge, \varphi\rangle = \langle u, \varphi^\wedge\rangle$ for $\varphi \in S$. The restriction $u^\wedge|Z$ gives our present functional, q.e.d.

Notice that $u^\wedge$, for $u \in D'$ in general is <u>not a distribution</u>, as defined in sec.2. It is a linear functional on Z, not on D.

Recall that for a function $f \in L^1_{loc}(\mathbb{R})$ with $f=0$ in $x<0$ and $f=O(e^{cx})$, some c, one commonly defines the <u>Laplace transform</u> by

$$(3.14) \qquad f^{\sim}(\zeta) = \int_0^\infty \bar{d}x e^{-ix\zeta} f(x) \ , \ \mathrm{Im}\,\zeta < -c \ ,$$

where the integral exists and defines a holomorphic function in the complex half-plane $\mathrm{Im}\,\zeta < -c$ (we have modified the standard definition, by a factor i). The inverse transform then is given by

$$(3.15) \qquad f(x) = \int_{-\infty+i\gamma}^{+\infty+i\gamma} \bar{d}\zeta e^{ix\zeta} f^{\sim}(\zeta) \ ,$$

with a complex curve integral along the parallel $\mathrm{Im}\,\zeta = \gamma < -c$.

We now will identify $f^{\sim}$ with the Fourier transform $f^\wedge \in Z'$ of the distribution $f \in D'$. For $\varphi \in D$, $\mathrm{supp}\,\varphi \subset \{|x| \le \eta\}$, we know that $\varphi^\vee$ is entire analytic, satisfying (3.8). For $\gamma < -c$ we have

$$(3.16) \qquad \int_{\mathrm{Im}\zeta=\gamma} f^{\sim}(\zeta)\varphi^\vee(\zeta)d\zeta = \int_0^\infty \bar{d}x f(x) \int_{\mathrm{Im}\zeta=\gamma} e^{-ix\zeta}\varphi^\vee(\zeta)d\zeta = \langle f, \varphi\rangle \ .$$

The integral $dx\,d|\zeta|$ exists absolutely, hence the interchange, by Fubini's theorem. Also, we get $\int_{\mathrm{Im}\zeta=\gamma} = \int_{\mathbb{R}}$, at right, by the properties of the (analytic) integrand. Then (3.16) follows from Fourier inversion for functions in D. Or, $f^\wedge \in Z'$ may be written as

$$(3.17) \qquad \langle f^\wedge, \varphi\rangle = \int_{\mathrm{Im}\zeta=\gamma} f^{\sim}(\zeta)\varphi(z)d\zeta \ , \ \varphi \in Z \ ,$$

where we must choose $\gamma < -c$ with c of (3.14) (for f).

Thus for a function $f \in L^1_{loc}(\mathbb{R})$ of exponential growth and $=0$ in $x<0$ the Fourier transform $f^\wedge$ is given as the complex integral (3.17) involving the Laplace transform $f^{\sim}$ of f.

<u>Problems</u>. 1) Obtain the Laplace transforms of the following functions (Each is extended zero for $x<0$). a) x^k $k=0,1,\ldots$, b) e^{ax}; c) $\cos bx$; d) $e^{ax}\sin bx$; e) $\frac{\sin x}{x}$. In each case, discuss the Fourier transform - i.e., the linear functional on Z. 2) Obtain the inverse Laplace transform of a) $\frac{1}{\sqrt{z-a}}$; b) $\log(1+\frac{1}{z^2})$. (In each case specify a branch of the (multi-valued) function well defined in a half-plane $\mathrm{Im}\,z < \gamma$.) 3) For $u \in D'(\mathbb{R}^n)$ with $\mathrm{supp}\,u \subset \{x_1 \ge 0\} =$

$\mathbb{R}^n_+$ and $e^{cx_1}u \in \mathcal{S}'$, for some c, show that $u^\wedge$ may be defined by a complex integral like (3.17), with $u^\sim$ replaced by $u^{\natural\sim}$, "$\natural$"="$\wedge$" with respect to $(x_2,\dots,x_n)$. 4) The convolution product w=u*v, so far defined for $u,v \in L^1(\mathbb{R}^n)$, by (1.19), may be defined for general distributions $u,v \in \mathcal{D}'(\mathbb{R}^n)$ under a support restriction -for example (i) if supp $u \subset\subset \mathbb{R}^n$, supp v general, or (ii) if supp $v \subset \{x_1 \ge 0\}$, supp $v \subset \{|x| \le cx_1\}$. One then defines $\langle w,\varphi\rangle = \iint dxdy u(x)v(y)\varphi(x+y)$,

with a distribution integral (for precise definition cf.[Schw_1], or, [C_1],I,(8.1)). Show that (1.21) is valid for this convolution product as well, assuming in case (ii) the cdns. of pbm.3 for u,v. 5) Let T_A be the space of all entire functions $\chi(z)$ satisfying (3.8) for some k. Show that $\chi\varphi \in \mathcal{Z}$, for $\varphi \in \mathcal{Z}$, $\chi \in \mathcal{T}_A$. Moreover, show that $f \in \mathcal{Z}'$ allows definition of a product $\chi f \in \mathcal{Z}'$, setting $\langle \chi f,\varphi\rangle = \langle f,\chi\varphi\rangle$, $\varphi \in \mathcal{Z}$. All polynomials p(x) belong to $\mathcal{T}_A$. 6) Show that (1.22) is valid for general distributions $u \in \mathcal{D}'(\mathbb{R}^n)$.

4. The Fourier-Laplace method; examples.

We now will discuss the 'Fourier-Laplace method' for 'free space'-problems of the following constant coefficient operators:

(4.1) $\Delta = \sum_{j=1}^n \partial_{x_j}{}^2$ (the Laplace operator) ,

(4.2) $\Delta + k^2$ (the Helmholtz operator) ,

(4.3) $H = \partial_{x_0} - \Delta = \partial_t - \Delta$ (the heat operator) ,

(4.4) $\square = \partial_{x_0}{}^2 - \Delta = \partial_t^2 - \Delta$ (the wave operator) ,

(4.5) $\square + m^2$ (the Klein-Gordon operator) .

The last 3 operators act on the n+1 variables $x_0=t$, $(x_1,\dots,x_n)=x$. The first two act on x only, distinguishing x_0 from the others.

The discussion around (1.23)-(1.26) was a formal attempt to solve constant coefficient PDE in free space (in all $\mathbb{R}^n$). We found $e = (\frac{1}{P(x)})^\vee$, for a P(D), of special interest. Now we are prepared to implicate this technique, called the Fourier-Laplace method.

Certain initial-boundary problems may be converted into free space problems: (a) An initial-value problem for (4.3),(4.4), or

(4.5) seeks to find solutions u of $P(D)u=f$ in some half-space, say, $t=x_0>0$, where f is given in $t\geq 0$, together with initial data of u at $t=0$. Such problem may be written as a free space problem by extending $u=0$ and $f=0$ into $t<0$, letting v and g be the extended functions. We will not have $P(D)v=g$ then, but, rather, $P(D)v=g+h$, with a distribution h, supp $h\subset\{t=0\}$, since normally v will jump at $t=0$. The initial conditions on u often are <u>well posed</u> if they allow to determine h, making the initial-value problem equivalent to the free space problem $P(D)v=g+h$, where $g+h$ is given.

(b) Another example: If $\Delta u=f$ (Δ of (4.1)) is to be solved in a half-space under Dirichlet bondary conditions - say, $\Delta u=f$ in $x_1>0$, $u=0$ as $x_1=0$, then consider the odd extensions of u and f to $\mathbb{R}^n$: $v(x_1,\dots,x_n)=u(x)$, $x_1>0$, $=-u(-x_1,x_2,\dots,x_n)$, $x_1<0$, similarly g extending f. It follows that $\Delta v=g$ in $\mathbb{R}^n$, again converting the half-space Dirichlet problem of Δ to a free space problem over $\mathbb{R}^n$. Similarly for the Neumann problem, using even extensions.

Technique (a) works as well for a more general initial surface $t=\Theta(x)$, $x\in\mathbb{R}^n$. Both techniqes may be combined to reduce certain initial-boundary problems to free space problems.

The above will emphasize the power of the Fourier Laplace method. (4.1)-(4.5) are a crossection of popular PDE's. We control parabolic and hyperbolic initial value problems, elliptic boundary problems and initial-boundary problems in half spaces, etc., with Green's (Riemann) functions, using results on special functions.

From now on interprete the equation $P(D)v=g$, $x\in\mathbb{R}^n$, as a PDE involving <u>distributions</u> $v,g\in\mathcal{D}'(\mathbb{R}^n)$. The Fourier transform exists without restrictions: Using 3,pbm.6, we get $P(x)v^\wedge=g^\wedge$, where $v^\wedge, g^\wedge\in\mathcal{Z}'$. If $e\in\mathcal{D}'(\mathbb{R}^n)$ solves $P(D)e=(2\pi)^{n/2}\delta$ we get $P(x)e^\wedge=1$.

In the cases corresponding to (4.1)-(4.5) we get, respectively,

$$(4.6)\quad P(x)=-|x|^2\ ,\ =k^2-|x|^2\ ,\ =it+|x|^2\ ,\ =|x|^2-t^2\ ,\ =m^2+|x|^2-t^2,$$

where $t=x_0$ again. Generally, $\frac{1}{P(x)}\notin L^1_{loc}$, except for (4.3) and (4.1), $n\geq 3$, due to zeros of P. Some pbm's of sec's 2,3 (and, more generally, [C1],II) discuss distributions p.v.a associated to a $\notin L^1_{loc}$. $P(x)z=1$ may have many solutions $z\in\mathcal{D}'$ (or $\in\mathcal{Z}'$). For (4.3)-(4.5) we will be interested in $z=e^\wedge$, $P(x)z=1$, with supp $e\subset\{t\geq 0\}$, because then supp $e*g\subset\{t\geq 0\}$ whenever supp $g\subset\{t\geq 0\}$, so that $u=(e*(g+h))|\{t\geq 0\}$ will solve the initial-value problem for $P(D)u=f$, $t\geq 0$, we started with in (a) above. Indeed, a proper z exists: In pbm's 1 and 3 of sec.2 we defined p.v.$\frac{1}{x}$, and $f_\pm$, all 3 distinct, $xf=1$. Only f_- has its inverse Fourier transform $=0$ for $x<0$.

For (4.3)-(4.5) we will construct such $e \in \mathcal{D}'(\mathbb{R}^{n+1})$ solving $P(D)e = \sqrt{2\pi}^{n+1}\delta$, supp $e \subset \{t \geq 0\}$, using the setup of sec.3, pbm.3: Such e, if $e^{cx}e \in \mathcal{S}'$, will have a Fourier transform in $x=(x_1,..)$ and a Laplace transform in $x_0 = t$. Accordingly we must seek an inverse Laplace transform of an inverse Fourier transform of a suitable solution z of $Pz=1$, or vice versa, in appropriate variables. Our proofs will be sketchy, in part, due to overflow of details.

The lemma below is convenient, due to spherical symmetry of P.

<u>Lemma 4.1</u>. Given a spherically symmetric function $f(x)=\omega(|x|)$, where $\omega(r) \in L^1(\mathbb{R}_+, r^{n-1}dr)$. Then the transforms $f^\wedge$ and $f^\vee$ are spherically symmetric as well: $f^\wedge(x)=f^\vee(x)=\chi(|x|)$, where $\chi(r)$ and $\omega(r)$ are related by the <u>Hankel transform</u> $\mathcal{H}_\nu$, $\nu=\frac{n}{2}-1$. In detail we have

$$(4.7)\qquad \begin{aligned} r^{(n-1)/2}\chi(r) &= \mathcal{H}_{n/2-1}(r^{(n-1)/2}\omega(r)) ,\\ r^{(n-1)/2}\omega(r) &= \mathcal{H}_{n/2-1}(r^{(n-1)/2}\chi(r)) ,\end{aligned}$$

where

$$(4.8)\qquad \mathcal{H}_\nu(\lambda(r))(\rho) = \int_0^\infty \sqrt{\rho r}\, J_\nu(r\rho)\, \lambda(r)\, dr \quad , \ \mathrm{Re}\,\nu > -\tfrac{1}{2} ,$$

with the Bessel function $J_\nu(z)$. The second formula (4.7) is valid if in addition $\chi \in L^1(\mathbb{R}_+, r^{n-1}dr)$.

<u>Proof</u>. For an orthogonal n×n-matrix O get $f^\wedge(O\xi) = \int d\!\!\!/x\, e^{-ix^T O\xi} f(\xi) = \int d\!\!\!/x\, e^{-i(O^T x)^T\xi}\omega(|O^T x|) = \int d\!\!\!/y\, e^{-iy\xi} f(y) = f^\wedge(\xi)$. Thus $f^\wedge$ has the same symmetry: $f^\wedge(\xi)=\chi(|\xi|)$, with some $\chi(\rho)$. We may set

$$(4.9)\qquad f^\wedge(\xi)=f^\wedge(|\xi|,0,\dots,0)=\int d\!\!\!/x\, e^{-i\rho x_1}\omega(r)=\kappa_n \int_0^\infty r^{n-1}\omega(r)\int e^{-ir\rho z_1} dS$$

where the inner integral I is over the unit sphere $|z|=1$. Evaluate this inner integral by converting it to an integral on the n-1-dimensional ball $|\lambda|^2 \leq 1$, setting $z=(z_1,\lambda)$. We know that $dS=\frac{d\lambda}{\sqrt{1-\lambda^2}}$. With a contribution from the upper and lower hemisphere where $z_1 = \pm\sqrt{1-\lambda^2}$, writing $d\lambda=\sigma^{n-2}d\sigma d\Sigma$, $\sigma=|\lambda|$, etc., we get

$$I=2\int \sigma^{n-2}\frac{d\sigma}{\sqrt{1-\sigma^2}}d\Sigma \cos(r\rho\sqrt{1-\sigma^2})=2a_{n-1}\int_0^1 \sigma^{n-2}\frac{d\sigma}{\sqrt{1-\sigma^2}}\cos(r\rho\sqrt{1-\sigma^2}) \quad .$$

A substitution $\sigma=\sin\theta$ of integration variable yields

$$(4.10)\qquad I = 2a_{n-1}\int_0^{\pi/2} d\theta\, \sin^{n-2}\theta\, \cos(r\rho\cos\theta) ,$$

with $a_{n-1}=\frac{2}{\Gamma((n-1)/2)}\pi^{(n-1)/2}$, the area of the n-1-dimensional unit sphere. Using Poisson's formula ([MOS],p.79) we get

$$\int_0^{\pi/2}\cos(r\rho\cos\theta)\,\sin^{n-2}\theta\,d\theta=2^{n/2-2}\sqrt{\pi}\Gamma(\tfrac{n-1}{2})J_{n/2-1}(r\rho)\ . \tag{4.11}$$

Substituting into (4.10) and I into (4.9) confirm (4.7). No change if $e^{-ix\xi}$ in (4.9) is replaced by $e^{ix\xi}$. Thus lemma 4.1 follows.

Recall also (1.26), now under the aspect of 3,pbm.4. In details, the convolution product $v*w$ of two distributions $v,w\in\mathcal{D}'(\mathbb{R}^n)$ may be defined by setting (with a distribution integral)

$$\langle v*w,\varphi\rangle=\int v(x)w(y)\varphi(x+y)dxdy=\langle v\otimes w,\psi\rangle\ ,\ \psi(x,y)=\varphi(x+y)\ , \tag{4.12}$$

for all $\varphi\in\mathcal{D}(\mathbb{R}^n)=C_0^\infty(\mathbb{R}^n)$, if v and w satisfy the support condition

$$K_a=(\text{supp } v\times\text{supp } w)\cap\{|x+y|\le a\}\text{ is compact for every }a>0\ . \tag{4.13}$$

Here supp v and supp w are regarded as subsets of $\mathbb{R}^n$ of the variables x and y, respectively (cf. [Schw_1] or [C_1], I,8).

The distribution $v=e*g$ is defined for e as constructed above whenever $g\in\mathcal{E}'$ for (4.1)-(4.3) , and $g\in\mathcal{D}'(\mathbb{R}^{n+1})$, $g=0$ as $t<0$, for (4.4) and (4.5) , since condition (4.13) holds under these assumptions. Moreover, $P(D)v=g$ follows, leading to a solution of the free space problem, and the related initial-boundary problems.

Now let us attempt a detailed construction of the proper e.

Ia) Consider the operator Δ of (4.1), i.e., the potential equation $\Delta u=f$. For $n\ge3$ the function $-\frac{1}{|x|^2}$ is L^1_{pol}, hence a distribution in $\mathcal{S}'$. This is a homogeneous distribution of degree -2 . Hence $e=-(\frac{1}{|x|^2})^\vee$ is homogeneous of degree 2-n . It is also spherically symmetric. Conclusion: $e(x)=c_n|x|^{2-n}$, with a constant c_n. Clearly $e\in L^1_{pol}$. The constant c_n may be evaluated by looking at

$$(2\pi)^{n/2}\varphi(0)=\langle e,\Delta\varphi\rangle=c_n\int\Delta\varphi(x)|x|^{2-n}dx=c_n\lim_{\varepsilon\to0}\int_{r=\varepsilon}dS_1\varphi\frac{d}{dr}r^{2-n}$$

$=c_na_n(2-n)\varphi(0)$. It follows that

$$e_n(x)=\frac{-1}{(n-2)\omega_n}|x|^{2-n}\quad,\quad\omega_n=\frac{2}{\Gamma(n/2)}\sqrt{\pi}^n\ . \tag{4.14}$$

For n=1 we first define a distribution

$$e^\wedge=-\text{p.f.}\frac{1}{x^2}=\frac{d}{dx}(\text{p.v.}\frac{1}{x})\ , \tag{4.15}$$

involving the distribution derivative and p.v.$\frac{1}{x}$ of pbm.1,sec.2. We confirm that $e^\wedge$ solves $-x^2e^\wedge=1$. Using $e=e^{\wedge\vee}$ one finds that

$$e(x)=\sqrt{\pi/2}|x|\ . \tag{4.16}$$

For n=2 we can define (cf.[C_1],II,(2.11), for l=2)

$$(4.17)\quad e^{\wedge}(x)=-\text{p.f.}\tfrac{1}{|x|^2}=-\textstyle\sum_{j=1}^{2}(\tfrac{1}{|x|^2}x_j\log|x|)_{|x_j}=-\tfrac{1}{2}\Delta((\log|x|)^2)\ .$$

again with distribution derivatives. However, it is easier to confirm directly that

$$(4.18)\qquad e(x) = \kappa_1\log|x|$$

is a spherically symmetric L^1_{loc}-function satisfying $\Delta e=\sqrt{2\pi}\delta$ (just evaluate the integral $\langle\Delta e,\varphi\rangle=\langle e,\Delta\varphi\rangle$, using partial integration).

Ib) Consider (4.2), i.e, the Helmholtz equation $(D+\lambda)u=f$, also known as the time-independent wave equation if $\lambda=k^2>0$, and as the resolvent equation of the Laplace operator Δ if $\lambda\in\mathbb{C}$, $\lambda\neq k^2$. In the latter case get $e^{\wedge}(x)=(\lambda-|x|^2)^{-1}\in C^{\infty}\cap L^1_{pol}$. An evaluation of e(x) is possible, using lemma 4.1, as long as $n\leq 3$. We get

$$(4.19)\qquad e(x) = |x|^{1-n/2}\int_0^{\infty}\rho^{n/2}\frac{d\rho}{\lambda-\rho^2}\,J_{n/2-1}(\rho|x|)\ .$$

For larger n this integral ceases to exist. However, it still will exist as improper integral in the sense of distributions - that is, as $\lim_{A\to\infty}\int_0^A$, where the limit exists in weak convergence of $\mathcal{D}'$ (i.e., $\langle\lim.,\varphi\rangle = \lim\langle.,\varphi\rangle$). For odd n the Bessel function $J_{n/2-1}$ may be expressed by trigononmetric functions. For example, in case n=3 we get $J_{1/2}(z) = \sqrt{2}\,\frac{\sin z}{\sqrt{\pi z}}$. Or,

$$(4.20)\qquad e(x) = \sqrt{\tfrac{2}{\pi}}\frac{1}{|x|}\int_0^{\infty}\frac{\rho d\rho}{\lambda-\rho^2}\sin\rho|x|\ .$$

We may write $\lambda = \kappa^2$, picking the root κ with $\operatorname{Im}\kappa>0$. Then $\int_0^{\infty}\frac{\rho d\rho}{\lambda-\rho^2}\sin\rho r = \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\rho}{\kappa-\rho}\sin\rho r = \frac{1}{4i}\int_{-\infty}^{\infty}e^{ir\rho}\frac{d\rho}{\kappa-\rho} = -\frac{\pi}{2}e^{i\kappa|x|}$. Hence

$$(4.21)\qquad e(x) = -\sqrt{\pi/2}\;e^{i\kappa|x|}/|x|\ .$$

(4.21) may be confirmed, noting that $e=e^{i\kappa r}/r$ solves $(\Delta+\lambda)e=2\pi\delta$.

For $\kappa=k$ real $P(x)=k^2-|x|^2$ vanishes at the set $|x|=k$, and $\frac{1}{P}$ is not L^1_{loc}. Then look at p.f.$(\frac{1}{P(x)})$. Or else, observe that

$$(4.22)\qquad \lim_{\varepsilon\to 0,\varepsilon>0} e_{k+i\varepsilon}(x) = e(x)\ ,\ e_{\kappa}(x) \text{ as } e(x) \text{ in (4.21)}\ ,$$

in the sense of distributions. This implies that

$$(4.23)\qquad e_{\pm}(x) = -\sqrt{\pi/2}\,e^{\pm ik|x|}/|x|$$

both will solve $(\Delta+k^2)e=(2\pi)^{3/2}\delta$. The proper sign may be chosen by imposing a 'radiation condition' at ∞ .

For general $n\geq 2$ we still may evaluate the integral (4.19). Using a formula by Sonine and Gegenbauer (cf. [MOS], p.105) we get

$$(4.24)\qquad e(x)=-\left(\frac{\kappa}{|x|}\right)^{n/2-1}K_{n/2-1}(\kappa|x|)\ ,\ \kappa^2=-\lambda\ ,\ \mathrm{Re}\ \kappa>0\ ,$$

with the modified Hankel function $K_\nu(z)$. Again get (4.24) more directly, observing that $e(x)=\gamma(|x|)$ solves $(\Delta+\lambda)e=0$, hence $\gamma(r)$ solves the ODE $\gamma''+\frac{n-1}{r}\gamma'-\kappa^2\gamma=0$. Substituting $\gamma=r^{-\nu}\delta$, $\nu=\frac{n}{2}-1$, we obtain the modified Bessel equation $\delta''+\frac{1}{r}\delta'-(1+\frac{\kappa^2}{r^2})\delta=0$, showing that the only spherically symmetric solutions of $(\Delta+\lambda)u=0$ in $\mathcal{S}'$ are the multiples of (4.24). A partial integration shows that $\int e(\Delta+\lambda)\varphi dx=\varphi(x)$ for all $\varphi\in\mathcal{D}$, fixing the remaining multiplicative constant.

II) In the case (4.3) of the heat equation $Hu=u_t-\Delta u=f$ we may use (4.24): Applying F_x^{-1} to $\frac{1}{p}$ for the second and third polynomial (4.6) gives the same result, if we set $\lambda=k^2=-\kappa^2=it$. That is, $\kappa=e^{-i\pi/4}\sqrt{t}$, Re $\kappa>0$, in (4.24) will define $F_x^{-1}(\frac{1}{p})$, and we then must obtain the inverse Laplace transform.

It is more practical, however, to first obtain $F_t^{-1}(\frac{1}{p})$. Note $\theta(\tau)=(i\tau+a)^{-1}$ has inverse Laplace transform $\theta^\Delta(t)=\sqrt{2\pi}e^{-at}$, $t\geq 0$, $=0$, $t<0$, calculating $\int_0^\infty e^{-at}e^{-i\tau t}dt$. For $e^\backprime=F^{-1}(\frac{1}{p})$ get

$$(4.25)\qquad e^\backprime(t)=(2\pi)^{1/2}e^{-t|x|^2}\ ,\ \text{as } t\geq 0\ ,\ e^\backprime(t)=0\ ,\ \text{as } t<0\ .$$

Recall that $(e^{-|x|^2/2})^\vee=e^{-|x|^2/2}$ (in n dimensions), by a complex integration. Also for the function $g_\sigma(x)=g(\sigma x)$ we get

$$(4.26)\qquad g_\sigma^\vee(x)=\sigma^{-n}g^\vee(x/\sigma)\ ,\qquad \sigma\in\mathbb{R}_+\ ,$$

as shown by an integral substitution. Choosing $\sigma=\sqrt{2t}$ we thus get

$$(4.27)\qquad e(t,x)=\frac{\sqrt{2\pi}}{(\sqrt{2t})^n}\,e^{-|x|^2/4t}\ ,\ t\geq 0\ ,\ =0\ ,\ t<0\ .$$

This is the well known <u>fundamental solution of the heat operator</u>. It is not $L^1(\mathbb{R}^{n+1})$. Use it to solve the <u>initial value problem</u>

$$(4.28)\qquad Hu=\partial_{x_0}u-\Delta u=f\ ,\ x_0\geq 0\ ,\ u(x_0,x)=\varphi(x)\ ,$$

where f,φ are given C_0^∞-functions. Setting u and f zero in $t<0$ to obtain functions v and g we get

$$(4.29) \qquad Hv = g + \delta(t)\otimes\varphi(x) = h \ .$$

Thus $v=e*h$, or,

$$(4.30) \quad u(t,x)=\kappa_{n+1}\int_{\tau\le t}d\tau dy\, e(t-\tau,x-y)f(y) + \kappa_{n+1}\int dy\, e(t,x-y)\varphi(y) .$$

III) Now we look at (4.4), or, the wave equation

$$(4.31) \qquad \square u = (\partial_t^2 - \Delta)u = f \ .$$

We apply the Fourier-Laplace method as for (II): The function

$$(4.32) \qquad \frac{1}{P} = \frac{1}{|x|^2-t^2} = \frac{1}{2|x|}\left\{\frac{1}{t+|x|} - \frac{1}{t-|x|}\right\}$$

has inverse Laplace transform (in t) given by

$$(4.33) \qquad \frac{\sqrt{2\pi}}{2i\,|x|}(e^{-it|x|}-e^{it|x|}) = \sqrt{2\pi}\,\frac{\sin(|x|t)}{|x|} \ , \ t\ge 0 \ ,$$

(and zero for $t<0$). Looking for F_x^{-1} of the function (4.33) we cannot apply a Fourier integral, since the function (4.33) is not L^1. First set $n=2$. Writing $F_x^{-1}w=w^\Delta$ for a moment, we have

$$(4.34) \quad \langle e,\varphi\rangle = \sqrt{2\pi}\,\langle \frac{\cos rt}{r^2} ,\varphi_{|t}{}^\Delta\rangle = \langle f,\varphi_{|t}{}^\Delta\rangle \ , \ \varphi\in \mathcal{D}(\mathbb{R}^n) \ , \ r=|x| \ ,$$

using that $(\cos |x|t)_{|t} = -|x| \sin |x|t$. Now the inverse Fourier integral of $f=\frac{\cos rt}{r^2}$ may be calculated as improper Riemann integral $\lim_{A\to\infty}\int_{|x|\le A}$, although still that function is not L^1 : Using (4.7), (4.8) - where $\nu=\frac{1}{2}$, $J_\nu(z)=\sqrt{2}\,\frac{\sin z}{\sqrt{\pi z}}$, for $n=3$ - we get

$$f^\Delta=\omega(|x|) \ , \ \omega(r)=(\sqrt{2\pi})(\tfrac{\sqrt{2}}{\sqrt{\pi}})\int_0^\infty \sqrt{r\rho}\,(\tfrac{\rho}{r})\,\frac{\sin r\rho}{\sqrt{r\rho}}\,\frac{\cos \rho t}{\rho^2}\,d\rho \ . \text{ Or,}$$

$$\omega(r) = \frac{2}{r}\int_0^\infty \sin r\rho \cos \rho t\,\frac{d\rho}{\rho} = \frac{1}{r}\int_0^\infty (\sin \rho(t-r) - \sin \rho(t+r))\,\frac{d\rho}{\rho}$$

$$= \frac{1}{r}\{\frac{\pi}{2}\operatorname{sgn}(t-r) - \frac{\pi}{2}\} \ . \text{ Conclusion:}$$

$$(4.35) \quad e(t,x) = \frac{\pi}{|x|}\,\partial_t H(t-|x|) \ , \text{ as } t>0 \ , =0 \ , \text{ as } t<0 \ , \ n=3 \ ,$$

with the distribution derivative ∂_t , and the Heaviside function $H(t)=1$, $t\ge 0$, $H(t)=0$, $t<0$. We are tempted to write $\partial_t H(t-|x|)$ as $\delta(t-|x|)$, but then must remember the proper interpretation.

Converting the Cauchy problem for the wave equation,