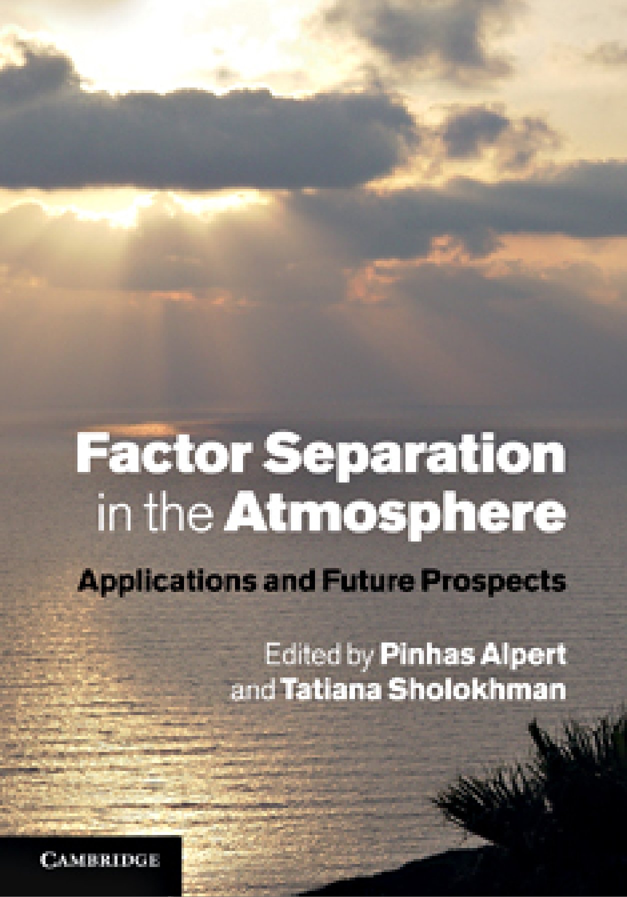

Factor Separation in the Atmosphere

Applications and Future Prospects

Edited by Pinhas Alpert and Tatiana Sholokhman

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FACTOR SEPARATION IN THE ATMOSPHERE

Applications and Future Prospects

Modeling atmospheric processes in order to forecast the weather or future climate change is an extremely complex and computationally intensive undertaking. One of the main difficulties is that there are a huge number of factors that need to be taken into account, some of which are still poorly understood. The Alpert–Stein Factor Separation (FS) Methodology is a computational procedure that helps deal with these nonlinear factors. Pinhas Alpert was the main pioneer of the FS method in meteorology, and in recent years many scientists have applied this methodology to a range of modeling problems, including paleoclimatology, limnology, regional climate change, rainfall analysis, cloud modeling, pollution, crop growth, and other forecasting applications. This book is the first to describe the fundamentals of the method, and to bring together its many applications in the atmospheric sciences, with chapters from many of the leading atmospheric modeling teams around the world. The main audience is researchers and graduate students using the FS method, but it is also of interest to advanced students, researchers, and professionals across the atmospheric sciences.

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Foreword

The Factor Separation method, pioneered in the now classic Stein and Alpert (1993) and Alpert *et al*. (1995) papers, provides a powerful, much-needed tool to assess both linear and nonlinear relationships among weather and climate forcings and feedbacks. As summarized in Chapter 1 of the book:

> The FS method provides the methodology to distinguish between the pure influence of each and every factor as well as their mutual influence or synergies, which come into play when several factors, at least two, are "switched on" together. The understanding of which factor, or what combination of factors is most significant for the final result, is often very interesting in atmospheric studies. Discovering the most dominant factors in a specific problem can guide us to the important physical mechanisms and also to potential improvements in the model formulations.

Before this analysis procedure was introduced, numerical models usually performed sensitivity studies by turning on one forcing at a time, and used these results to decide what are the most important factors affecting a particular model simulation. However, we now recognize that such a linear type of analysis is incomplete and can even lead to the incorrect answer, as is illustrated in several chapters in this book.

This book provides a range of examples that illustrate the power of this analysis methodology for a range of spatial and temporal scales. The next step, besides applying the Alpert–Stein Factor Separation Methodology to additional atmospheric studies, should be to broaden it to include other geophysical disciplines.

Roger A. Pielke Sr.

Preface

This book is the result of almost two decades of research initiated in 1991 by an idea that sensitivity studies in the atmosphere were not being performed in the proper manner. My Ph.D. student Uri Stein was simulating the effects of two central factors on the rainfall over the Eastern Mediterranean with the Mesoscale Model MM4: Mediterranean Sea fluxes and topography. I called Uri, on a Friday in 1991, suggesting that the three simulations he was performing were not enough to capture the potential synergy or interaction between these two factors and that an additional simulation should be performed. Consequently, we developed the Factor Separation Methodology that allows for the separation of four potential contributions in our two-factor problem and 2^n simulations for any n-factor problem. This separation included specifically the double synergy term which is the net result of interaction between the two factors. As I had expected, we found immediately that the synergy term plays a central role in the atmosphere, one that is often larger than the net contribution of any singular factor's contribution.

When our paper was submitted in 1992 to the *Journal of Atmospheric Sciences*, I did not know what to expect. It seemed to me that the method was so basic that I could not believe we were the first to apply it in atmospheric sciences. However, our extensive literature search did not show any similar publications.

We were thrilled when the paper was accepted for publication, and I began to talk about it in a lecture series at several well-known institutions including the University of Oklahoma, where the lecture was attended by Doug Lilly and the late Tzvi Gal-Chen. I also spoke at CSU where the lecture was attended by Roger A. Pielke Sr. and Bill Cotton, and I presented the idea at the following four conferences over the years 1992–3:

- the International Workshop on Mediterranean Cyclone Studies, Trieste, Italy, 1992
- the Mesoscale Modeling Workshop, El-Paso, Texas, 1992
- the Yale Mintz Memorial Symposium, Jerusalem, Israel, 1992
- the U.S. Army Mesomet Panel Meeting, Monterey, CA, USA, 1993.

On all these occasions, the response was enthusiastic, which served to strengthen our faith in the great potential of our new methodology. I wish to particularly mention the outstandingly strong words of support from senior meteorologists including T. T. Warner (PSU, at that time), Don Johnson (Wisconsin), R. P. Pearce (Reading), D. Lilly (University of Oklahoma), R. A. Pielke Sr. (CSU), B. Cotton (CSU) A. Berger (Louvain-la-Neuve), T. N. Krishnamurti (FSU, Tallahassee University), T. Gal-Chen (University of Oklahoma) and late J. Neumann (Hebrew University, Jerusalem).

D. Lilly informed me that he decided immediately to apply our method to a turbulence study he was performing at that time with his student, L. Deng. In fact, Deng and Lilly presented their factor separation study in the same year (1992), even before our paper was published in 1993. Their reference is: L. Deng and D. K. Lilly (1992) Helicity effect on turbulent decay in a rotating frame, *10th Symposium on Turbulence and Diffusion*, Portland, OR, pp. 338–341, AMS.

This new application illustrated the relative ease of applying the new methodology to different applications – my presentation in Oklahoma was in May 1992 and their proceeding publication appeared just a few months later.

Meanwhile, we submitted more papers on the method including a study focusing on the mechanisms related to the Genoa cyclogenesis, which analysed four factors and required 16 simulations including one quadruple, four triple, and six double synergies. In 1995 we submitted a paper entitled, 'Synergism in weather and climate' by P. Alpert, U. Stein, M. Tsidulko, and B. U. Neeman. In response, we received the following beautiful and most encouraging words from Rainer Bleck, who was one of the referees:

Unless there are difficulties with this method that have yet to come to light and may limit its broad use, the synergistic analysis method developed by Stein and Alpert is likely to become an indispensable tool in our field. After reading this paper, very few investigators trying to isolate the effect of various physical factors on climate and circulation through numerical simulation will be able to argue that they are exempt from using this method.

I cannot think of any aspect in the paper requiring further work; in other words, the paper is essentially ready to be published in J. Climate as is. A few minor editorial suggestions are spelled out in the manuscript which I am returning to the editor.

This is one of the few cases where one waits to hear from the journal editor that the paper has been accepted, so that one can start referencing it in one's own work, talking about it in class, etc. (Rainer Bleck)

Interestingly, the paper was rejected by the Editor stating that it was not appropriate for that particular journal.

A few years after our initial efforts on factor separation, I established the Factor Separation Group email list which grew within a year or two to about 50 users all over the world; many of whom can be found in my acknowledgements list below because they have initiated and been part of many interesting discussions and developments over the years. The incorporation of the method into various and diverse atmospheric topics was quick, and some of those works provide the basis for the different applications in the following chapters of the present book.

A few words on the order of the chapters in the book are appropriate here. Following the introduction (Chapter 1) and the mathematical formulation of the method (Chapter 2), some analytical functions are analyzed (Chapter 3) based on the Master Thesis of Tatiana Sholokhman, my co-author of the present book. Following these initial chapters are eleven chapters on various applications with the general order decreasing from the macro-scale to the micro-scale. Therefore, the following chapters and applications are: Paleoclimate (Chapter 4), Mesometeorology (Chapter 5), Regional climate (Chapter 6), Heavy rainfall and Cyclogenesis (Chapter 7), Clouds (Chapter 8), Limnology (Chapter 9), Pollution (Chapter 10), Crop growth (Chapter 11), Sea breeze (Chapter 12) and two different forecasting applications (Chapters 13, 14). Chapter 15 discusses some difficulties and prospects of the methodology, including a comparison to a similar but different method applied mainly in biotechnology experiments. Chapter 14 by T. N. Krishnamurti also suggests and discusses a similar but different method for factor separation. Chapter 16 provides a summary. An important addition at the end is an Appendix, which lists, to the best of our knowledge, all articles and publications employing the Alpert–Stein Factor Separation Methodology at the time of publication. I wish to emphasize that up until the last moment we were informed of new publications using the method that we had not heard about before.

It should be noted that for consistency throughout the chapters of the book we have suggested that the method be referred to as the ‘Alpert–Stein Methodology’ or the ‘Alpert–Stein Factor Separation Methodology’.

Special thanks go to the US–Israel BiNational Science Foundation (BSF) jointly with T. T. Warner (then at PSU), which funded our initial research on cyclogenesis over the Mediterranean that yielded the Factor Separation Methodology. Also, thanks go to the German–Israel Foundation (GIF), which continued the funding to our cyclogenesis study jointly with J. Egger (Munich University).

Cambridge University Press is to be congratulated on carefully bringing the book to its final stage with highly professional foresight and good advice.

I am indebted to a large number of colleagues and students for their interest and many suggestions through the years. In particular, I am very grateful to the following colleagues who contributed to the factor separation discussion during its various stages of evolution: Sergio Alonso, Andre Berger, Reiner Bleck, Bob Bornstein, Itsik Carmona, Bill (W. R.) Cotton, Eric Deleersnijder, C. A. Doswell III, Lenny Druyan, Joe Egger, Hubert Gallee, the late Tzvi Gal-Chen, Víctor Homar, Agustí Jansá, Don Johnson, Alexander Khain, Simon (S. O.) Krichak, Krish (T. N.) Krishnamurti, Gad Levy, Doug (D. K.) Lilly, Barry Lynn, Benny U. Neeman, the late Jehuda Neumann, Dev Niyogi, Bob (R. P.) Pearce, Natalie Perlin, Roger (R. A.) Pielke, Sr. Clemente Ramis, Romu (R.) Romero, Dorita Rostkier-Edelstein, Moti Segal, Uri Stein, Dave (D. J.) Stensrud, Alan Thorpe, Marina Tsidulko, and Tom (T. T.) Warner.

Pinhas Alpert

1

Introduction

P. Alpert

1.1 Background

Numerical models provide a powerful tool for atmospheric research. One of the most common ways of utilizing a model is by performing sensitivity experiments. Their purpose is to isolate the effects of different factors on certain atmospheric fields in one or more case studies. Factors that have been tested in sensitivity studies include, for example, surface sensible and latent heat fluxes, latent heat release, horizontal and vertical resolution, sea surface temperatures, horizontal diffusion, surface stress, initial and boundary conditions, topography, surface moisture, atmospheric stability, and radiation. Sensitivity studies are performed either with real-data case studies or with idealized atmospheric situations.

Sensitivity studies often evaluate the influence of only one factor such as topography (Tibaldi *et al.*, 1980; Dell'Osso, 1984, McGinley and Goerss, 1986), but many investigations test several factors, and try to estimate their relative importance. One common method of evaluating the contribution of a specific factor is by analyzing the difference in fields between a control run and a simulation where this factor is switched off. The difference map is, in general, more illustrative than the presentation of the two individual simulations, and therefore has often been used (e.g., Tibaldi *et al.*, 1980; Mesinger and Strickler, 1982; Lannici *et al.*, 1987; Leslie *et al.*, 1987; Uccellini *et al.*, 1987; Mullen and Baumhafner, 1988; Kuo and Low-Nam, 1990). Presentation of a map showing the difference between two simulations is also a common procedure (Maddox *et al.*, 1981; Chang *et al.*, 1982, 1984; Chen and Cotton, 1983; Kenney and Smith, 1983; Alpert and Neumann, 1984; Benjamin and Carlson, 1986; McGinley and Goerss, 1986; Orlansky and Katzfey, 1987; Zack and Kaplan, 1987; Mailhot and Chouinard, 1989). It will be shown, however, that the difference map for two simulations, when more than one factor

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are involved, does not have a simple meaning, and in fact may be quite misleading. Suppose that the effects of two factors are investigated: the topography and the surface fluxes. Three simulations are then performed (as in many of the aforementioned studies): CON, the control simulation; NOT, the no-terrain simulation; and NOF, the no-fluxes simulation. What is the meaning of the difference between the simulated fields of CON and NOT? It shows the effect of the topography, and also of the joint effect (interaction or synergy) of topography with fluxes, because both the terrain and its synergy effects vanish when the terrain is switched off. In the same way, the difference between CON and NOF includes the effects of both the fluxes and the interaction between fluxes and topography. If the interaction factor is not isolated, the difference maps CON – NOT and CON – NOF cannot then be simply interpreted, as is commonly attempted.

Although the aforementioned interactions between factors are usually neglected, their significant role in some cases has been pointed out (e.g., Uccellini *et al.*, 1987; Mailhot and Chouinard, 1989). To the best of our knowledge, no sensitivity studies had been proposed to isolate these interactions before our Factor Separation methodology was introduced in 1993. The method presented in this book with many applications shows a consistent and quite simple approach for isolating the resulting fields due to any interactions among factors, as well as that due to the pure factors, using linear combinations of a number of simulations.

1.2 What is the concept behind the Factor Separation method?

It is close to two decades since the publication of our basic methodology in "Factor Separation in Numerical Simulation" (henceforth FS) by U. Stein and P. Alpert (1993), and today this method provides a powerful tool for atmospheric research. A relatively large number of studies in the past two decades have been devoted to various applications of this method in atmospheric research.

The present book provides first a theoretical investigation of this methodology, and defines common principles and analytical explorations of the factors and their synergies. It should be noted that most of the studies employing the Alpert–Stein Factor Separation Methodology used the numerical modeling framework, which commonly does not allow an analytical investigation of the methodology. Hence, we present the results of the method application with simple mathematical functions that, at first, may have no clear physical meaning. From these results we create the basis for investigation of simple functions that describe complicated processes.

How does the method work, and what do we need it for? The next chapter (Chapter 2) provides the full mathematical description of the method, for which we propose a basic intuitive explanation. The starting point for investigating the problem is the inclusion of several factors that are assumed to influence the final result. It is important to note that both physical and geophysical values used in the model are often approximated, or they may change very quickly by the function of the chosen factors. For example, the albedo of the surface depends on the structure of the surface, and can change rapidly with the weather. Another consideration is that the temporal variation is very important for modeling the problem, but it adds complexity as it includes additional factors. The FS method can identify the most important factors and their combinations that henceforth are also referred to as *synergisms*. The synergism is defined in the dictionary as "the action of two or more factors to achieve an effect of which each is individually incapable."

An additional problem that the FS method addresses is the stability of the chosen factor in the problem. This means, how does the result react to small changes in the factors' values? The FS standard method works on the principle of "on–off". In other words, the factors are switched off (zeroed) one-by-one, and the intermediate result is investigated independently for each case. The case where all the chosen factors are switched off is the basic case, which obviously does not depend on the factors and their various combinations. The opposite case, with all of the factors switched "on", is the "control" or "full" result, which includes all the factors and their synergistic contributions. The FS method provides the methodology to distinguish between the pure influence of each and every factor as well as their mutual influence or synergies, which come into play when several factors, at least two, are "switched on" together. The understanding of which factor, or what combination of factors, is most significant for the final result, is often very interesting in atmospheric studies. Discovering the most dominant factors in a specific problem can guide us to the underlying physical mechanisms and to potential improvement in the model formulations.

1.3 The philosophy of synergy

What is the simple meaning of synergism? Let us take, for example, the following real-life situation. A worker needs to push a load along X meters. Let us assume that the push lasts time T. This time may not be equal to the time that two workers require to push the same load along double the distance, i.e., $2X$ meters. If the workers are in good coordination, i.e., positive synergy between them, they will push the load to a greater distance than $2X$ meters during the same period of time. But if they do not coordinate well, i.e., negative synergy, the distance will be less than $2X$ meters. In this case, each worker may be considered as a factor in the problem, and the result is the distance over which the load will be transported during the specific time T. Another example of negative synergy for the above workers could result from external limits, for instance, if the total distance is limited to $1.5X$ meters. Then, even if the two workers coordinate very well, the synergism must be negative by at least $0.5X$. This can correspond in the atmosphere to the very common case of saturation, where, for example, it is manifested as humidity saturation for rainfall. In other words, processes in life are often not linear, and the situation is similar or even more pronounced in the atmosphere. Probably there are no factors in the real atmosphere that are not correlated at all. This infinite chain of interactions among factors, or some small subset of them, is commonly investigated by scientists, and involves a finite number of assumptions.

A most attractive feature of the FS approach is the ability to quantify synergies or the interaction processes that were found to play a central role in many atmospheric processes, as found by several of the applications discussed in this book. There seems to be some basic psychological tendency in human thinking to present results linearly, with the hidden assumption that the synergies are small or can be ignored. Nonlinearities in the atmosphere, however, are often significant, and therefore need to be calculated and separated from the pure contributions of each factor, as shown in the many very different FS applications presented in this book.

Another attractive feature of the FS methodology is the fact that the system is closed in the sense that the sum of all contributions of the pure factors and their synergies always yields the result given by the control run, in which all factors are switched on.

In some of the following chapters, the synergies can be given a very clear physical meaning that also yields a much better understanding of the complex phenomena being investigated.

The next chapter presents the full mathematical derivation of the FS methodology.

2

The Factor Separation Methodology and the fractional approach

T. Sholokhman and P. Alpert

2.1 Following Stein and Alpert (1993) (SA, in brief)

2.1.1 The proposed method for two factors

The value of any predicted field f depends on the initial and boundary conditions, as well as the model itself. If a continuous change is made in any factor ψ, the resulting field f in general changes in a continuous manner as well. This can be mathematically formulated as follows.

Let the factor ψ be multiplied by a changing coefficient c so that

$$\psi(c) = c\psi \tag{2.1}$$

The resulting field f is a continuous function of c:

$$f = f(c) \tag{2.2}$$

so that $f(1)$ is the value of f in the control simulation, and $f(0)$ is the value of f in the simulation where the factor ψ is omitted. In the notation that follows, f_0 and f_1 are used for $f(0)$ and $f(1)$, respectively.

It is always possible to decompose any function $f(c)$ into a constant part, $\hat{f}_0$, that is independent of c, and a c-dependent component, $\hat{f}(c)$, such that $\hat{f}(0) = 0$.

In this simple example,

$$\hat{f}_0 = f_0 \tag{2.3}$$

and

$$\hat{f}(c) = f(c) - f_0 \tag{2.4}$$

It is important to understand the meaning of $\hat{f}_0$ and $\hat{f}_1$, the latter being a short form for $\hat{f}(1)$. The term $\hat{f}_1$ represents that fraction of f that is induced by the factor ψ, while $\hat{f}_0$ is the remaining part that does not depend on factor ψ. In order to obtain $\hat{f}_0$ and $\hat{f}_1$, two simulations must be performed, one with the ψ factor included (control) that results in f_1, and the other with ψ excluded, for obtaining the result for f_0:

$$f_1 = \hat{f}_0 + \hat{f}_1 \tag{2.5}$$

$$f_0 = \hat{f}_0 \tag{2.6}$$

Solution of the above equations for $\hat{f}$ in terms of the output field f yields

$$\hat{f}_0 = f_0 \tag{2.7}$$

$$\hat{f}_1 = f_1 - f_0 \tag{2.8}$$

The last equation shows that subtraction of the field f_0 (factor ψ excluded) from field f_1 (control run) results in that part of f that is solely induced by the factor ψ. This is how the method works for a single factor, and exemplifies the more general rule that is developed below.

2.1.2 Generalization of the method for *n* factors

Let the field f depend on n factors ψ_i, where $i = 1, 2, \dots, n$. Each factor is multiplied by a coefficient c_i, where

$$f = f(c_1, c_2, c_3, \dots, c_n) \tag{2.9}$$

By way of a Taylor series expansion, the function f can be decomposed as follows:

$$\begin{aligned} f(c_1, c_2, \dots, c_n) = \hat{f}_0 + \sum_{i=1}^{n} \hat{f}_i(c_i) + \sum_{i,j=1,2}^{n-1,n} \hat{f}_{ij}(c_i, c_j) \\ + \sum_{i,j,k=1,2,3}^{n-2,n-1,n} \hat{f}_{ijk}(c_i, c_j, c_k) + \dots + \hat{f}_{123\dots n}(c_1, c_2, c_3, \dots, c_n) \end{aligned} \tag{2.10}$$

Here, $\sum_{i,j=1,2}^{n-1,n}$ is the sum of all sorted pairs, and $\sum_{i,j,k=1,2,3}^{n-2,n-1,n}$ is the sum of all sorted trios, and so on.

A more complicated case, a fractional approach to the factor separation method, where c_i can obtain values between 0 and 1, is discussed below, Section 2.2.

For the simple case, factor ψ_i is fully switched on or off as c_i changes from 1 to 0, respectively. Each function $\hat{f}_{ijk\ldots}(c_i, c_j, c_k, \ldots)$ becomes identically zero if any of its variables c_i is zero. Employing a symbol system in which f_{ij} is the value of f in a simulation with $c_i = c_j = 1$, while all the rest of the coefficients are zero, and setting $c_i (i = 1, 2, \ldots, n)$ to either 1 or 0 in (2.10), yields

$$f_0 = f(0, 0, \ldots, 0) = \hat{f}_0 \tag{2.11}$$

$$f_i = \hat{f}_i + \hat{f}_0 \tag{2.12}$$

$$f_{ij} = \hat{f}_{ij} + \hat{f}_i + \hat{f}_j + \hat{f}_0 \tag{2.13}$$

$$f_{ijk} = \hat{f}_{ijk} + \hat{f}_{ij} + \hat{f}_{jk} + \hat{f}_{ik} + \hat{f}_i + \hat{f}_j + \hat{f}_k + \hat{f}_0 \tag{2.14}$$

$$f_{123\ldots n} = \hat{f}_{123\ldots n} + \cdots + \sum_{i,j,k=1,2,3}^{n-2,n-1,n} \hat{f}_{ijk} + \sum_{i,j=1,2}^{n-1,n} \hat{f}_{ij} + \sum_{i=1}^{n} \hat{f}_i + \hat{f}_0 \tag{2.15}$$

$f_{ij} = f_{ij}(1, 1)$, and the same applies for all other terms.

Equations (2.11)–(2.15) contain $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ equations respectively. Hence, Eqs. (2.11)–(2.15) contain a total of 2^n equations for 2^n unknowns $\hat{f}_0, \hat{f}_1, \ldots, \hat{f}_n, \hat{f}_{12}, \ldots, \hat{f}_{n-1,n}, \ldots, \hat{f}_{123\ldots n}$. This set of equations is solved by a recursive elimination of $\hat{f}_i$ from (2.12), then $\hat{f}_{ij}$ from (2.13), and so forth. The general solution then becomes

$$\hat{f}_{i_1 i_2 i_3 \ldots i_l} = \sum_{m=0}^{l} (-1)^{l-m} \left[\sum_{j_1, j_2, j_3, \ldots, j_m = i_1, i_2, i_3, \ldots, i_m}^{i_{l-m+1}, i_{l-m+2}, \ldots, i_l} f_{j_1 j_2 j_3 \cdots j_m} \right] \tag{2.16}$$

where the sum $\sum_{j_1, j_2, j_3, \ldots, j_m = i_1, i_2, i_3, \ldots, i_m}^{i_{l-m+1}, i_{l-m+2}, \ldots, i_l}$ is over all groups of m sorted indices chosen from l indices $i_1, i_2, i_3, \ldots, i_l$, where $0 \leqslant l \leqslant n$.

Let us take the case of two factors and then three factors.

First case: two factors

run	factor 1	factor 2		
f_{12}	on	on	$= \hat{f}_0 + \hat{f}_1 + \hat{f}_2 + \hat{f}_{12}$	(2.17)
f_1	on	off	$= \hat{f}_0 + \hat{f}_1$	(2.18)
f_2	off	on	$= \hat{f}_0 + \hat{f}_2$	(2.19)
f_0	off	off	$= \hat{f}_0$	(2.20)

The solution of (2.17)–(2.20) yields the following four terms:

$$\text{Unrelated to factors 1 and 2 } \hat{f}_0 = f_0 \tag{2.21}$$

$$\text{Induced by the factor 1 (independent of 2) } \hat{f}_1 = f_1 - f_0 \tag{2.22}$$

$$\text{Induced by the factor 2 (independent of 1) } \hat{f}_2 = f_2 - f_0 \tag{2.23}$$

$$\text{Induced by the synergism of factors 1 and 2 } \hat{f}_{12} = f_{12} - (f_1 + f_2) + f_0 \tag{2.24}$$

Second case: three factors

In the case of three factors, (2.16) yields eight (2^n) equations.

$$\hat{f}_0 = f_0 \tag{2.25}$$

$$\hat{f}_1 = f_1 - f_0 \tag{2.26}$$

$$\hat{f}_2 = f_2 - f_0 \tag{2.27}$$

$$\hat{f}_3 = f_3 - f_0 \tag{2.28}$$

$$\hat{f}_{12} = f_{12} - (f_1 + f_2) + f_0 \tag{2.29}$$

$$\hat{f}_{13} = f_{13} - (f_1 + f_3) + f_0 \tag{2.30}$$

$$\hat{f}_{23} = f_{23} - (f_2 + f_3) + f_0 \tag{2.31}$$

$$\hat{f}_{123} = f_{123} - (f_{12} + f_{13} + f_{23}) + (f_1 + f_2 + f_3) - f_0 \tag{2.32}$$

Hence, eight simulations are necessary with three factors.

The result is then not only the separation of the factors for $\hat{f}_1, \hat{f}_2, \hat{f}_3$, but also all the possible combinations of these factors, i.e. $\hat{f}_{12}, \hat{f}_{23}, \hat{f}_{13}$, and $\hat{f}_{123}$. The factor $\hat{f}_{123}$, for instance, is the contribution due to the triple interaction among the three factors under evaluation. Next, the method is analyzed in a study of the fractional approach.

2.2 The fractional approach: following Krichak and Alpert (2002) (KA, in brief)

Let the system of the model equations with both factors excluded be considered as a base system (BS). When the factors are included, the total time change of the model variable f at a particular point contains the contribution of each factor under consideration, as well as the contributions due to interactions among the factors. In addition, the f time change also contains contributions of the interactions of each factor being analyzed by the BS.

Let a and b be the two factors (physical effects) under deliberations 1 and 2, respectively.

In the case when both the a and b factors are taking part in the analysis:

$$f_0 = \hat{f}_0 \tag{2.33}$$

$$f_{01} = \hat{f}_0 + \hat{f}_1 + \hat{f}_{01} \tag{2.34}$$

$$f_{02} = \hat{f}_0 + \hat{f}_2 + \hat{f}_{02} \tag{2.35}$$

$$f_{012} = \hat{f}_0 + \hat{f}_1 + \hat{f}_2 + \hat{f}_{12} + \hat{f}_{01} + \hat{f}_{02} + \hat{f}_{012} \tag{2.36}$$

Here:

- f_0 is the time variation of a model characteristic f obtained by integration of the model with the chosen factors a and b excluded (BS);
- f_{01} is the time variation of a model characteristic f obtained by integration of the model with factor a included;
- f_{02} is the same, but for the b factor; and
- f_{012} is the same, but when both factors are included.

The $|\hat{}|$ terms of the $\hat{f}_\psi$ type represent fractions of f that are contributed by the ψ factor:

- $\hat{f}_{01}$ is the contribution of interaction of the factor a with the BS;
- $\hat{f}_{02}$ is the same, but for factor b;
- $\hat{f}_{012}$ is the contribution of the joint interaction of the two (a and b) factors with the BS;
- $\hat{f}_0$ is the pure contribution of the BS solution to the time variation of f.
- $\hat{f}_1$ is the pure contribution of the factor a to the time variation of f;
- $\hat{f}_2$ is the same as $\hat{f}_1$, but for factor b; and
- $\hat{f}_{12}$ is the pure contribution due to the a and b interaction – synergy a and b.

Solving the system (2.33)–(2.36) for the pure contributions $(\hat{f}_0, \hat{f}_1, \hat{f}_2, \hat{f}_{12})$ yields:

$$\hat{f}_0 = f_0 \tag{2.37}$$

$$\hat{f}_1 = f_{01} - f_0 - \hat{f}_{01} \tag{2.38}$$

$$\hat{f}_2 = f_{02} - f_0 - \hat{f}_{02} \tag{2.39}$$

$$\hat{f}_{12} = f_{012} - (f_{01} + f_{02}) + f_0 - \hat{f}_{012} \tag{2.40}$$

In many cases, the role of the BS-related contributions $(\hat{f}_{01}, \hat{f}_{02}, \hat{f}_{012})$ may be neglected. This allows using the standard FS formulation as suggested in the previous chapter. In this case, four model simulations are sufficient to evaluate the role of two chosen factors (a and b) and that of their synergic interaction. These are the simulations with the a and b factors excluded; the a factor is excluded, while b is included; the a factor is included, but b is excluded; and both a and b are included.

Thereby, after a demonstration of the methodology, we pass to mathematics. By applying the FS method to the simple mathematical functions, the trivial and basic rules are derived.