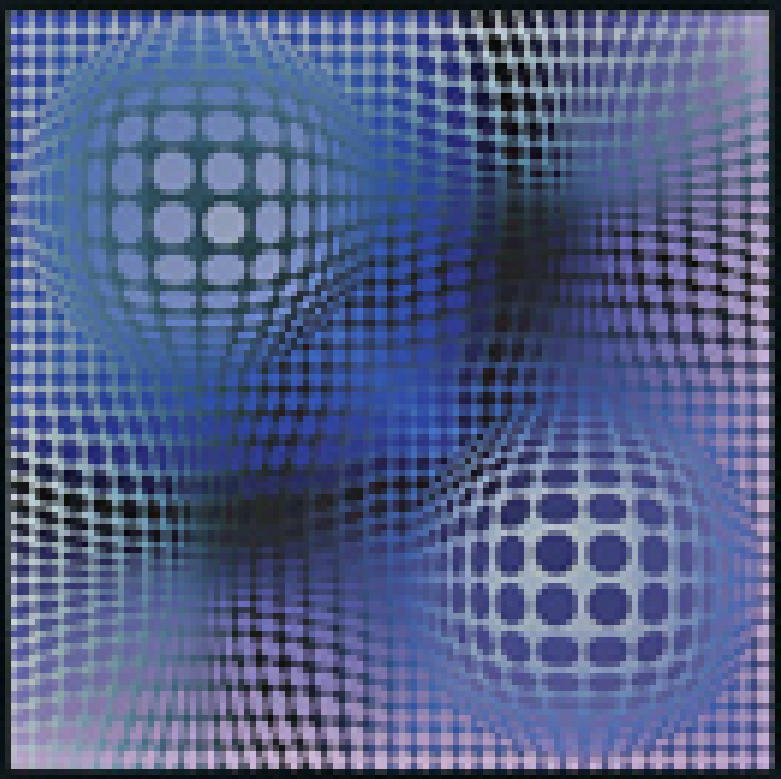
Introduction to

Optical Quantum Information Processing

Pieter Kok and Brendon W. Lovett



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Introduction to Optical Quantum Information Processing

Quantum information processing offers fundamental improvements over classical information processing, such as computing power, secure communication, and high-precision measurements. However, the best way to create practical devices is not yet known. This textbook describes the techniques that are likely to be used in implementing optical quantum information processors.

After developing the fundamental concepts in quantum optics and quantum information theory, this book shows how optical systems can be used to build quantum computers according to the most recent ideas. It discusses implementations based on single photons and linear optics, optically controlled atoms and solid-state systems, atomic ensembles, and optical continuous variables.

This book is ideal for graduate students beginning research in optical quantum information processing. It presents the most important techniques of the field using worked examples and over 120 exercises.

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Introduction to Optical Quantum Information Processing

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To Rose, Xander and Janet

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Preface

The field of quantum information processing has reached a level of maturity, and spans such a wide variety of topics, that it merits further specialization. In this book, we consider quantum information processing with optical systems, including quantum communication, quantum computation, and quantum metrology. Optical systems are the obvious choice for quantum communication, since photons are excellent carriers of quantum information due to their relatively slow decoherence. Indeed, many aspects of quantum communication have been demonstrated to the extent that commercial products are now available. The importance of optical systems for quantum communication leads us to ask whether we can construct integrated systems for communication and computation in which all processing takes place in optical systems. Recent developments indicate that while full-scale quantum computing is still extremely challenging, optical systems are one of the most promising approaches to a fully functional quantum computer.

This book is aimed at beginning graduate students who are starting their research career in optical quantum information processing, and it can be used as a textbook for an advanced master's course. The reader is assumed to have a background knowledge in classical electrodynamics and quantum mechanics at the level of an undergraduate physics course. The nature of the topic requires familiarity with quantized fields, and since this is not always a core topic in undergraduate physics, we derive the quantum mechanical formulation of the free electromagnetic field from first principles. Similarly, we aim to present the topics in quantum information theory in a self-contained manner.

The book is organized as follows: in Part I, we develop the quantum theory of light, give an introduction to quantum communication and computation, and we present a number of advanced quantum mechanical techniques that are essential for the understanding of optical quantum information processing. In Part II, we consider quantum information processing using single photons and atoms. We first develop the theory of photodetection and explore what we mean by photon sources, followed by an exposition of quantum communication with single photons, quantum computation with single photons and linear optics, and quantum computing where the information carriers, the qubits, are encoded in atoms. In Part III, we explore quantum information processing in many-body systems. We revisit linear optical quantum communication and computation, but now in the context of quantum continuous variables, rather than qubits. We discuss how atomic ensembles can be used as quantum memories and repeaters, and we study in detail how to define robust qubits in solid-state systems such as quantum dots and crystal defects. The last chapter of the book deals with quantum metrology, where we explore how quantum states of light can be exploited to attain a measurement precision that outperforms classical metrology. As is inevitable in a book of this nature, a number of important topics have been omitted due

to length restrictions. We have not included quantum information processing in ion traps, photonic band-gap materials, optical lattices, and Bose–Einstein condensates. We have also omitted the topic of quantum imaging.

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PART I

QUANTUM OPTICS AND QUANTUM INFORMATION

1 The quantum theory of light

Classically, light is an electromagnetic phenomenon, described by Maxwell's equations. However, under certain conditions, such as low intensity or in the presence of certain nonlinear optical materials, light starts to behave differently, and we have to construct a 'quantum theory of light'. We can exploit this quantum behaviour of light for quantum information processing, which is the subject of this book. In this chapter, we develop the quantum theory of the free electromagnetic quantum field. This means that we do not yet consider the interaction between light and matter; we postpone that to Chapter 7. We start from first principles, using the canonical quantization procedure in the Coulomb gauge: we derive the field equations of motion from the classical Lagrangian density for the vector potential, and promote the field and its canonical momentum to operators and impose the canonical commutation relations. This will lead to the well-known creation and annihilation operators, and ultimately to the concept of the photon. After quantization of the free electromagnetic field we consider transformations of the mode functions of the field. We will demonstrate the intimate relation between these linear mode transformations and the effect of beam splitters, phase shifters, and polarization rotations, and show how they naturally give rise to the concept of squeezing. Finally, we introduce coherent and squeezed states.

The first two sections of this chapter are quite formal, and a number of subtleties arise when we quantize the electromagnetic field, such as the continuum of modes, the gauge freedom, and the definition of the creation and annihilation operators with respect to the classical modes. Readers who have not encountered field quantization procedures before may find these sections somewhat daunting, but most of the subtleties encountered here have very little bearing on the later chapters. We mainly include the full derivation from first principles to give the field of optical quantum information processing a proper physical foundation, and derive the annihilation and creation operators of the discrete optical modes from the continuum of modes that is the electromagnetic field.

1.1 The classical electromagnetic field

Classical electrodynamics is the theory of the behaviour of electric and magnetic fields in the presence of charge and current distributions. It was shown by James Clerk Maxwell (1831–1879) that the equations of motion for electric and magnetic fields, the Maxwell equations, allow for electromagnetic waves. In vacuum, these waves propagate with a velocity $c = 299\,792\,458\ \mathrm{ms}^{-1}$, and Maxwell therefore identified these waves in a certain

frequency range with light. In this section, we define the electric and magnetic fields in terms of the scalar and vector potentials, and construct the field Lagrangian density in the presence of charge and current distributions. Variation of the Lagrangian density with respect to the potentials then leads to Maxwell's equations. Subsequently, we consider the Maxwell equations for the vacuum, and derive the wave equation and its plane-wave solutions. The source-free Lagrangian density is then used to define the canonical momenta to the potentials, which in turn allow us to give the Hamiltonian density for the free field. These are the ingredients we need for the canonical quantization procedure in Section 1.2.

The electric and magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ are related to a scalar and a vector potential $\Phi(\mathbf{r},t)$ and $\mathbf{A}(\mathbf{r},t)$:

$$\mathbf{E}(\mathbf{r},t) = -\nabla\Phi(\mathbf{r},t) - \frac{\partial \mathbf{A}(\mathbf{r},t)}{\partial t} \qquad \text{and} \qquad \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)\,. \tag{1.1}$$

The most elegant way to construct a classical field theory is via the Lagrangian density. We can use the potentials as the dynamical variables of our classical field theory, which means that we can write the Lagrangian density $\mathscr{L}$ as a function of the potentials and their time derivatives

$$\mathscr{L} = \mathscr{L}(\Phi, \dot{\Phi}; \mathbf{A}, \dot{\mathbf{A}})\,. \tag{1.2}$$

The equations of motion for the potentials Φ and $\mathbf{A}$ are then given by the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\delta\mathscr{L}}{\delta\dot{\Phi}} - \frac{\delta\mathscr{L}}{\delta\Phi} = 0 \tag{1.3}$$

and

$$\frac{d}{dt}\frac{\delta\mathscr{L}}{\delta\dot{A}_j} - \frac{\delta\mathscr{L}}{\delta A_j} = 0\,. \tag{1.4}$$

Here δ denotes the functional derivative, since the potentials are themselves functions of space and time, and each component of $\mathbf{A}$, denoted by A_j, obeys a separate Euler–Lagrange equation.

In the presence of a charge density $\rho(\mathbf{r},t)$ and a current density $\mathbf{J}(\mathbf{r},t)$ the general Lagrangian density of classical electrodynamics can be written as

$$\mathscr{L} = \mathbf{J}(\mathbf{r},t)\cdot\mathbf{A}(\mathbf{r},t) - \rho(\mathbf{r},t)\Phi(\mathbf{r},t) + \frac{\varepsilon_0}{2}E^2(\mathbf{r},t) - \frac{1}{2\mu_0}B^2(\mathbf{r},t)\,, \tag{1.5}$$

where $E^2 \equiv |\mathbf{E}|^2$ and $B^2 \equiv |\mathbf{B}|^2$ depend on Φ and $\mathbf{A}$ according to Eq. (1.1). When the Lagrangian density is varied with respect to Φ we obtain the Euler–Lagrange equation in Eq. (1.3), which can be written as Gauss' law

$$-\varepsilon_0\nabla\cdot\mathbf{E}(\mathbf{r},t) + \rho(\mathbf{r},t) = 0\,. \tag{1.6}$$

When we vary the Lagrangian density with respect to the components of $\mathbf{A}$, we find the Euler–Lagrange equations in Eq. (1.4). These can be reformulated as the Maxwell–Ampère law

$$\mathbf{J}(\mathbf{r},t) + \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}(\mathbf{r},t) - \frac{1}{\mu_0}\nabla\times\mathbf{B}(\mathbf{r},t) = 0\,. \tag{1.7}$$

The relations in Eq. (1.1) and Eqs. (1.6) and (1.7) are equivalent to Maxwell's equations, as can be seen by taking the curl of **E** in Eq. (1.1):

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}\,. \tag{1.8}$$

The last Maxwell equation, $\nabla\cdot\mathbf{B} = 0$, is implicit in $\mathbf{B} = \nabla\times\mathbf{A}$ since the divergence of any curl vanishes.

It is well known that we have a gauge freedom in defining the potentials Φ and **A** that constitute the fields **E** and **B**. Since we are interested in radiation, it is convenient to adopt the Coulomb, or radiation, gauge

$$\nabla\cdot\mathbf{A} = 0 \quad \text{and} \quad \Phi = 0\,. \tag{1.9}$$

In addition to the gauge choice, in this chapter we consider only the vacuum solutions of the electromagnetic fields:

$$\rho = 0 \quad \text{and} \quad \mathbf{J} = 0\,. \tag{1.10}$$

When we now write Eq. (1.7) in terms of the potentials, we obtain the homogeneous wave equation for **A**

$$\nabla^2\mathbf{A} - \varepsilon_0\mu_0\frac{\partial^2\mathbf{A}}{\partial t^2} = 0\,. \tag{1.11}$$

The classical solutions to this equation can be written as

$$\mathbf{A}(\mathbf{r},t) = \sum_\lambda \int \frac{d\mathbf{k}}{\sqrt{\varepsilon_0}} \frac{A_\lambda(\mathbf{k})\boldsymbol{\epsilon}_\lambda(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t}}{\sqrt{(2\pi)^3 2\omega_\mathbf{k}}} + \text{c.c.}, \tag{1.12}$$

where $A_\lambda(\mathbf{k})$ denotes the amplitude of the mode with wave vector **k** and polarization λ, and c.c. denotes the complex conjugate. The vector $\boldsymbol{\epsilon}_\lambda$ gives the direction of the polarization, which we will discuss in Section 1.3. The dispersion relation for the free field is given by

$$|\mathbf{k}|^2 - \varepsilon_0\mu_0\,\omega_\mathbf{k}^2 \equiv k^2 - \frac{\omega_\mathbf{k}^2}{c^2} = 0\,, \tag{1.13}$$

where c is the phase velocity of the wave with frequency $\omega_\mathbf{k}$. Any well-behaved potential $\mathbf{A}(\mathbf{r},t)$ that can be expressed as a superposition of Fourier components is a solution to the wave equation. This is exemplified by the fact that we can see different shapes, colours, etc., rather than just uniform plane waves.

Finally, the Lagrangian density can be used to find the Hamiltonian density of the field. To this end, we define the canonical momenta of Φ and **A** as

$$\Pi_\Phi \equiv \frac{\delta\mathscr{L}}{\delta\dot{\Phi}} \quad \text{and} \quad \Pi_\mathbf{A} \equiv \frac{\delta\mathscr{L}}{\delta\dot{\mathbf{A}}}\,. \tag{1.14}$$

We can now take the Legendre transform of the Lagrangian density with respect to the dynamical variables $\dot{\Phi}$ and $\dot{\mathbf{A}}$ to obtain the Hamiltonian density of the free electromagnetic field

$$\mathcal{H}(\Phi, \mathbf{\Pi}_\Phi; \mathbf{A}, \mathbf{\Pi}_\mathbf{A}) = \mathbf{\Pi}_\Phi \dot{\Phi} + \mathbf{\Pi}_\mathbf{A} \cdot \dot{\mathbf{A}} - \mathscr{L} \,. \tag{1.15}$$

In the Coulomb gauge, the canonical momenta are

$$\Pi_\Phi = 0 \quad \text{and} \quad \Pi_\mathbf{A} = \varepsilon_0 \dot{\mathbf{A}} \,. \tag{1.16}$$

This leads to the Hamiltonian density for the free field

$$\mathcal{H} = \Pi_\mathbf{A} \cdot \dot{\mathbf{A}} - \mathscr{L}|_{\rho=\mathbf{J}=0} = \frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \,. \tag{1.17}$$

We now have all the necessary ingredients to proceed with the quantization of the electromagnetic field.

Exercise 1.1: Derive the homogeneous wave equation in Eq. (1.11) and show that the solutions are given by Eq. (1.12).

1.2 Quantization of the electromagnetic field

We are now ready to quantize the classical electromagnetic field. First, we have to decide which of the fields **A**, **E** (or **B**) we wish to quantize. In later chapters, we discuss the coupling between light and matter, and it is most convenient to express that coupling in terms of the vector potential **A**. We therefore apply the quantization procedure to **A**, rather than to **E**. In the quantization procedure we have to ensure that the quantum fields obey Maxwell's equations in the classical limit, and this leads to the introduction of a modified Dirac delta function. After the formal quantization, we explore the properties of the mode functions and the mode operators that result from the quantization procedure, and establish a fundamental relationship between them. We then construct eigenstates of the Hamiltonian, and define the discrete, physical modes. This leads to the concept of the photon. The final part of this section is devoted to the construction of the quantum mechanical field observables associated with single modes.

1.2.1 Field quantization

We denote the difference between classical and quantum mechanical observables by writing the latter with a hat. In the quantum theory of light, **A** and $\mathbf{\Pi}_\mathbf{A}$ then become operators satisfying the equal-time commutation relations. In index notation these are written as

$$\left[\hat{A}^j(\mathbf{r}, t), \hat{A}^k(\mathbf{r}', t)\right] = \left[\hat{\Pi}^j_\mathbf{A}(\mathbf{r}, t), \hat{\Pi}^k_\mathbf{A}(\mathbf{r}', t)\right] = 0 \,. \tag{1.18}$$

The field consists of four variables: three from $\mathbf{A}$ and one from Φ. We again work in the Coulomb gauge, where $\Phi = 0$ and $\nabla \cdot \mathbf{A} = 0$ ensures that we end up with only two dynamical variables.

Standard canonical quantization prescribes that, in addition to Eq. (1.18), we impose the following commutation relation:

$$\left[\hat{A}_i(\mathbf{r}, t), \hat{\Pi}^j_{\mathbf{A}}(\mathbf{r}', t)\right] = i\hbar\delta_{ij}\,\delta^3(\mathbf{r} - \mathbf{r}')\,, \tag{1.19}$$

where we must remember the difference between upper and lower indices, $A_j = -A^j$, because electrodynamics is, at heart, a relativistic theory. Unfortunately, given that in the Coulomb gauge $\Pi^k_{\mathbf{A}} \propto E^k$, this commutation relation is not compatible with Gauss' law in vacuum: $\nabla \cdot \mathbf{E} = 0$. If we take the divergence with respect to the variable $\mathbf{r}'$ on both sides of Eq. (1.19), the left-hand side will be zero, but the divergence of the delta function does not vanish. We therefore have to modify the delta function such that its divergence does vanish. For the ordinary Dirac delta function we use the following definition:

$$\delta_{ij}\delta^3(\mathbf{r} - \mathbf{r}') \equiv \int \frac{d\mathbf{k}}{(2\pi)^3}\,\delta_{ij}\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,. \tag{1.20}$$

We have included the Kronecker delta δ_{ij}, because after the redefinition of the delta function the internal degree of freedom j and the external degree of freedom $\mathbf{r}$ may no longer be independent (in fact, they will not be). Taking the divergence of Eq. (1.20) with respect to $\mathbf{r}$ yields

$$\sum_i \partial_i\,\delta_{ij}\,\delta^3(\mathbf{r} - \mathbf{r}') = i\int \frac{d\mathbf{k}}{(2\pi)^3}\,k_j e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,. \tag{1.21}$$

Therefore, we have to subtract something like this from the redefined delta function. We write

$$\Delta_{ij}(\mathbf{r} - \mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^3}\,\delta_{ij}\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} - i\int \frac{d\mathbf{k}}{(2\pi)^3}\,\alpha_i k_j e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\,, \tag{1.22}$$

and we want to find α_i such that $\partial_i\Delta_{ij}(\mathbf{r} - \mathbf{r}') = 0$:

$$\begin{aligned}\sum_i \partial_i\Delta_{ij}(\mathbf{r} - \mathbf{r}') &= \sum_i \partial_i \int \frac{d\mathbf{k}}{(2\pi)^3}\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\left(\delta_{ij} - ik_j\alpha_i\right)\\ &= \sum_i \int \frac{d\mathbf{k}}{(2\pi)^3}\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\left[ik_i\delta_{ij} - (ik_j)(ik_i)\alpha_i\right]\\ &= 0\,.\end{aligned} \tag{1.23}$$

We therefore have that

$$\sum_i \left(ik_i\delta_{ij} + k_i k_j\alpha_i\right) = 0\,, \quad \text{or} \quad \alpha_i = -i\frac{k_i}{|\mathbf{k}|^2}\,, \tag{1.24}$$

and the 'transverse' delta function $\Delta_{ij}(\mathbf{r}-\mathbf{r}')$ becomes

$$\Delta_{ij}(\mathbf{r}-\mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right). \tag{1.25}$$

Using this modified delta function, we can complete the quantization procedure by imposing the equal-time canonical commutation relation

$$\left[\hat{A}_i(\mathbf{r},t), \hat{\Pi}^j_{\mathbf{A}}(\mathbf{r}',t)\right] = i\hbar\,\Delta_{ij}(\mathbf{r}-\mathbf{r}')\,. \tag{1.26}$$

That this leads to the correct covariant Hamiltonian and momentum is shown, for example, in Bjorken and Drell (1965). We can now write the three space components of the quantum field as

$$\hat{A}_j(\mathbf{r},t) = \sum_{\lambda=1}^{2} \int d\mathbf{k} \sqrt{\frac{\hbar}{\varepsilon_0}} \left[\epsilon_{\lambda j}(\mathbf{k})\hat{a}_\lambda(\mathbf{k})u(\mathbf{k};\mathbf{r},t) + \epsilon^*_{\lambda j}(\mathbf{k})\hat{a}^\dagger_\lambda(\mathbf{k})u^*(\mathbf{k};\mathbf{r},t)\right], \tag{1.27}$$

where the $u(\mathbf{k};\mathbf{r},t)$ are mode functions that are themselves solutions to the wave equation in Eq. (1.11), and λ again indicates the polarization of the electromagnetic field. The classical amplitudes $A_\lambda(\mathbf{k})$ are replaced by the operators $\hat{a}_\lambda(\mathbf{k})$, and $\hat{\mathbf{A}}$ is now a *quantum* field. Note that $\hat{\mathbf{A}}$ is now an operator, and unlike its classical counterpart does not directly represent a particular vector potential. Specific quantum mechanical vector potentials are represented by quantum states.

The equal-time commutation relation in Eq. (1.26) determines the commutation relation for $\hat{a}_\lambda(\mathbf{k})$ and $\hat{a}^\dagger_\lambda(\mathbf{k})$, given the mode functions $u(\mathbf{k};\mathbf{r},t)$ and $u^*(\mathbf{k},\mathbf{r},t)$. From Eq. (1.12) we can read off the plane-wave solutions with continuum normalization:

$$u(\mathbf{k};\mathbf{r},t) = \frac{e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t}}{\sqrt{(2\pi)^3 2\omega_\mathbf{k}}}, \tag{1.28}$$

where $\mathbf{k}$ is the wave vector of a wave with frequency $\omega_\mathbf{k}$. Plane waves are of constant intensity throughout space and time, and are therefore unphysical. However, they are mathematically very convenient. When the mode functions are the plane waves defined in Eq. (1.28), we find explicitly that

$$\left[\hat{a}_\lambda(\mathbf{k}), \hat{a}^\dagger_{\lambda'}(\mathbf{k}')\right] = \delta_{\lambda\lambda'}\delta^3(\mathbf{k}-\mathbf{k}')\,, \tag{1.29}$$

and

$$\left[\hat{a}_\lambda(\mathbf{k}), \hat{a}_{\lambda'}(\mathbf{k}')\right] = \left[\hat{a}^\dagger_\lambda(\mathbf{k}), \hat{a}^\dagger_{\lambda'}(\mathbf{k}')\right] = 0\,. \tag{1.30}$$

The operator $\hat{a}_\lambda(\mathbf{k})$ and its Hermitian conjugate are the 'mode operators' of the quantized electromagnetic field. In the next section we will see that any operators that obey these commutation relations are good mode operators.

We are now done with the quantization of the classical electromagnetic field, and the remainder of this section is devoted to the exploration of the direct consequences of this procedure.

Exercise 1.2: Derive the commutation relations in Eqs. (1.29) and (1.30).

1.2.2 Mode functions and mode operators

We will now discuss some of the fundamental properties of the mode functions $u(\mathbf{k};\mathbf{r},t)$ and $u^*(\mathbf{k};\mathbf{r},t)$, and the mode operators $\hat{a}_\lambda(\mathbf{k})$ and $\hat{a}^\dagger_\lambda(\mathbf{k})$. In order to study the mode functions of the field (its 'shape', if you like), we must first define a scalar product that allows us to talk about orthogonal mode functions. This is given by the 'time-independent scalar product'

$$(\phi,\psi) \equiv i\int d\mathbf{r}\,\phi^* \overset{\leftrightarrow}{\partial}_t \psi = i\int d\mathbf{r}\,\left[\phi^*(\partial_t\psi) - (\partial_t\phi^*)\psi\right]. \tag{1.31}$$

From the general structure of the scalar product in Eq. (1.31) we see that

$$(\phi,\psi)^* = (\psi,\phi) \quad \text{and} \quad (\phi^*,\psi^*) = -(\psi,\phi). \tag{1.32}$$

This scalar product finds its origin in the continuity equation of the field, which determines the conserved currents (see Bjorken and Drell, 1965). It is therefore time-independent. The completeness relation of the mode functions $u(\mathbf{k};\mathbf{r},t)$ is then derived as follows: consider a function $f(\mathbf{r},t)$ that is a superposition of different mode functions

$$f = \int d\mathbf{k}\,\left[\alpha(\mathbf{k})u(\mathbf{k}) + \beta(\mathbf{k})u^*(\mathbf{k})\right], \tag{1.33}$$

where we have suppressed the dependence on $\mathbf{r}$ and t in $f(\mathbf{r},t)$ and $u(\mathbf{k};\mathbf{r},t)$ for notational brevity. Using the orthogonality of the mode functions defined by the scalar product, we can write the coefficients $\alpha(\mathbf{k})$ and $\beta(\mathbf{k})$ as

$$\alpha(\mathbf{k}) = (u(\mathbf{k}),f) \quad \text{and} \quad \beta(\mathbf{k}) = -(u^*(\mathbf{k}),f). \tag{1.34}$$

This leads to an expression for f

$$f = \int d\mathbf{k}\,\left[(u(\mathbf{k}),f)\,u(\mathbf{k}) - (u^*(\mathbf{k}),f)\,u^*(\mathbf{k})\right]. \tag{1.35}$$

For a second superposition of mode functions g the scalar product (g,f) can be written as

$$(g,f) = \int d\mathbf{k}\,\left[(g,u(\mathbf{k}))(u(\mathbf{k}),f) - (g,u^*(\mathbf{k}))(u^*(\mathbf{k}),f)\right]. \tag{1.36}$$

This constitutes the 'completeness relation' for the mode functions $u(\mathbf{k};\mathbf{r},t)$, and it holds only if the mode functions are, in fact, complete.

Using the definition of the time-independent scalar product, we can show that plane-wave solutions are orthonormal:

$$\begin{aligned}(u_{\mathbf{k}},u_{\mathbf{k}'}) &\equiv i\int d\mathbf{r}\,\frac{e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_{\mathbf{k}}t}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}} \overset{\leftrightarrow}{\partial}_t \frac{e^{i\mathbf{k}'\cdot\mathbf{r}-i\omega_{\mathbf{k}'}t}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}'}}} \\ &= \int \frac{d\mathbf{r}}{(2\pi)^3}\,\frac{(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'})e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}+i(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})t}}{2\sqrt{\omega_{\mathbf{k}}\omega_{\mathbf{k}'}}} \\ &= \delta^3(\mathbf{k}-\mathbf{k}').\end{aligned} \tag{1.37}$$

We also find by direct evaluation that $(u_{\mathbf{k}}, u^*_{\mathbf{k}'}) = 0$. This can be understood physically as the orthogonality of waves moving forward in time, and waves moving backwards in time and in opposite directions. The plane waves therefore form a complete orthonormal set of mode functions.

We can define a new set of mode functions $v(\boldsymbol{\kappa}; \mathbf{r}, t)$, which are a linear combination of plane waves

$$v(\boldsymbol{\kappa}; \mathbf{r}, t) = \int d\mathbf{k}\, V(\boldsymbol{\kappa}, \mathbf{k}) u(\mathbf{k}; \mathbf{r}, t) = \int d\mathbf{k}\, V(\boldsymbol{\kappa}, \mathbf{k}) \frac{e^{i\mathbf{k}\cdot\mathbf{r} - i\omega_{\mathbf{k}} t}}{\sqrt{(2\pi)^3 2\omega_{\mathbf{k}}}}\,. \tag{1.38}$$

Two symbols are needed for wave vectors here, namely $\mathbf{k}$ and $\boldsymbol{\kappa}$. We emphasize that we will normally reserve $\mathbf{k}$ for describing wave vectors. When $V(\boldsymbol{\kappa}, \mathbf{k})$ is unitary, the new mode functions $v(\boldsymbol{\kappa}; \mathbf{r}, t)$ are also orthonormal. When we express $\hat{\mathbf{A}}$ in terms of the new mode functions, we should also change the operators $\hat{a}_\lambda(\mathbf{k})$ to $\hat{b}_{\lambda'}(\boldsymbol{\kappa})$, since the mode operators depend on $\mathbf{k}$ and will generally change due to the transformation $V(\boldsymbol{\kappa}, \mathbf{k})$. The field operator then becomes

$$\hat{A}_j(\mathbf{r}, t) = \sum_{\lambda'=1}^{2} \int d\boldsymbol{\kappa} \sqrt{\frac{\hbar}{\varepsilon_0}} \left[\epsilon_{\lambda' j}(\boldsymbol{\kappa})\, \hat{b}_{\lambda'}(\boldsymbol{\kappa}) v(\boldsymbol{\kappa}; \mathbf{r}, t) + \epsilon^*_{\lambda' j}(\boldsymbol{\kappa})\, \hat{b}^\dagger_{\lambda'}(\boldsymbol{\kappa}) v^*(\boldsymbol{\kappa}; \mathbf{r}, t) \right]. \tag{1.39}$$

Note that here we have also included a possible change in the polarizarion degree of freedom λ', which can be incorporated straightforwardly in the time-independent scalar product.

Exercise 1.3: Prove the orthonormality of $v(\boldsymbol{\kappa}; \mathbf{r}, t)$ if V is unitary.

We next explore the precise relationship between mode functions and mode operators. The mode operators $\hat{a}_\lambda(\mathbf{k})$ and $\hat{a}^\dagger_\lambda(\mathbf{k})$ are related to the mode functions $u(\mathbf{k}; \mathbf{r}, t)$ and $u^*(\mathbf{k}; \mathbf{r}, t)$ via the time-independent scalar product

$$\begin{aligned} \hat{a}_{\lambda'}(\mathbf{k}) &\equiv \sqrt{\frac{\varepsilon_0}{\hbar}} \left(u(\mathbf{k}) \boldsymbol{\epsilon}_{\lambda'}, \hat{\mathbf{A}} \right) \\ &= i \sqrt{\frac{\varepsilon_0}{\hbar}} \int d\mathbf{r}\, u^*(\mathbf{k}; \mathbf{r}, t) \overleftrightarrow{\partial}_t \boldsymbol{\epsilon}^*_{\lambda'}(\mathbf{k}) \cdot \hat{\mathbf{A}}(\mathbf{r}, t)\,. \end{aligned} \tag{1.40}$$

We can then extract the operator $\hat{b}_{\lambda'}(\boldsymbol{\kappa})$, associated with the mode function $v(\boldsymbol{\kappa}; \mathbf{r}, t)$, using the procedure

$$\hat{b}_{\lambda'}(\boldsymbol{\kappa}) \equiv \sqrt{\frac{\varepsilon_0}{\hbar}} \left(v(\boldsymbol{\kappa}) \boldsymbol{\epsilon}_{\lambda'}, \hat{\mathbf{A}} \right) = i \sqrt{\frac{\varepsilon_0}{\hbar}} \int d\mathbf{r}\, v^*(\boldsymbol{\kappa}; \mathbf{r}, t) \overleftrightarrow{\partial}_t \boldsymbol{\epsilon}^*_{\lambda'}(\boldsymbol{\kappa}) \cdot \hat{\mathbf{A}}(\mathbf{r}, t)\,. \tag{1.41}$$

This is a *definition* of the operator $\hat{b}_{\lambda'}(\boldsymbol{\kappa})$, and is completely determined by the mode function $v(\boldsymbol{\kappa})$ and polarization vector $\boldsymbol{\epsilon}_{\lambda'}(\boldsymbol{\kappa})$. Now suppose that we have an expression for $\hat{\mathbf{A}}(\mathbf{r}, t)$ in terms of mode functions $u(\mathbf{k})$, mode operators $\hat{a}_\lambda(\mathbf{k})$, and polarization vectors $\boldsymbol{\epsilon}_\lambda(\mathbf{k})$ given in Eq. (1.27). The mode operator $\hat{b}_{\lambda'}(\boldsymbol{\kappa})$ then becomes

$$\hat{b}_{\lambda'}(\boldsymbol{\kappa}) = \sum_\lambda \int d\mathbf{k} \left[\boldsymbol{\epsilon}^*_{\lambda'}(\boldsymbol{\kappa}) \cdot \boldsymbol{\epsilon}_\lambda(\mathbf{k})\, (v, u)\, \hat{a}_\lambda(\mathbf{k}) + \boldsymbol{\epsilon}^*_{\lambda'}(\boldsymbol{\kappa}) \cdot \boldsymbol{\epsilon}^*_\lambda(\mathbf{k}) \left(v, u^*\right) \hat{a}^\dagger_\lambda(\mathbf{k}) \right], \tag{1.42}$$

where we express the mode operator $\hat{b}$ in terms of mode operators $\hat{a}$ and $\hat{a}^\dagger$. The spatial integration in the scalar products (v,u) and (v,u^*) must be evaluated before the integration over $\mathbf{k}$ in order to make the scalar product of the two polarization vectors $\boldsymbol{\epsilon}_{\lambda'}(\boldsymbol{\kappa}) \cdot \boldsymbol{\epsilon}_\lambda(\mathbf{k})$ definite. This demonstrates that the mode *operators* have a notion of orthogonality that is directly inherited from the orthogonality of the mode *functions*. Up to addition of a complex constant, Eq. (1.42) is the most general linear transformation of the mode operators, and is called the 'Bogoliubov transformation'. In principle it can mix the mode operators with their adjoints when the scalar product (v,u^*) is non-zero.

Exercise 1.4: Using Eq. (1.42), show that

$$\left[\hat{b}_\lambda(\boldsymbol{\kappa}), \hat{b}^\dagger_{\lambda'}(\boldsymbol{\kappa}')\right] = \delta_{\lambda\lambda'}\,\delta^3(\boldsymbol{\kappa}-\boldsymbol{\kappa}')\,, \tag{1.43}$$

and

$$\left[\hat{b}_\lambda(\boldsymbol{\kappa}), \hat{b}_{\lambda'}(\boldsymbol{\kappa}')\right] = \left[\hat{b}^\dagger_\lambda(\boldsymbol{\kappa}), \hat{b}^\dagger_{\lambda'}(\boldsymbol{\kappa}')\right] = 0 \tag{1.44}$$

These are the expected commutation relations for the mode operators.

1.2.3 Photons as excitations of the electromagnetic field

The revolutionary aspect of the quantum mechanical description of the electromagnetic field is the notion that the field can deliver its energy only in discrete amounts. This leads to the concept of the 'photon'. In order to derive this from the quantum theory, we first consider the Hamiltonian and momentum operators for the quantum field. We then construct energy eigenstates, and regularize them to obtain well-behaved physical states of the electromagnetic field.

From the quantum mechanical version of Eq. (1.17) we can formally derive the Hamiltonian operator $\mathcal{H}$ of the free field as

$$\begin{aligned}\mathcal{H} &= \sum_\lambda \int d\mathbf{k}\, \frac{\hbar\omega_\mathbf{k}}{2}\left[\hat{a}^\dagger_\lambda(\mathbf{k})\hat{a}_\lambda(\mathbf{k}) + \hat{a}_\lambda(\mathbf{k})\hat{a}^\dagger_\lambda(\mathbf{k})\right] \\ &\equiv \sum_\lambda \int d\mathbf{k}\, \mathcal{H}_\lambda(\mathbf{k})\,,\end{aligned} \tag{1.45}$$

where $\mathcal{H}_\lambda(\mathbf{k})$ will be called the 'single-mode Hamiltonian operator'. Similarly, the 'field momentum operator' is

$$\hat{\mathbf{P}} = \sum_\lambda \int d\mathbf{k}\, \frac{\hbar\mathbf{k}}{2}\left[\hat{a}^\dagger_\lambda(\mathbf{k})\hat{a}_\lambda(\mathbf{k}) + \hat{a}_\lambda(\mathbf{k})\hat{a}^\dagger_\lambda(\mathbf{k})\right]. \tag{1.46}$$

This operator is similar to $\mathcal{H}$, but $\omega_\mathbf{k}$ is replaced by $\mathbf{k}$. Therefore, the properties we derive for the Hamiltonian can easily be translated into properties for the momentum. The field momentum can be formally derived from Maxwell's stress tensor, but this is beyond the scope of this book.

The Hamiltonian and the operators $\hat{a}_\lambda(\mathbf{k})$ and $\hat{a}^\dagger_\lambda(\mathbf{k})$ obey the following commutation relations:

$$\left[\mathcal{H}, \hat{a}_\lambda(\mathbf{k})\right] = -\hbar\omega_\mathbf{k}\,\hat{a}_\lambda(\mathbf{k}) \quad \text{and} \quad \left[\mathcal{H}, \hat{a}^\dagger_\lambda(\mathbf{k})\right] = \hbar\omega_\mathbf{k}\,\hat{a}^\dagger_\lambda(\mathbf{k})\,. \tag{1.47}$$

We can define $|\psi_n\rangle$ as an eigenstate of the Hamiltonian $\mathcal{H}$ such that

$$\mathcal{H}\,|\psi_n\rangle = E_n\,|\psi_n\rangle\ , \tag{1.48}$$

with E_n the corresponding energy eigenvalue (which depends on $\omega_\mathbf{k}$). From the commutation relations in Eq. (1.47) we find

$$\mathcal{H}\hat{a}^\dagger_\lambda(\mathbf{k})\,|\psi_n\rangle = \hat{a}^\dagger_\lambda(\mathbf{k})\,(\mathcal{H} + \hbar\omega_\mathbf{k})\,|\psi_n\rangle = (E_n + \hbar\omega_\mathbf{k})\,\hat{a}^\dagger_\lambda(\mathbf{k})\,|\psi_n\rangle\ . \tag{1.49}$$

This means that $\hat{a}^\dagger_\lambda(\mathbf{k})\,|\psi_n\rangle$ is again an eigenstate of the Hamiltonian $\mathcal{H}$ with energy $E_n+\hbar\omega_\mathbf{k}$. Similarly, $\hat{a}_\lambda(\mathbf{k})\,|\psi_n\rangle$ is again an eigenstate of the Hamiltonian $\mathcal{H}$ with energy $E_n - \hbar\omega_\mathbf{k}$. The eigenvalues of $\mathcal{H}$ must be bounded from below, and there is therefore a ground state $|\psi_0\rangle$ for which $\hat{a}_\lambda(\mathbf{k})\,|\psi_0\rangle = 0$ for any $\mathbf{k}$ and λ. We call this state the 'vacuum state' for that mode. This leads to the vacuum energy

$$\mathcal{H}\,|\psi_0\rangle = \frac{\hbar}{2}\sum_\lambda \int d\mathbf{k}\,\omega_\mathbf{k}\hat{a}_\lambda(\mathbf{k})\hat{a}^\dagger_\lambda(\mathbf{k})\ |\psi_0\rangle\ . \tag{1.50}$$

Using the commutation relation in Eq. (1.29), we find that this expression diverges. The vacuum energy is therefore infinite, and needs to be subtracted in all practical calculations of energy expectation values. However, we cannot completely ignore the vacuum energy, since it leads to so-called vacuum fluctuations. It has observable consequences, such as the Lamb shift and vacuum noise. This will become important in certain applications of optical quantum information processing, e.g., in Chapters 8 and 9.

For every mode denoted by $\mathbf{k}$ and λ, we can construct a set of basis states that are eigenstates of the Hamiltonian by repeatedly applying the operator $\hat{a}^\dagger_\lambda(\mathbf{k})$ to the ground state $|\psi_0\rangle$. After removing the infinite vacuum energy, we can find the relative energy of each mode:

$$E_n(\mathbf{k}) = n\,\hbar\omega_\mathbf{k}\,, \tag{1.51}$$

and we can write the energy eigenstate of mode $\{\mathbf{k}, \lambda\}$ as $|n\rangle_{\mathbf{k},\lambda}$ or $|n_{\mathbf{k},\lambda}\rangle$. These states are commonly known as 'Fock states'.

We can similarly define the operator

$$\hat{n} = \sum_\lambda \int d\mathbf{k}\,\hat{a}^\dagger_\lambda(\mathbf{k})\hat{a}_\lambda(\mathbf{k})\,. \tag{1.52}$$

The Fock states are clearly also eigenstates of $\hat{n}$:

$$\hat{n}|n\rangle_{\mathbf{k}',\lambda} = n|n\rangle_{\mathbf{k}',\lambda}\,, \tag{1.53}$$

and for obvious reasons $\hat{n}$ is called the number operator. We have seen that $a_\lambda^\dagger(\mathbf{k})$ and $a_\lambda(\mathbf{k})$ move us up and down the ladder of Fock states, and are therefore called the 'creation' and 'annihilation' operators, respectively.

However, the state created by applying the creation operator $\hat{a}_\lambda^\dagger(\mathbf{k})$, associated with a plane wave, to the vacuum is not normalizable, and therefore unphysical. For example, the scalar product of two single-excitation Fock states is

$$\langle 1_{\mathbf{k}',\lambda'} | 1_{\mathbf{k},\lambda} \rangle = \delta_{\lambda,\lambda'}\, \delta(\mathbf{k} - \mathbf{k}') \,. \tag{1.54}$$

In order to define physical states, we must construct well-behaved mode functions. Earlier, we have seen that we can superpose plane waves

$$f(\mathbf{r},t) = \int d\mathbf{k} \left[\alpha^*(\mathbf{k})\, u(\mathbf{k};\mathbf{r},t) + \beta^*(\mathbf{k})\, u^*(\mathbf{k};\mathbf{r},t)\right] , \tag{1.55}$$

which can be normalized according to

$$(f,f) = 1 \quad \Rightarrow \quad \int d\mathbf{k} \left[|\alpha(\mathbf{k})|^2 - |\beta(\mathbf{k})|^2\right] = 1 \,. \tag{1.56}$$

In general, f can be any well-behaved function, but in practice we often assume that f is sharply peaked around a central wave vector $\mathbf{k}_0$. This can then be considered a normalizable frequency mode, even though it is strictly an approximation.

Using the definition of the mode operators in Eq. (1.40), we can then write the mode operator for the mode f with polarization λ as

$$\hat{b}_{f\lambda} = \sqrt{\frac{\varepsilon}{\hbar}} \left(\boldsymbol{\epsilon}_\lambda f, \hat{\mathbf{A}}\right) = \int d\mathbf{k} \left[\alpha(\mathbf{k})\, \hat{a}_\lambda(\mathbf{k}) + \beta(\mathbf{k})\, \hat{a}_\lambda^\dagger(\mathbf{k})\right] . \tag{1.57}$$

By definition, the mode operators $\hat{b}_{f\lambda}$ and $\hat{b}_{f\lambda}^\dagger$ obey the commutation relation

$$\left[\hat{b}_{f\lambda}, \hat{b}_{f\lambda'}^\dagger\right] = \delta_{\lambda\lambda'} \,, \tag{1.58}$$

and if f is part of an orthonormal set of mode functions $\{f_1, f_2, \ldots\}$, the commutation relations become

$$\left[\hat{b}_{j\lambda}, \hat{b}_{k\lambda'}^\dagger\right] = \delta_{\lambda\lambda'}\, \delta_{jk} \quad \text{and} \quad \left[\hat{b}_{j\lambda}, \hat{b}_{k\lambda'}\right] = \left[\hat{b}_{j\lambda}^\dagger, \hat{b}_{k\lambda'}^\dagger\right] = 0 \,, \tag{1.59}$$

where the subscripts j and k indicate the mode functions f_j and f_k, respectively. In the remainder of this book, we will mostly consider these well-behaved, discretized mode functions with their associated mode operators.

Assuming that $\beta(\mathbf{k}) = 0$, a single excitation in mode f_j with polarization λ can be written as

$$\left|1_{j\lambda}\right\rangle = \hat{b}_{j\lambda}^\dagger \left|0\right\rangle = \int d\mathbf{k}\, \alpha_j^*(\mathbf{k})\, \hat{a}_\lambda^\dagger(\mathbf{k}) \left|0\right\rangle \,, \tag{1.60}$$

where $\alpha_j(\mathbf{k})$ is associated with f_j, c.f., Eq. (1.55). This is the state of a 'wave packet', albeit of infinite duration. It follows that

$$
\begin{aligned}
{}_{j\lambda}\langle 1\,|1\rangle_{j\lambda'} &= \delta_{\lambda,\lambda'} \int d\mathbf{k}\, d\mathbf{k}'\, \alpha_j^*(\mathbf{k})\alpha_j(\mathbf{k}')\hat{a}_\lambda(\mathbf{k})\hat{a}_{\lambda'}^\dagger(\mathbf{k}') \\
&= \delta_{\lambda,\lambda'} \int d\mathbf{k}\, |\alpha_j(\mathbf{k})|^2 \equiv \delta_{\lambda,\lambda'}\,,
\end{aligned}
\tag{1.61}
$$

where we used Eq. (1.56). Wave packets with higher numbers of excitations are easily generated by applying the discrete creation operator to the vacuum several times

$$
|n\rangle_{j\lambda} = \frac{(\hat{b}_{j\lambda}^\dagger)^n}{\sqrt{n!}}\,|0\rangle\,. \tag{1.62}
$$

The factor $\sqrt{n!}$ is found by normalizing the states and using the commutation relations. We can now be precise about the action of the discrete operators as follows

$$
\begin{aligned}
\hat{b}_{j\lambda}\,|n\rangle_{j\lambda} &= \sqrt{n}\,|n-1\rangle_{f\lambda} \\
\hat{b}_{j\lambda}^\dagger\,|n\rangle_{j\lambda} &= \sqrt{n+1}\,|n+1\rangle_{j\lambda}\,.
\end{aligned}
\tag{1.63}
$$

These relations strongly suggest an interpretation of $|n\rangle_{j\lambda}$ as a state of 'n particles'. The particle is created by the action of $\hat{b}_{j\lambda}^\dagger$ and annihilated by the action of $\hat{b}_{j\lambda}$, and is called the 'photon'. Photons in sharply peaked frequency modes have well-defined energy and momentum.

The interpretation of $|1\rangle_{j\lambda}$ as a physical particle is not without problems. The photon is massless, and is therefore the ultimate relativistic particle, travelling at the speed of light. As a consequence, there is no position operator for the photon, unlike for traditional particles. Another subtlety is that the superposition f can in principle be chosen with non-zero $\beta(\mathbf{k})$, which means that $\hat{b}$ is a superposition of annihilation and creation operators $\hat{a}(\mathbf{k})$ and $\hat{a}^\dagger(\mathbf{k})$. The concept of a photon therefore depends on the mode functions and the coordinate system. We will encounter this mixing of creation and annihilation operators later in this chapter, when we describe optical squeezing.

1.2.4 Quadrature operators

We now return to the continuous mode operators $\hat{a}(\mathbf{k})$ and $\hat{a}^\dagger(\mathbf{k})$, and momentarily suppress the polarization degree of freedom λ for notational brevity. The creation and annihilation operators are not Hermitian, and are therefore not associated directly with physical observables. However, they can be used to construct Hermitian operators. For example, for any operator $\hat{F}$ we can construct a Hermitian operator $\hat{F}^\dagger\hat{F}$, and with $\hat{F} = \hat{a}$ this led to the number operator $\hat{a}^\dagger\hat{a}$. We can also construct the Hermitian operator $\hat{F} + \hat{F}^\dagger$. Including an extra phase freedom, this leads us to define the 'quadrature operator' $\hat{x}_\zeta$ of the general form

$$
\hat{x}_\zeta(\mathbf{k}) = \frac{e^{-i\zeta}\,\hat{a}(\mathbf{k}) + e^{i\zeta}\,\hat{a}^\dagger(\mathbf{k})}{\sqrt{2}}\,, \tag{1.64}
$$

for a particular mode $\mathbf{k}$. The natural observables associated with $\hat{x}_\zeta(\mathbf{k})$ are not dimensionless, and in order to find the constant of proportionality, we use the well-known fact that the free field single-mode Hamiltonian operator is formally identical to the Hamiltonian of the simple harmonic oscillator. It can therefore be written as

$$\mathcal{H}_\lambda(\mathbf{k}) = \omega_\mathbf{k}^2\, \hat{q}_\lambda^2(\omega_\mathbf{k}) + \hat{p}_\lambda^2(\omega_\mathbf{k})\,. \tag{1.65}$$

This must be of the same form as the definition in Eq. (1.45). It is not difficult to show that

$$\hat{q}_\lambda(\mathbf{k}) = \sqrt{\frac{\hbar}{2\omega_\mathbf{k}}}\left[\hat{a}_\lambda(\mathbf{k}) + \hat{a}_\lambda^\dagger(\mathbf{k})\right] \tag{1.66}$$

and

$$\hat{p}_\lambda(\mathbf{k}) = -i\sqrt{\frac{\hbar\omega_\mathbf{k}}{2}}\left[\hat{a}_\lambda(\mathbf{k}) - \hat{a}_\lambda^\dagger(\mathbf{k})\right]. \tag{1.67}$$

These operators obey the commutation relation

$$\left[\hat{q}_\lambda(\mathbf{k}), \hat{p}_{\lambda'}(\mathbf{k}')\right] = i\hbar\,\delta_{\lambda\lambda'}\,\delta^3(\mathbf{k}-\mathbf{k}')\,, \tag{1.68}$$

and the operators $\hat{q}$ and $\hat{p}$ are therefore often called the 'position' and 'momentum' quadratures. They are not really associated with the position and momentum of a particle; they merely have the same commutation relations.

The quadratures are Hermitian by construction, and this therefore raises the question as to how these observables should be interpreted physically. To answer this, consider a single mode of the quantum field $\hat{A}$ given in Eq. (1.27):

$$\hat{A}_\mathbf{k}(\mathbf{r},t) = \sqrt{\frac{\hbar}{2(2\pi)^3\varepsilon_0\omega_\mathbf{k}}}\left[\hat{a}(\mathbf{k})\,e^{i\mathbf{k}\cdot\mathbf{r}-i\omega_\mathbf{k}t} + \hat{a}^\dagger(\mathbf{k})\,e^{-i\mathbf{k}\cdot\mathbf{r}+i\omega_\mathbf{k}t}\right], \tag{1.69}$$

where we have suppressed the polarization degree of freedom λ by considering just one linear polarization direction. At a given time t and position $\mathbf{r}$, we can write $\zeta = \omega_\mathbf{k}t - \mathbf{k}\cdot\mathbf{r}$, and the field becomes

$$\hat{A}_\mathbf{k} = \sqrt{\frac{\hbar}{(2\pi)^3\varepsilon_0\omega_\mathbf{k}}}\left[\frac{\hat{a}(\mathbf{k})\,e^{-i\zeta} + \hat{a}^\dagger(\mathbf{k})\,e^{i\zeta}}{\sqrt{2}}\right] = \sqrt{\frac{\hbar}{(2\pi)^3\varepsilon_0\omega_\mathbf{k}}}\,\hat{x}_\zeta(\mathbf{k})\,. \tag{1.70}$$

Similarly, the canonically conjugate momentum to $\hat{A}_\mathbf{k}$ is $\Pi_{\hat{A}} = \varepsilon_0\partial_t\hat{A}_\mathbf{k}$, which can be written as

$$\hat{\Pi}_{\hat{A}_\mathbf{k}} = -i\sqrt{\frac{\hbar\varepsilon_0\omega_\mathbf{k}}{(2\pi)^3}}\left[\frac{\hat{a}(\mathbf{k})\,e^{-i\zeta} - \hat{a}^\dagger(\mathbf{k})\,e^{i\zeta}}{\sqrt{2}}\right] = \sqrt{\frac{\hbar\varepsilon_0\omega_\mathbf{k}}{(2\pi)^3}}\,\hat{x}_{\zeta+\pi/2}(\mathbf{k})\,. \tag{1.71}$$

So the quadratures correspond to the physical observables that are the single-mode field amplitudes of $\mathbf{A}$ and $\mathbf{E} \propto \mathbf{\Pi}_\mathbf{A}$.

Exercise 1.5: Show that the commutation relations of the quadrature operators are

$$\left[\hat{x}_\zeta(\mathbf{k}), \hat{x}_{\zeta+\pi/2}(\mathbf{k}')\right] = i\delta^3(\mathbf{k}-\mathbf{k}') \quad \text{and} \quad \left[\hat{x}_\zeta(\mathbf{k}), \hat{x}_\zeta(\mathbf{k}')\right] = 0\,. \tag{1.72}$$

For any discrete mode, the expectation values of the quadrature operators $\hat{x}_\zeta(\mathbf{k})$ are dimensionless, whereas the expectation values of position and momentum quadratures $\hat{q}$ and $\hat{p}$ have dimensions $\sqrt{M}L$ and $\sqrt{M}LT^{-1}$, respectively.

1.3 Mode functions and polarization

In this section we study in more detail some aspects of the mode functions and the polarization. This is essentially a classical study, since the mode functions and the polarization are inherited directly from classical electrodynamics. We first describe the structure of the polarization of the field, and subsequently derive the orthonormal transverse mode functions in terms of Hermite–Gaussian and Laguerre–Gaussian mode shapes.

1.3.1 Polarization

One of the most popular degrees of freedom for quantum information processing with light is polarization. Usually, the polarization is defined in terms of the vector behaviour of the electric field $\mathbf{E}$. Here, however, we treat the vector potential $\hat{\mathbf{A}}$ as the fundamental quantized field. Since the electric field is proportional to the canonical momentum of the vector potential, and the time derivative of $\hat{\mathbf{A}}$ does not change the vector behaviour, we can define the polarization based on the vector potential without problems.

We already established that polarization is closely related to the vector character of $\hat{\mathbf{A}}(\mathbf{r}, t)$, carried by the vector $\boldsymbol{\epsilon}_\lambda$, and we will now explore this in more detail. Using the plane-wave expansion and the Coulomb gauge condition that $\nabla \cdot \hat{\mathbf{A}} = 0$, it follows that

$$\boldsymbol{\epsilon}_\lambda(\mathbf{k}) \cdot \mathbf{k} = 0 \qquad \text{for each } \lambda\,. \tag{1.73}$$

This means that the vectors $\boldsymbol{\epsilon}_\lambda$ point transversely to the direction of propagation $\mathbf{k}$. Furthermore, since the two dynamical variables indicated by λ are independent, we can choose

$$\boldsymbol{\epsilon}_\lambda(\mathbf{k}) \cdot \boldsymbol{\epsilon}^*_{\lambda'}(\mathbf{k}) = \delta_{\lambda\lambda'}\,. \tag{1.74}$$

The vectors $\boldsymbol{\epsilon}_\lambda$ have unit length, and are now identified with the polarization of the field. They may in general have complex components, since the vector potential will still be Hermitian if the term with the creation operator contains the complex conjugate of $\boldsymbol{\epsilon}_\lambda$.

Suppose, without loss of generality, that the direction of propagation of the light is in the z direction: $\mathbf{k} = k\hat{\mathbf{z}}$. We can then immediately construct two orthogonal vectors $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$

according to

$$\boldsymbol{\epsilon}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\epsilon}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \tag{1.75}$$

Furthermore, we can rotate these vectors around the z axis over an angle θ to find a new set of orthonormal vectors. In addition, the third component of the vector is always zero, and we therefore omit it from the description. The two independent polarization vectors then become

$$\boldsymbol{\epsilon}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad \text{and} \quad \boldsymbol{\epsilon}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}. \tag{1.76}$$

These are real vectors, and they are associated with 'linear polarization' since they point in an unambiguous spatial direction.

What happens when the vectors $\boldsymbol{\epsilon}_\lambda$ have complex components? Two possible orthonormal polarization vectors are

$$\boldsymbol{\epsilon}_L = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \boldsymbol{\epsilon}_R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}. \tag{1.77}$$

Due to the imaginary entries $\pm i$, it is not straightforward to interpret the spatial direction of these polarization vectors. It turns out that the imaginary entries cause a rotation of the polarization in time. The polarization vectors in Eq. (1.77) therefore describe 'circular' polarization, and the two orthonormal vectors correspond to left-handed and right-handed rotations. When the magnitudes of the real and imaginary components become unequal, we speak of 'elliptical polarization'. We will discuss the polarization of single photons in detail in Chapter 5.

1.3.2 Transverse mode functions

The mode functions of the electromagnetic field, or in this case the vector potential, are intrinsically a continuum, which has associated difficulties of normalization and precision in addressing. For quantum information processing purposes it is essential that we have good control over all possible degrees of freedom. We will derive an expression for properly confined modes in the transverse direction, instead of infinitely extending plane waves, and we then construct complete orthonormal sets of transverse mode functions. Finally, we use these sets to construct the quantum mechanical creation and annihilation operators via the procedure in Eq. (1.40).

First, we return to plane waves. The exact translational symmetry in the transverse plane of the plane wave means that the propagation of the plane wave in the z direction is exact. In reality, however, beams of light do not stretch out to infinity in the transverse directions. As a result, translational invariance is lost, and the direction of propagation is no longer strictly in the z direction. One way to understand this intuitively is to note that the localization of the intensity at the position $x = x_0$ and $y = y_0$ requires some uncertainty in the transverse momentum k_x and k_y. We next introduce the paraxial wave approximation, and then find

the minimum uncertainty localized transverse mode. These modes can, in turn, be used as a generating function when we construct the Hermite–Gaussian and Laguerre–Gaussian transverse modes.

When the transverse momentum is small we can treat the beam in the 'paraxial approximation'. We start with the Helmholtz equation for the jth spatial component of the classical vector potential in the frequency domain $A_j(\mathbf{r},\omega)$, given by the Fourier transform

$$A_j(\mathbf{r},t) = \int_0^\infty d\omega\, A_j(\mathbf{r},\omega) e^{-i\omega t}. \tag{1.78}$$

Using $\omega = ck$, this leads to the wave equation for the Fourier components

$$\left(\nabla^2 + k^2\right) A_j(\mathbf{r},\omega) = 0\,. \tag{1.79}$$

This is the Helmholtz equation. For a wave propagating in the z direction, we use the following 'Ansatz' for A_j:

$$A_j(\mathbf{r},\omega) = \Psi_j(\mathbf{r},\omega)\, e^{ikz}\,. \tag{1.80}$$

The differential operator acting on A_j can then be written as

$$\nabla^2 A_j = \partial_x^2 \Psi_j\, e^{ikz} + \partial_y^2 \Psi_j\, e^{ikz} + \partial_z \left[\left(\partial_z \Psi_j + ik\Psi_j\right) e^{ikz}\right]. \tag{1.81}$$

Substituting into Eq. (1.79), this leads to

$$\partial_x^2 \Psi_j + \partial_y^2 \Psi_j + 2ik\partial_z \Psi_j + \partial_z^2 \Psi_j = 0\,. \tag{1.82}$$

Our discussion has been exact up to this point. We can now make the paraxial approximation when $2ik\partial_z \Psi_j \gg \partial_z^2 \Psi_j$, that is, when the variation of the field in the direction of propagation z is much smaller than the wavelength of the light. In addition, we require $\partial_z^2 \Psi_j \ll \partial_x^2 \Psi_j,\ \partial_y^2 \Psi_j$. The second derivative with respect to z can be dropped, leading to the 'paraxial wave equation':

$$\frac{\partial^2}{\partial x^2}\Psi_j + \frac{\partial^2}{\partial y^2}\Psi_j = -2ik\frac{\partial}{\partial z}\Psi_j\,. \tag{1.83}$$

We will almost exclusively use the paraxial approximation in this book.

We note something interesting about the paraxial wave equation: when we treat z as a time variable, Eq. (1.83) is formally identical to the Schrödinger equation. Consequently, we can use well-known results from quantum mechanics in our discussion of the transverse mode functions. In particular, it will allow us to derive the orbital angular momentum properties of paraxial light beams.

Gaussian modes

Rather than constructing solutions to the paraxial wave equation directly, we will instead derive a complete set of orthonormal mode functions for the classical case, and subsequently

show that they are solutions to Eq. (1.83). We first derive the mode function of the lowest-order Gaussian mode, and only then make the paraxial approximation.

In the scalar field approximation[1] (where the polarization degree of freedom has been suppressed, and A has become a scalar) we can write the transverse components for an arbitrary solution of the wave equation with frequency ω in Cartesian coordinates

$$\begin{aligned} A(\mathbf{r},\omega) &= \int d\mathbf{k} \sqrt{\frac{\hbar}{\varepsilon_0}}\, \delta\left(\frac{\omega^2}{c^2} - \mathbf{k}^2\right) \alpha(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}} + \text{c.c.} \\ &= \int d\mathbf{k} \sqrt{\frac{\hbar}{\varepsilon_0}}\, \frac{\delta\left(k_z - \sqrt{k^2 - |\mathbf{k}_\perp|^2}\right)}{|2\sqrt{k^2 - |\mathbf{k}_\perp|^2}|} \alpha(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}} + \text{c.c.} \\ &= \int d\mathbf{k}_\perp \sqrt{\frac{\hbar}{\varepsilon_0}} f(\mathbf{k}_\perp)\, e^{ik_x x + ik_y y + i\sqrt{k^2 - k_x^2 - k_y^2}\, z} + \text{c.c.}. \end{aligned} \tag{1.84}$$

In the first line the delta function is introduced to constrain the $\mathbf{k}$ vector to the 'dispersion shell' $k^2 = \omega^2/c^2$. In the second line we have simplified the delta function according to

$$\delta\left(h(x)\right) = \frac{\delta(x_0)}{|h'(x_0)|}, \tag{1.85}$$

where h' is the derivative of h with respect to x, and x_0 is defined by $h(x_0) = 0$. There are two solutions x_0 in the above case, and we have selected the term that corresponds to propagation in the positive z direction. In the last line of Eq. (1.84) we evaluated the integral over k_z and absorbed the factor $|2\sqrt{k^2 - |\mathbf{k}_\perp|^2}|^{-1}$ into the amplitude function f, yielding

$$f(\mathbf{k}_\perp) \equiv \frac{\alpha\left(\mathbf{k}_\perp, \sqrt{k^2 - |\mathbf{k}_\perp|^2}\right)}{2\left|\sqrt{k^2 - |\mathbf{k}_\perp|^2}\right|}, \tag{1.86}$$

where $\mathbf{k}_\perp = (k_x, k_y)$ and $d\mathbf{k}_\perp = dk_x\, dk_y$. For convenience we define the mode function u as

$$u(\mathbf{k},\mathbf{r}) = f(\mathbf{k}_\perp)\, \exp\left(i\mathbf{k}_\perp \cdot \mathbf{r}_\perp + i\sqrt{k^2 - k_\perp^2}\, z\right). \tag{1.87}$$

Even though this expression is still completely general, it will be most useful when we consider propagation in the z direction, such that $f(\mathbf{k}_\perp)$ becomes small very quickly when $|\mathbf{k}_\perp|$ becomes large.

The beam defined by the mode functions $f(\mathbf{k}_\perp)$ will have a certain divergence, or 'spread' of the amplitudes $f(\mathbf{k}_\perp)$ in k-space. This means that the transverse mode area of the beam will become larger (or smaller) with increasing z. The divergence is given by the variance in the transverse wave vector

$$(\Delta\mathbf{k}_\perp)^2 = \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} \left(k_x^2 + k_y^2\right) |f(\mathbf{k}_\perp)|^2, \tag{1.88}$$

[1] We will often call A a field, rather than a potential, because we do not want to call A the scalar potential (which strictly means Φ), and in the scalar approximation it is inappropriate to call A a *vector* potential.

where the domain of integration is now taken as the entire transverse wave vector space $-\infty < k_x, k_y < \infty$. At the same time, the transverse extension in real space can be written as

$$(\Delta \mathbf{r}_\perp)^2 = \frac{\varepsilon_0}{\hbar} \int dx\, dy \left(x^2 + y^2\right) |A(\mathbf{r})|^2 = \int \frac{d\mathbf{k}_\perp}{(2\pi)^2} \left(\left| \frac{\partial f}{\partial k_x} \right|^2 + \left| \frac{\partial f}{\partial k_y} \right|^2 \right), \tag{1.89}$$

where the last equality follows from Eq. (1.84) and integrating over x and y. We use the Cauchy–Schwarz inequality for square-integrable complex functions g and h:

$$\int |g(\mathbf{a})|^2\, d\mathbf{a} \cdot \int |h(\mathbf{a})|^2\, d\mathbf{a} \geq \left| \int d\mathbf{a}\, g(\mathbf{a}) \cdot h^*(\mathbf{a}) \right|^2 \tag{1.90}$$

to obtain the classical 'uncertainty relations'

$$\Delta k_x\, \Delta x \geq \frac{\|f\|^2}{4\pi} \quad \text{and} \quad \Delta k_y\, \Delta y \geq \frac{\|f\|^2}{4\pi}, \tag{1.91}$$

where $\|f\|^2$ is the norm of the mode function:

$$\|f\|^2 = \int d\mathbf{k}_\perp\, |f(\mathbf{k}_\perp)|^2 . \tag{1.92}$$

For the minimum uncertainty in the beam (maximum transverse localization and minimum divergence) the equality in Eq. (1.91) holds. This can only happen when $k_i f$ and $\partial f / \partial k_i$ differ by a constant factor. It is straightforward to verify that in this case f must be a Gaussian function of k_i. The (exact) minimum uncertainty mode function $u_0^{(\mathrm{ex})}(\mathbf{k}, \mathbf{r})$ therefore becomes

$$u_0^{(\mathrm{ex})}(\mathbf{k}, \mathbf{r}) = f(\mathbf{k}_\perp)\, e^{i\mathbf{k}\cdot\mathbf{r}} = \exp\left(i\mathbf{k}_\perp \cdot \mathbf{r}_\perp + ik_z z - \frac{w_0^2}{4} |\mathbf{k}_\perp|^2 \right), \tag{1.93}$$

for some real number w_0. Indeed, it is clear that $f(\mathbf{k}_\perp)$ drops to zero exponentially fast as $|\mathbf{k}_\perp|$ becomes large. We now make the paraxial approximation that $|\mathbf{k}_\perp|$ is small compared to k, and that led to the paraxial wave equation in Eq. (1.83):

$$k_z = \sqrt{k^2 - |\mathbf{k}_\perp|^2} \simeq k - \frac{|\mathbf{k}_\perp|^2}{2k} . \tag{1.94}$$

This leads to the following expression for the approximate mode function $u_0(\mathbf{k}, \mathbf{r})$:

$$\begin{aligned} u_0(\mathbf{k}, \mathbf{r}) &= \exp\left[i\mathbf{k}_\perp \cdot \mathbf{r}_\perp + ikz - \left(\frac{w_0^2}{4} + \frac{iz}{2k} \right) |\mathbf{k}_\perp|^2 \right] \\ &= \exp\left[i\mathbf{k}_\perp \cdot \mathbf{r}_\perp + ikz - \frac{s^2(z)}{4} |\mathbf{k}_\perp|^2 \right] . \end{aligned} \tag{1.95}$$

The parameter w_0 is the beam waist and we have defined the complex parameter s:

$$s^2(z) \equiv w_0^2 + \frac{2iz}{k} , \tag{1.96}$$

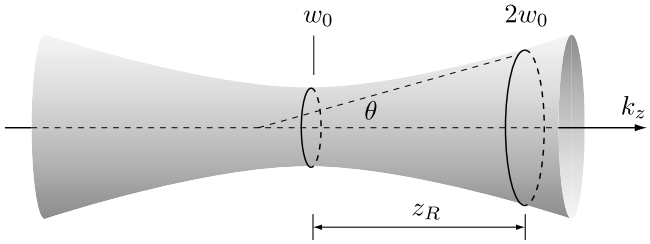

Fig. 1.1. **The mode shape of a beam of light localized in the transverse direction. The beam waist is w_0 and the Rayleigh length is z_R. The angle of divergence is θ, which must be small for the paraxial approximation to hold.**

which can be interpreted as the curvature of the wave front. Since the divergence of the beam is non-zero, the surfaces of constant phase are no longer planes of constant z extending in the xy direction; they are curved surfaces. The surfaces of constant phase can be extracted by writing the complex mode function $u_0(\mathbf{k}, \mathbf{r})$ in polar notation and setting the phase equal to a constant. The mode has a characteristic length in the direction of propagation, called the 'Rayleigh length':

$$z_R = \frac{k w_0^2}{2} = \frac{\pi w_0^2}{\lambda}, \tag{1.97}$$

which is the distance along the beam in which the waist becomes twice as large (in linear dimensions), as shown in Fig. 1.1. Here, $\lambda = 2\pi/k$ is the wavelength of the light. The (scalar) field $A_0(\mathbf{r}, \omega)$ in the fundamental Gaussian transverse mode can now be written as $(\omega = ck)$

$$\begin{aligned} A_0(\mathbf{r}, \omega) &= \int \frac{d\mathbf{k}_\perp}{\sqrt{\varepsilon_0}} u_0(\mathbf{k}, \mathbf{r}) + \text{c.c.} \\ &= \frac{4\pi}{\sqrt{\varepsilon_0}\, s^2(z)} \exp\left(ikz - \frac{x^2 + y^2}{s^2(z)} \right) + \text{c.c.}\,. \end{aligned} \tag{1.98}$$

Exercise 1.6: Show that Eq. (1.98) is a solution to the paraxial wave equation.

In the quantum theory, the single-field mode in Eq. (1.98) is associated with the annihilation operator $\hat{a}(k)$. The Gaussian function then describes the appropriate amplitude function in the transverse to the beam direction. A physical state of a photon would require a further regularization of the continuum of modes in the direction of propagation. For a mode function strongly peaked around wave vector k_0 we can then assign the photon a frequency $\omega_0 = ck_0$.

Hermite–Gaussian modes

The Gaussian beam is only one possible transverse mode shape. In particular, we expect that we can construct basis functions that can be used to describe *any* transverse amplitude distribution. For example, such a basis can be given in terms of Hermite polynomials.

We can write a general transverse mode shape of a field mode with (central) frequency ω as a power expansion in k_x and k_y:

$$u(\mathbf{k},\mathbf{r}) = \sum_{n,m=0}^{\infty} c_{nm} k_x^n k_y^m \, u_0(\mathbf{k},\mathbf{r}). \tag{1.99}$$

Suppose that a particular $c_{nm} = 1$, and all the other coefficients are zero. The higher-order mode A_{nm} of the field can then be written as

$$\begin{aligned} A_{nm}(\mathbf{r},\omega) &= \int \frac{d\mathbf{k}_\perp}{\sqrt{\varepsilon_0}} (ik_x)^n (ik_y)^m u_0(\mathbf{k},\mathbf{r}) + \text{c.c.} \\ &= \int \frac{d\mathbf{k}_\perp}{\sqrt{\varepsilon_0}} \frac{\partial^{n+m}}{\partial x^n \partial y^m} u_0(\mathbf{k},\mathbf{r}) + \text{c.c.} \\ &= \frac{\partial^{n+m}}{\partial x^n \partial y^m} A_0(\mathbf{r},\omega)\,. \end{aligned} \tag{1.100}$$

Given Eq. (1.98), it is clear that the field can be written in terms of Hermite polynomials:

$$H_q(x) = (-1)^q \, e^{x^2} \frac{d^q}{dx^q} \, e^{-x^2}. \tag{1.101}$$

This leads to a frequency component of the field in mode n, m

$$A_{nm}(\mathbf{r},\omega) = \frac{4\pi(-1)^{n+m}}{\sqrt{\varepsilon_0}\, s^{n+m+2}(z)} H_n\left(\frac{x}{s}\right) H_m\left(\frac{y}{s}\right) \exp\left[ikz - \frac{x^2+y^2}{s^2(z)}\right] + \text{c.c.}\,. \tag{1.102}$$

If we take the Fourier transform in the frequency domain again, we find that the real scalar field becomes

$$A_{nm}(\mathbf{r},t) = \int dk \, u_{nm}^{\text{HG}}(k;\mathbf{r},t) + \text{c.c.}, \tag{1.103}$$

with

$$u_{nm}^{\text{HG}}(k;\mathbf{r},t) = \frac{4\pi(-1)^{n+m}}{s^{n+m+2}(z)} H_n\left(\frac{x}{s}\right) H_m\left(\frac{y}{s}\right) \exp\left[ikz - i\omega_k t - \frac{x^2+y^2}{s^2(z)}\right], \tag{1.104}$$

the Hermite–Gaussian mode functions.

We can now construct the quantum theory by associating an annihilation operator $\hat{a}_{nm}(k)$ with the functions $u_{nm}^{\text{HG}}(k;\mathbf{r},t)$ using Eq. (1.40)

$$\hat{A}(\mathbf{r},t) = \sqrt{\frac{\hbar}{\varepsilon_0}} \sum_{n,m=0}^{\infty} \int dk \, \left[\hat{a}_{nm}(k)\, u_{nm}^{\text{HG}}(k;\mathbf{r},t) + \hat{a}_{nm}^\dagger(k)\, u_{nm}^{\text{HG}*}(k;\mathbf{r},t)\right]. \tag{1.105}$$

The modes $u_{nm}^{\text{HG}}(k;\mathbf{r},t)$ are complete and orthonormal (see Hochstadt, 1971), and depend intrinsically on the waist dimension w_0^2 (or, equivalently, the wave-front curvature $s^2(z)$). Some of the lower-order modes are shown in Fig. 1.2.

The argument of the Hermite polynomials in $u^{\mathrm{HG}}_{nm}(k;\mathbf{r},t)$ is complex because $s^2(z)$ is complex. These are the so-called elegant Hermite–Gaussian modes. There are also 'standard' Hermite–Gaussian modes, in which the argument of the Hermite polynomials is real. The elegant modes can be written in terms of the standard modes, and the imaginary part is then captured by a phase factor, called the 'Gouy phase'. Most lasers create fields that are near standard Hermite–Gaussian modes. For the translation from the elegant to the standard modes, we refer the reader to Enderlein and Pampaloni (2004).

Laguerre–Gaussian modes

The Hermite–Gaussian functions were found using a power expansion of the wave vectors k_x and k_y of the transverse mode function in Cartesian coordinates. We can also choose a rotated coordinate system x' and y' and expand the transverse mode functions in terms of $k_{x'}$ and $k_{y'}$, which leads simply to rotated Hermite–Gaussian modes. However, instead of Cartesian coordinates in the transverse plane, we can also use 'polar' coordinates (r,ϕ):

$$\begin{aligned} x = r\cos\phi \quad &\text{and} \quad y = r\sin\phi \quad \text{or} \\ x + iy = re^{i\phi} \equiv \sigma \quad &\text{and} \quad x - iy = re^{-i\phi} \equiv \bar{\sigma}\,. \end{aligned} \tag{1.106}$$

The transverse mode function can then be expanded as

$$u(\mathbf{k},\mathbf{r}) \propto \sum_{l,m=0}^{\infty} d_{lm} k_+^l k_-^{l+m}\, u_0(\mathbf{k},\mathbf{r})\,, \tag{1.107}$$

with $k_\pm = k_x \pm ik_y$. We have chosen the powers l and $l+m$ such that our equations will simplify later on. In polar coordinates the fundamental Gaussian mode becomes

$$A_0(\mathbf{r},\omega) = \frac{1}{s^2(z)} \exp\left(ikz - \frac{\sigma\bar{\sigma}}{s^2(z)}\right). \tag{1.108}$$

We can then write for the field with a particular l and m:

$$\begin{aligned} A_{lm}(\mathbf{r},\omega) &\propto \int d\mathbf{k}_\perp\, (ik_+)^l (ik_-)^{l+m}\, u_0(\mathbf{k},\mathbf{r}) \\ &= \int d\mathbf{k}_\perp\, \frac{\partial^{2l+m}}{\partial\sigma^l \partial\bar{\sigma}^{l+m}}\, u_0(\mathbf{k},\mathbf{r}) \\ &= \frac{\partial^{2l+m}}{\partial\sigma^l \partial\bar{\sigma}^{l+m}} A_0(\mathbf{r},\omega)\,. \end{aligned} \tag{1.109}$$

Using the definition of the generalized Laguerre polynomials

$$L_l^m(\xi) = \frac{e^\xi \xi^{-m}}{l!} \frac{d^l}{d\xi^l}\left(e^\xi\, \xi^{l+m}\right) \tag{1.110}$$

this leads, after some algebra, to the *elegant* Laguerre–Gaussian mode functions

$$A_{lm}(\mathbf{r},\omega) = \frac{4\pi(-1)^{l+m}l!}{\sqrt{\varepsilon_0}\,s^{2(l+m+1)}(z)} r^{|m|} e^{im\phi} L_l^m\left(\frac{r^2}{s^2(z)}\right) \exp\left(ikz - \frac{r^2}{s^2(z)}\right) + \text{c.c.}\,. \quad (1.111)$$

Taking the Fourier transform over the frequency domain, this leads to the classical field

$$A_{lm}(\mathbf{r},t) = \int dk\, u_{lm}^{\mathrm{LG}}(k;\mathbf{r},t) + \text{c.c.}, \quad (1.112)$$

with

$$u_{lm}^{\mathrm{LG}}(k;\mathbf{r},t) = \frac{4\pi(-1)^{l+m}l!}{s^{2(l+m+1)}(z)} r^{|m|} e^{im\phi} L_l^m\left(\frac{r^2}{s^2(z)}\right) \exp\left(ikz - i\omega_k t - \frac{r^2}{s^2(z)}\right). \quad (1.113)$$

Analogous to the Hermite–Gaussian case, we can now construct the quantum field for the elegant Laguerre–Gaussian modes:

$$\hat{A}(\mathbf{r},t) = \sqrt{\frac{\hbar}{\varepsilon_0}} \sum_{l,m=0}^{\infty} \int dk \left[\hat{a}_{lm}(k)\, u_{lm}^{\mathrm{LG}}(k;\mathbf{r},t) + \hat{a}_{lm}^{\dagger}(k)\, u_{lm}^{\mathrm{LG}*}(k;\mathbf{r},t)\right]. \quad (1.114)$$

The Laguerre–Gaussian mode functions are orthogonal and complete (see, for example, Hochstadt, 1971), and the lowest-order modes are shown in Fig. 1.2.

As noted before, the paraxial wave equation has a similar form to the Schrödinger equation, where the time parameter is replaced with z. This allows us to use key results from quantum mechanics. Specifically, we can construct an angular momentum operator $\hat{L}_z$ using the transverse position and momentum operators:

$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = \frac{\hbar}{i}\frac{\partial}{\partial\phi}. \quad (1.115)$$

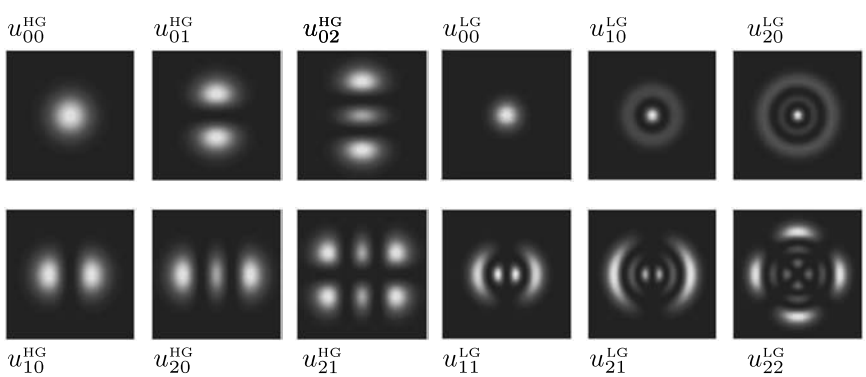

Fig. 1.2. The lowest-order transverse mode functions.

It is immediately clear from Eq. (1.113) that the Laguerre–Gaussian mode functions u_{lm}^{LG} are eigenfunctions of $\hat{L}_z$:

$$\hat{L}_z\, u_{lm}^{\mathrm{LG}}(\mathbf{k},\mathbf{r}) = m\hbar\, u_{lm}^{\mathrm{LG}}(\mathbf{k},\mathbf{r})\,. \tag{1.116}$$

The Laguerre–Gaussian modes therefore carry orbital angular momentum in that the beam exerts a torque on the medium it travels through. These are sometimes also called 'helical' modes. The angular momentum is quantized (with quantum numbers l and m) such that each photon carries an orbital angular momentum of $m\hbar$. This effect is different from the spin angular momentum, or polarization of the light. In particular, the orbital angular momentum is not restricted to $\pm\hbar$. In some circumstances it is difficult to separate spin and orbital angular momentum, since both are determined by the direction and phase of the electric and magnetic fields in the beam.

1.4 Evolution of the field operators

So far, we have concentrated mostly on the spatial aspects of the electromagnetic quantum field. However, it is important for all physical applications, and for quantum information processing in particular, to find the time evolution of the quantum fields. There are two equivalent ways of achieving this. We can either take the time dependence into account when we give a description of the quantum state (the Schrödinger picture), or we can describe the time dependence in terms of the operators (the Heisenberg picture). Later in the book we will adopt a hybrid approach, where the free evolution of the systems is described in the state, and the deviation from this is described by the interaction Hamiltonian (the interaction picture). For now, we will use the Heisenberg picture.

1.4.1 The Heisenberg equations of motion

In the Heisenberg picture an arbitrary Hermitian operator A evolves in time under the influence of a unitary operator $U(t)$ such that

$$A(t) = U(t)AU^{\dagger}(t) \quad \text{with} \quad U(t) = \exp\left(-\frac{i}{\hbar}\mathcal{H}t\right), \tag{1.117}$$

where t is the time and $\mathcal{H}$ the Hamiltonian. In the exponential form of $U(t)$, the Hamiltonian is often called the 'generator' of U. For convenience, we define $\mu = -it/\hbar$, such that we have

$$A(\mu) = \exp(\mu\mathcal{H})A\exp(-\mu\mathcal{H})\,. \tag{1.118}$$

A general Taylor expansion of $A(\mu)$ around $\mu = 0$ can be written as

$$A(\mu) = A(0) + \mu\left.\frac{dA}{d\mu}\right|_{\mu=0} + \frac{\mu^2}{2!}\left.\frac{d^2A}{d\mu^2}\right|_{\mu=0} + \cdots \tag{1.119}$$

Next, we evaluate the derivatives of A to μ in Eq. (1.119) using Eq. (1.118) and

$$i\hbar\frac{dU(t)}{dt} = \mathcal{H}U(t)\,. \tag{1.120}$$

In terms of the commutators with $\mathcal{H}$ the derivatives of A become:

$$\frac{dA}{d\mu} = [\mathcal{H}, A] \quad \text{and} \quad \frac{d^2A}{d\mu^2} = [\mathcal{H}, [\mathcal{H}, A]]\,, \tag{1.121}$$

and so forth. This leads to the Baker–Campbell–Hausdorff relation

$$A(\mu) = A(0) + \mu[\mathcal{H}, A(0)] + \frac{\mu^2}{2!}[\mathcal{H}, [\mathcal{H}, A(0)]] + \cdots \tag{1.122}$$

This equation is true for all Hermitian operators $\mathcal{H}$ and complex parameters μ. Hermiticity of A is not required. In compact notation it reads

$$e^{\mu B}Ae^{-\mu B} = A + \mu[B, A] + \frac{\mu^2}{2!}[B, [B, A]] + \cdots\,, \tag{1.123}$$

where B is Hermitian. Taking the time derivative of Eq. (1.117) we find the Heisenberg equation of motion for the operator $A(t)$:

$$\frac{dA(t)}{dt} = \frac{i}{\hbar}[\mathcal{H}, A(t)] + \frac{\partial A(t)}{\partial t}\,. \tag{1.124}$$

The Heisenberg equations of motion for the creation and annihilation operators are then

$$\frac{d\hat{a}_\lambda(\mathbf{k})}{dt} = \frac{i}{\hbar}\left[\mathcal{H}, \hat{a}_\lambda(\mathbf{k})\right] \quad \text{and} \quad \frac{d\hat{a}^\dagger_\lambda(\mathbf{k})}{dt} = \frac{i}{\hbar}\left[\mathcal{H}, \hat{a}^\dagger_\lambda(\mathbf{k})\right]\,, \tag{1.125}$$

where we have exploited the fact that the annihilation and creation operators do not have an explicit time dependence. Using the expression in Eq. (1.47) we can solve the differential equations for the creation and annihilation operators to get

$$\hat{a}_\lambda(\mathbf{k}, t) = \hat{a}_\lambda(\mathbf{k})\, e^{-i\omega_\mathbf{k} t} \quad \text{and} \quad \hat{a}^\dagger_\lambda(\mathbf{k}, t) = \hat{a}^\dagger_\lambda(\mathbf{k})\, e^{i\omega_\mathbf{k} t}\,. \tag{1.126}$$

The time-dependent annihilation operator is obtained by integrating over all wave vectors $\mathbf{k}$:

$$\hat{a}_\lambda(t) = \int d\mathbf{k}\, \hat{a}_\lambda(\mathbf{k})\, e^{-i\omega_\mathbf{k} t}\,, \tag{1.127}$$

and the time-dependent number operator can be written as

$$\hat{n}_\lambda(t) = \hat{a}^\dagger_\lambda(t)\hat{a}_\lambda(t) = \int d\mathbf{k} \int d\mathbf{k}'\, \hat{a}^\dagger_\lambda(\mathbf{k})\, \hat{a}_\lambda(\mathbf{k}')\, e^{-i(\omega_{\mathbf{k}'} - \omega_\mathbf{k})t}\,. \tag{1.128}$$

Exercise 1.7: Derive the Heisenberg equations of motion in Eq. (1.124).

As an example, suppose we have a wave packet containing exactly one photon with polarization λ, defined in Eq. (1.60)

$$|1_{j,\lambda}\rangle = \int d\mathbf{k}\, \alpha_j(\mathbf{k}) \hat{a}^\dagger_\lambda(\mathbf{k})|0\rangle\,, \tag{1.129}$$

where, as in Section 1.2 the state is normalized according to a mode function $\alpha_j(\mathbf{k})$ and $|0\rangle$ is the global vacuum state (i.e., no excitations in any mode). The number density $\langle\hat{n}\rangle$ at time t is readily calculated to be

$$\langle 1_{j,\lambda}|\hat{n}_\lambda(t)|1_{j,\lambda}\rangle = \left|\int d\mathbf{k}\, \alpha_j(\mathbf{k})\, e^{-i\omega_\mathbf{k} t}\right|^2 . \tag{1.130}$$

This is a function in t that depends on the amplitudes $\alpha_j(\mathbf{k})$. We can construct temporally localized, single-photon wave packets by choosing a suitable $\alpha_j(\mathbf{k})$ in Eq. (1.129). For example, a Lorentzian wave packet is defined by

$$\alpha_L(\mathbf{k}) = \frac{1}{\sqrt{\pi}} \frac{\sqrt{\gamma}}{\gamma + i(\omega_\mathbf{k} - \omega_0)} \tag{1.131}$$

with median ω_0, and γ parameterizes the width of the distribution. Alternatively, Gaussian wave packets can be written as

$$\alpha_G(\mathbf{k}) = \frac{1}{\sqrt[4]{2\pi\sigma^2}} \exp\left[-\frac{(\omega_\mathbf{k} - \omega_0)^2}{4\sigma^2}\right], \tag{1.132}$$

which amounts to a wave packet with central frequency ω_0 and spectral width 2σ. These modes must be broadband, and when we speak of 'the frequency' of the wave packet, we mean the central (mean) frequency, or median in the case of the Lorentzian.

The evolution of the quadrature operators is similarly determined by the Heisenberg equation of motion

$$\frac{d\hat{x}_\zeta(\mathbf{k})}{dt} = \frac{i}{\hbar}\left[\mathcal{H}, \hat{x}_\zeta(\mathbf{k})\right], \tag{1.133}$$

the solution of which is given by

$$\hat{x}_\zeta(\mathbf{k}, t) = \frac{e^{-i(\zeta+\omega_\mathbf{k} t)}\, \hat{a}(\mathbf{k}) + e^{i(\zeta+\omega_\mathbf{k} t)}\, \hat{a}^\dagger(\mathbf{k})}{\sqrt{2}}\,. \tag{1.134}$$

Integrating over all wave vectors then yields the time-dependent quadrature operator:

$$\hat{x}_\zeta(t) = \frac{e^{-i\zeta}\, \hat{a}(t) + e^{i\zeta}\, \hat{a}^\dagger(t)}{\sqrt{2}}\,. \tag{1.135}$$

The position and momentum quadratures are then given by

$$\hat{q}(t) = \sqrt{\frac{\hbar}{2\omega_\mathbf{k}}}\left[\hat{a}(t) + \hat{a}^\dagger(t)\right] \quad \text{and} \quad \hat{p}(t) = -i\sqrt{\frac{\hbar\omega_\mathbf{k}}{2}}\left[\hat{a}(t) - \hat{a}^\dagger(t)\right]. \tag{1.136}$$

1.4.2 Time-bin mode operators

We have shown how to construct temporally localized mode functions, with the Lorentzian and Gaussian wave packets as examples. Here, we will construct mode operators that create and annihilate photons in finite time intervals, or 'time bins', which are important in quantum communication. It is most convenient to construct the time-bin operators from discrete, one-dimensional plane-wave operators $\hat{a}_n$ in a cavity of length L. The index n relates to the discrete frequency $\omega_n = n\pi c/L$, and the corresponding wave number is $k_n = n\pi/L$. The field operator in terms of discrete modes is then given by

$$\hat{A} = \frac{1}{\sqrt{N+1}}\sqrt{\frac{\hbar}{\varepsilon_0}}\sum_{n=0}^{N}\left[\hat{a}_n u_n(\mathbf{r},t) + \hat{a}_n^\dagger u_n^*(\mathbf{r},t)\right], \tag{1.137}$$

where we have suppressed the polarization for notational simplicity, and we cut off the frequency at some sufficiently high value.

The finite time interval τ is determined by the cut-off frequency ω_N, above which the modes are unoccupied. This allows us to truncate the plane-wave expansion, and set

$$\tau = \frac{2L}{c(N+1)} \quad \text{and} \quad \omega_n\tau = \frac{2\pi n}{N+1}\,. \tag{1.138}$$

We define the time-bin annihilation operator as

$$\hat{b}_\mu = \frac{1}{\sqrt{N+1}}\sum_{n=0}^{N} e^{-i\mu\omega_n\tau}\,\hat{a}_n\,, \tag{1.139}$$

which has the inverse

$$\hat{a}_n = \frac{1}{\sqrt{N+1}}\sum_{\mu=0}^{N} e^{i\mu\omega_n\tau}\,\hat{b}_\mu\,. \tag{1.140}$$

The integer μ indicates the time bin. To see that this can really be interpreted as a time-bin operator, we relate $\hat{b}_\mu$ to the annihilation operator $\hat{a}(t)$ in the time domain

$$\begin{aligned}\hat{a}(t) &= \sum_{n=0}^{\infty} e^{-i\omega_n t}\,\hat{a}_n = \frac{1}{\sqrt{N+1}}\sum_{n=0}^{\infty}\sum_{\mu=0}^{N} e^{-i\omega_n(t-\mu\tau)}\,\hat{b}_\mu \\ &= \sum_{\mu=0}^{N}\left(\frac{1}{\sqrt{N+1}}\sum_{n=0}^{\infty} e^{-i\omega_n(t-\mu\tau)}\right)\hat{b}_\mu \\ &= \sum_{\mu=0}^{N}\alpha_\mu(t)\,\hat{b}_\mu\,,\end{aligned} \tag{1.141}$$

where, using $\omega = \pi c/L$

$$\alpha_\mu(t) = \frac{1}{\sqrt{N+1}}\left[1 - e^{-i\omega(t-\mu\tau)}\right]^{-1} \tag{1.142}$$

are the time-bin expansion functions. They depend only on $t - \mu\tau$ and are discrete translations of $\alpha_\mu(0)$ with period τ: this is the requirement for $\hat{b}_\mu$ to be a time-bin operator. Since we constructed the time-bin operators from (a discrete set of) infinite plane waves, the $\alpha_\mu(t)$ are highly singular when $t \to \mu\tau$. Discrete mode functions, which have localized transverse and longitudinal profiles and a regularized frequency distribution, can be used to remove the singularity. The time-bin expansion function that is most natural for a given application depends on the mode functions that describe the photon source.

We still need to verify if the operator $\hat{b}_\mu$ and its Hermitian adjoint $\hat{b}_\mu^\dagger$ are valid mode operators. In other words, we need to verify that the commutation relations hold. Assuming that $[\hat{a}_n, \hat{a}_m^\dagger] = \delta_{nm}$, we can write

$$\begin{aligned}\left[\hat{b}_\mu, \hat{b}_\nu^\dagger\right] &= \frac{1}{N+1} \sum_{n,m=0}^{N} e^{-i\mu\omega_n\tau + i\nu\omega_m\tau} \left[\hat{a}_n, \hat{a}_m^\dagger\right] \\ &= \frac{1}{N+1} \sum_{n=0}^{N} e^{-i(\mu-\nu)\omega_n\tau} \\ &= \frac{1}{N+1} \sum_{n=0}^{N} e^{-2\pi i(\mu-\nu)n/(N+1)} \\ &= \delta_{\mu\nu}\,, \end{aligned} \tag{1.143}$$

which is indeed the correct commutation relation for photons. The last equation can be regarded as a definition of the Kronecker delta symbol. The remaining commutation relations are also satisfied by the time-bin operators.

1.4.3 Mode transformations and optical elements

We have seen that the mode operators are defined with respect to mode functions, and that linear mode transformations lead to Bogoliubov transformations of the mode operators. The question is then how we generate Bogoliubov transformations. Since they are unitary, they must be generated by Hermitian Hamiltonians. This will lead to a physical interpretation of these Hamiltonians in terms of ideal beam splitters and phase shifters. In the remainder of this chapter we will use discrete physical modes.

Let us assume that the mode functions are sharply peaked such that the discrete modes are labelled with j, and can be associated with central wave vectors $\mathbf{k}_j$, and label the operators $\hat{a}_{j\lambda}$ and $\hat{a}_{j\lambda}^\dagger$. The time evolution of a discrete single-mode operator depends only on the discrete free-field Hamiltonian $\mathcal{H}_{j\lambda} = \hbar\omega_{\mathbf{k}} \hat{a}_{j\lambda}^\dagger \hat{a}_{j\lambda}$, and this generates the natural 'phase shift' $\exp(-i\omega_j t)$ accumulated by the free evolution. This gives us a clue how to describe lossless phase shifts in optical modes that are induced by materials with a refractive index n_r. When a dielectric of length ℓ is inserted in the optical beam, the phase shift is $\phi = n_r k \ell$, where $k = |\mathbf{k}|$ is the wave number of the optical mode. We can construct a dimensionless

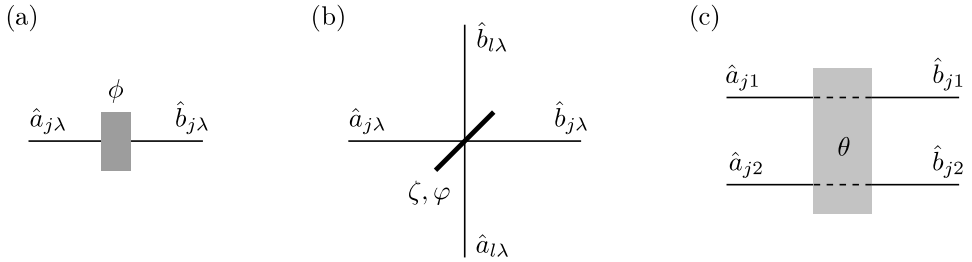

Fig. 1.3. **Graphical representations of optical elements: (a) the phase shifter; (b) the beam splitter; and (c) the polarization rotation.**

single-mode 'interaction' Hamiltonian[2] $\mathcal{H}_{j\lambda}$

$$\mathcal{H}_{j\lambda}(\phi) \equiv \hbar\phi\, \hat{a}^{\dagger}_{j\lambda}\hat{a}_{j\lambda} \quad \text{and} \quad U(\phi) = \exp\left[-\frac{i}{\hbar}\mathcal{H}_{j\lambda}(\phi)\right]. \tag{1.144}$$

To stress that this is not ordinary time evolution, we have removed the parameter t from the unitary operator and replaced it with ϕ. This will induce the transformation

$$\hat{b}_{j\lambda} = \exp\left[-\frac{i}{\hbar}\mathcal{H}_{j\lambda}(\phi)\right]\hat{a}_{j\lambda}\,\exp\left[\frac{i}{\hbar}\mathcal{H}_{j\lambda}(\phi)\right] = \hat{a}_{j\lambda}\,e^{-i\phi}\,. \tag{1.145}$$

The interaction Hamiltonian in Eq. (1.144) can be written in terms of the number operator for mode $\mathbf{k}_j$, and the number operator is therefore the generator of phase shifts. See Fig. 1.3a for a graphical representation of the phase shift.

Similarly, we can construct an interaction Hamiltonian for two modes $\mathbf{k}_j$ and $\mathbf{k}_l$ and fixed polarization λ

$$\mathcal{H}_{jl}(\zeta,\varphi) = \hbar\zeta e^{i\varphi}\,\hat{a}^{\dagger}_{j\lambda}\hat{a}_{l\lambda} + \hbar\zeta e^{-i\varphi}\,\hat{a}_{j\lambda}\hat{a}^{\dagger}_{l\lambda}\,. \tag{1.146}$$

Both terms must be present to ensure that $\mathcal{H}_{jl}(\zeta,\varphi)$ is Hermitian. If we interpret $\mathcal{H}_{jl}(\zeta,\varphi)$ physically, it describes the creation of a photon in mode $\{\mathbf{k}_j,\lambda\}$ and the annihilation of a photon in mode $\{\mathbf{k}_l,\lambda\}$, and vice versa. We therefore expect that $\mathcal{H}_{jl}(\zeta,\varphi)$ is the interaction Hamiltonian for a beam splitter mixing modes $\mathbf{k}_j$ and $\mathbf{k}_l$. Indeed, when we calculate the transformations of $\hat{a}_{j\lambda}$ and $\hat{a}_{l\lambda}$ using the Baker–Campbell–Hausdorff relation, we find that

$$\begin{aligned}
\hat{b}_{j\lambda} &= e^{\frac{i}{\hbar}\mathcal{H}_{jl}}\hat{a}_{j\lambda}e^{-\frac{i}{\hbar}\mathcal{H}_{jl}} = \cos\zeta\;\hat{a}_{j\lambda} - ie^{i\varphi}\sin\zeta\;\hat{a}_{l\lambda}\,,\\
\hat{b}_{l\lambda} &= e^{\frac{i}{\hbar}\mathcal{H}_{jl}}\hat{a}_{l\lambda}e^{-\frac{i}{\hbar}\mathcal{H}_{jl}} = -ie^{-i\varphi}\sin\zeta\;\hat{a}_{j\lambda} + \cos\zeta\;\hat{a}_{l\lambda}\,.
\end{aligned} \tag{1.147}$$

[2] It is not an interaction Hamiltonian in the standard sense of interactions, since we are considering free fields. There is a microscopic interaction of the field with the dielectric, but the details are averaged out in this Hamiltonian.

This can be expressed in terms of a unitary matrix:

$$\begin{pmatrix} \hat{b}_{j\lambda} \\ \hat{b}_{l\lambda} \end{pmatrix} = \begin{pmatrix} \cos\zeta & -ie^{i\varphi}\sin\zeta \\ -ie^{-i\varphi}\sin\zeta & \cos\zeta \end{pmatrix} \begin{pmatrix} \hat{a}_{j\lambda} \\ \hat{a}_{l\lambda} \end{pmatrix}. \tag{1.148}$$

A physical beam splitter is typically described with $\varphi = 0$ or $\varphi = \pi/2$. When $\zeta = \pi/4$, we have a 50:50 beam splitter. The two-mode transformation given here is therefore a generalized beam-splitter transformation. The beam-splitter and phase-shifter elements are shown in Fig. 1.3b.

In the above example of a beam splitter we chose two modes with identical polarization λ, but different wave vectors $\mathbf{k}_j$ and $\mathbf{k}_l$. However, we can also vary the polarization ($\lambda \in \{1, 2\}$) and keep the wave vector constant:

$$\mathcal{H}_j(\theta) = \hbar\theta e^{i\varphi}\hat{a}_{j1}^\dagger \hat{a}_{j2} + \hbar\theta e^{-i\varphi}\hat{a}_{j1}\hat{a}_{j2}^\dagger . \tag{1.149}$$

This will lead to a polarization rotation

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -ie^{i\varphi}\sin\theta \\ -ie^{-i\varphi}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \tag{1.150}$$

where we have suppressed the dependence on $\mathbf{k}_j$ for notational brevity. The graphical representation of a polarization rotation is given in Fig. 1.3c.

If we also include a relative phase shift ϕ in the two modes a_1 and a_2, the corresponding matrix

$$U(\phi, \theta, \varphi) = \begin{pmatrix} e^{i\phi/2}\cos\theta & -ie^{i\varphi}\sin\theta \\ -ie^{-i\varphi}\sin\theta & e^{-i\phi/2}\cos\theta \end{pmatrix} \tag{1.151}$$

is the most general two-mode unitary transformation. In other words, any 2×2 unitary matrix can be written as a particular trio of angles (ϕ, θ, φ).

The beam splitter and the polarization rotation described here are lossless, and are therefore idealizations of real beam splitters and polarization rotations, as implemented by e.g., half-silvered mirrors and quarter wave plates. Nevertheless, we will see later that we can account for photon loss quite easily in this formalism, and considering that the losses must be low for quantum information processing anyway, this is an excellent approximation.

As a practical example of linear mode transformations, consider the Mach–Zehnder interferometer (Fig. 1.4). It consists of two input and output modes, two 50:50 beam splitters BS_1 and BS_2, and a relative phase shift ϕ between the internal arms in the interferometer. Suppose the input modes are $\hat{a}_1$ and $\hat{a}_2$, and the output modes are $\hat{b}_1$ and $\hat{b}_2$. The mode transformations are

$$\begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix},$$

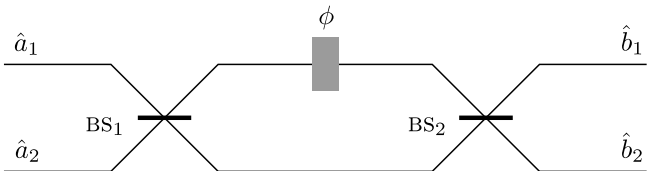

Fig. 1.4. The Mach–Zehnder interferometer.

where the first matrix represents BS_2, the second is the phase shift, and the third represents BS_1. The intensities (proportional to the number operators of the modes) in the output can be related to the intensities in the input:

$$\hat{b}_1^\dagger \hat{b}_1 - \hat{b}_2^\dagger \hat{b}_2 = \cos\phi \left(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2\right) - i \sin\phi \left(\hat{a}_1^\dagger \hat{a}_2 + \hat{a}_2^\dagger \hat{a}_1\right),$$
$$\hat{b}_1^\dagger \hat{b}_1 + \hat{b}_2^\dagger \hat{b}_2 = \hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 \,. \tag{1.152}$$

Taking the expectation values on the right-hand side with respect to the input state gives the expectation value of the intensity sum and difference in the output modes. The second equality expresses photon number conservation in the Mach–Zehnder interferometer. Measuring the intensity difference of the output modes gives information about the (unknown) phase shift ϕ. We will return to this topic in Chapter 13.

1.4.4 Normal modes

A beam splitter is an example of the mixing of two optical modes. Often, we can think of the modes as two Gaussian beams that have near-perfect overlap on the beam splitter. It is then a good approximation to treat the system as an 'interferometer' of two discrete modes, described by a unitary 2×2 matrix. This can be generalized to many modes. For N discrete modes, the transformation is an $N \times N$ unitary matrix. The physical system is sometimes called an 'N-port interferometer' (Fig. 1.5). The question is now whether we can create any conceivable N-port interferometer out of regular beam splitters and phase shifters, or whether we need more arbitrary M-point beam splitters ($2 < M \leq N$) that are not reducible to beam splitters and phase shifters.

Reck *et al.* (1994) proved that every N-mode unitary transformation can be constructed from at most $N(N-1)/2$ beam splitters and phase shifters. A sketch of their proof runs as follows: let's denote the unitary transformation that describes the N-port interferometer by $U(N)$. In addition, we define the matrix T_{nm} as the N-dimensional identity operator $I(N)$ with the elements I_{nn}, I_{nm}, I_{mn}, and I_{mm} replaced by the four elements of a 2×2 unitary matrix. We can then use the T_{nm} to reduce the size of $U(N)$:

$$U(N) \cdot T_{N,N-1} \cdot T_{N,N-2} \cdots T_{N,1} = U(N-1) \oplus e^{i\phi} \,. \tag{1.153}$$

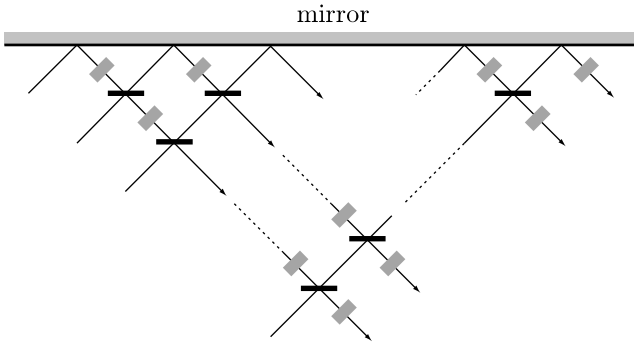

Fig. 1.5. The *N*-port interferometer in terms of generalized beam splitters.

In block schematics this is

$$\left(\begin{array}{c} \\ U(N) \\ \\ \end{array}\right) \cdot T_{N,N-1} \cdots T_{N,1} = \left(\begin{array}{c|c} U(N-1) & 0 \\ \hline 0 & e^{i\phi_N} \end{array}\right). \tag{1.154}$$

We therefore need $N-1$ matrices T to reduce the unitary $U(N)$ to $U(N-1)$. We can repeat this procedure to reduce $U(N-1)$ to $U(N-2)$, and so on. In total, we have to use at most $N(N-1)/2$ matrices T to reduce $U(N)$ to a diagonal matrix with only phases as non-zero elements:

$$U(N) \cdot T_{N,N-1} \cdot T_{N,N-2} \cdots T_{2,1} = \begin{pmatrix} e^{i\phi_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\phi_N} \end{pmatrix}. \tag{1.155}$$

Since the T matrices are unitary (and therefore invertible), we can write $U(N)$ as a series of $N(N-1)/2$ matrices acting on the diagonal matrix in Eq. (1.155). Because every T_{nm} matrix is associated with a beam splitter and phase shift on modes n and m, this means that any N-port interferometer can be constructed with only beam splitters and phase shifters.

Since $U(N)$ can be deconstructed into a series of beam splitters and phase shifters on discrete modes a_n, we can write its generator in terms of a Hamiltonian

$$\mathcal{H}_{1\ldots N}(B) = \hbar \sum_{n,m=1}^{N} \hat{a}_n^\dagger B_{nm} \hat{a}_m \,, \tag{1.156}$$

where B is a Hermitian matrix. In the following argument, we will suppress the subscripts and argument of $\mathcal{H}$ for brevity. Under the influence of this Hamiltonian a mode operator $\hat{a}_n$ becomes

$$\hat{a}_n \to \hat{b}_n = e^{\frac{i}{\hbar}\mathcal{H}} \hat{a}_n \, e^{-\frac{i}{\hbar}\mathcal{H}} = \sum_{m=1}^{N} U_{nm} \hat{a}_m \,. \tag{1.157}$$

The U_{nm} are elements of the unitary matrix $U(N)$. We can diagonalize $\mathcal{H}$ by diagonalizing B using a unitary matrix V:

$$\begin{aligned} V\mathcal{H}V^\dagger &= \hbar \sum_{j,n,m=1}^{N} V_{jn}\hat{a}_n^\dagger V_{nj}^* \, V_{jn}B_{nm}V_{mj}^* \, V_{jm}\hat{a}_m V_{mj}^* \\ &= \hbar \sum_{j=1}^{N} \hat{b}_j^\dagger \, D_{jj} \, \hat{b}_j \,, \end{aligned} \tag{1.158}$$

where D is a diagonal matrix. This gives us the 'normal modes' $\hat{b}_n$ given in Eq. (1.157) with $U_{nm} = V_{nm}V_{mn}^*$. By construction, these modes diagonalize the Hamiltonian, which in turn describes a set of non-interacting modes. Since beam splitters can be transformed away using linear Bogoliubov transformations, their operation does not constitute a real interaction between modes (and by implication between photons). This will have profound implications for optical quantum information processing.

1.4.5 Non-photon-number-preserving transformations

So far, the 'interaction' Hamiltonians have generated transformations of annihilation operators into other annihilation operators (and creation operators into other creation operators). However, we have seen in Eq. (1.42) that the most general mode transformations allow for the transformation of annihilation operators into creation operators, and vice versa. The question is therefore which Hamiltonian can generate such Bogoliubov transformations.

There is one type of quadratic Hamiltonian that we have not yet considered, namely the Hamiltonians that are proportional to $\hat{a}^2$ and $\hat{a}^{\dagger 2}$. First, we consider the Hamiltonian

$$\mathcal{H}_j(\xi,\varphi) = \hbar\xi e^{i\varphi}\,\hat{a}_{j\lambda}^2 + \hbar\xi e^{-i\varphi}\,\hat{a}_{j\lambda}^{\dagger 2} \,. \tag{1.159}$$

We use the Baker–Campbell–Hausdorff relation from Eq. (1.123) to calculate the transformed mode operators

$$\begin{aligned} \hat{b}_{j\lambda} &= \cosh(2\xi)\,\hat{a}_{j\lambda} - ie^{-i\varphi}\sinh(2\xi)\,\hat{a}_{j\lambda}^\dagger \\ \hat{b}_{j\lambda}^\dagger &= ie^{i\varphi}\sinh(2\xi)\,\hat{a}_{j\lambda} + \cosh(2\xi)\,\hat{a}_{j\lambda}^\dagger \,. \end{aligned} \tag{1.160}$$

In matrix form this is

$$\begin{pmatrix} \hat{b}_{j\lambda} \\ \hat{b}_{j\lambda}^\dagger \end{pmatrix} = \begin{pmatrix} \cosh(2\xi) & -ie^{-i\varphi}\sinh(2\xi) \\ ie^{i\varphi}\sinh(2\xi) & \cosh(2\xi) \end{pmatrix} \begin{pmatrix} \hat{a}_{j\lambda} \\ \hat{a}_{j\lambda}^\dagger \end{pmatrix} . \tag{1.161}$$

We now observe that the quadratic Hamiltonian in Eq. (1.159) mixes the creation and annihilation operators of the mode $a_{j\lambda}$. This is called 'single-mode squeezing'.

Alternatively, we can define 'two-mode squeezing' as the mode transformation that is induced by the Hamiltonian

$$\mathcal{H}_{jl}(\xi,\varphi) = \hbar\xi e^{i\varphi}\,\hat{a}_{j\lambda}\hat{a}_{l\lambda'} + \hbar\xi e^{-i\varphi}\,\hat{a}^\dagger_{j\lambda}\hat{a}^\dagger_{l\lambda'}\,. \tag{1.162}$$

The mode transformation then becomes

$$\begin{aligned}\hat{b}_{j\lambda} &= \cosh\xi\,\hat{a}_{j\lambda} - ie^{i\varphi}\sinh\xi\,\hat{a}^\dagger_{l\lambda'}\\ \hat{b}_{l\lambda'} &= -ie^{-i\varphi}\sinh\xi\,\hat{a}^\dagger_{j\lambda} + \cosh\xi\,\hat{a}_{l\lambda'}\,.\end{aligned} \tag{1.163}$$

In terms of the matrix representation this transformation can be written as

$$\begin{pmatrix}\hat{b}_{j\lambda}\\ \hat{b}^\dagger_{j\lambda}\\ \hat{b}_{l\lambda'}\\ \hat{b}^\dagger_{l\lambda'}\end{pmatrix} = \begin{pmatrix}\cosh\xi & 0 & 0 & -ie^{i\varphi}\sinh\xi\\ 0 & \cosh\xi & ie^{-i\varphi}\sinh\xi & 0\\ 0 & ie^{i\varphi}\sinh\xi & \cosh\xi & 0\\ -ie^{-i\varphi}\sinh\xi & 0 & 0 & \cosh\xi\end{pmatrix}\begin{pmatrix}\hat{a}_{j\lambda}\\ \hat{a}^\dagger_{j\lambda}\\ \hat{a}_{l\lambda'}\\ \hat{a}^\dagger_{l\lambda'}\end{pmatrix}.$$

We now have all the ingredients for a general theory of discrete linear mode transformations: a general Bogoliubov transformation is given by

$$\begin{aligned}\hat{b}_i &= \sum_j A_{ij}\,\hat{a}_j + B_{ij}\,\hat{a}^\dagger_j\\ \hat{b}^\dagger_i &= \sum_j B^*_{ij}\,\hat{a}_j + A^*_{ij}\,\hat{a}^\dagger_j\,,\end{aligned} \tag{1.164}$$

where the labels i and j include the polarization degree of freedom. The operators $\hat{b}_i$ and $\hat{b}^\dagger_i$ must again be proper annihilation and creation operators, which means they have to obey the commutation relations

$$\left[\hat{b}_i,\hat{b}^\dagger_j\right] = \delta_{ij} \quad \text{and} \quad \left[\hat{b}_i,\hat{b}_j\right] = \left[\hat{b}^\dagger_i,\hat{b}^\dagger_j\right] = 0\,. \tag{1.165}$$

This leads to the following restrictions on A and B:

$$AB^T = \left(AB^T\right)^T \quad \text{and} \quad AA^\dagger = BB^\dagger + \hat{\mathbb{I}}. \tag{1.166}$$

The second equation indicates that $AA^\dagger$ and $BB^\dagger$ are simultaneously diagonalized, for example by a matrix U. The matrices A and B are then diagonalized according to the singular value decomposition theorem:

$$A = UA_DV^\dagger \quad \text{and} \quad B = UB_DW^\dagger, \tag{1.167}$$

where A_D and B_D are diagonal matrices. To determine the relation between V and W, we consider the inverse transformation:

$$\hat{a}_i = \sum_j A^*_{ij}\,\hat{b}_j - B^*_{ij}\,\hat{b}^\dagger_j\,. \tag{1.168}$$

Following the same argumentation as before, we find the restriction

$$A^\dagger B = \left(A^\dagger B\right)^T \quad \text{and} \quad A^\dagger A = \left(B^\dagger B\right)^T + \hat{\mathbb{I}}. \tag{1.169}$$

This leads to the relation $V^* = W$, and

$$A = UA_DV^\dagger \quad \text{and} \quad B = UB_DV^T. \tag{1.170}$$

When we write the Bogoliubov transformation in matrix notation with $\vec{a} = (\hat{a}_1, \ldots, \hat{a}_N)$ and $\vec{b} = (\hat{b}_1, \ldots, \hat{b}_N)$, we obtain

$$\begin{aligned}\begin{pmatrix}\vec{b}\\ \vec{b}^\dagger\end{pmatrix} &= \begin{pmatrix}U & 0\\ 0 & U^*\end{pmatrix}\begin{pmatrix}A_D & B_D\\ B_D^* & A_D^*\end{pmatrix}\begin{pmatrix}V^\dagger & 0\\ 0 & V^T\end{pmatrix}\begin{pmatrix}\vec{a}\\ \vec{a}^\dagger\end{pmatrix}\\ &= \begin{pmatrix}A & B\\ B^* & A^*\end{pmatrix}\begin{pmatrix}\vec{c}\\ \vec{c}^\dagger\end{pmatrix},\end{aligned} \tag{1.171}$$

which agrees with Eq. (1.170). The three matrices in the top line are three successive components in a nonlinear optical interferometer. The two block-diagonal matrices in the first line do not mix creation and annihilation operators, and therefore describe linear optical interferometers. The central matrix *does* mix creation and annihilation operators. However, the components A_D, B_D, A_D^*, and B_D^* are themselves diagonal matrices, and the entire matrix therefore corresponds to a set of single-mode squeezers. This procedure is called the 'Bloch–Messiah reduction', and is shown in Fig. 1.6.

The generator that is responsible for the Bogoliubov transformation in Eq. (1.164) is, in terms of the normal modes,

$$\mathcal{H} = \frac{\hbar}{2}\sum_i \hat{b}_i^\dagger\,\hat{b}_i\,. \tag{1.172}$$

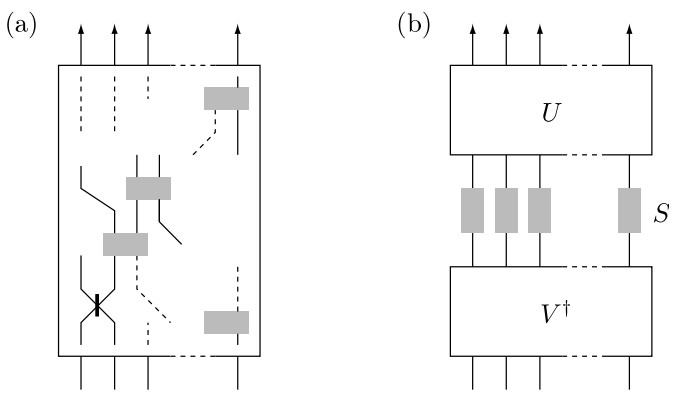

Fig. 1.6. The Bloch–Messiah reduction: (a) a general multi-port interferometer including multi-mode squeezing; (b) the reduction of the interferometer in two linear optical interferometers U and $V^\dagger$, and a set of single-mode squeezers S.

We can now substitute the mode transformation in Eq. (1.164) into $\mathcal{H}$ in order to obtain

$$\begin{aligned}\mathcal{H} &= \frac{\hbar}{2}\sum_{ijk}\left[\left(A_{ij}\,\hat{a}_j + B_{ij}\,\hat{a}_j^\dagger\right)\left(B_{ik}^*\,\hat{a}_k + A_{ik}^*\,\hat{a}_k^\dagger\right)\right] \\ &= \frac{\hbar}{2}\sum_{ijk}\left[\hat{a}_j A_{ij} B_{ik}^* \hat{a}_k + \hat{a}_j^\dagger B_{ij} A_{ik}^* \hat{a}_k^\dagger + \hat{a}_j A_{ij} A_{ik}^* \hat{a}_k^\dagger + \hat{a}_j^\dagger B_{ij} B_{ik}^* \hat{a}_k\right] \\ &= \frac{\hbar}{2}\sum_{jk}\left[\hat{a}_j F_{jk} \hat{a}_k + 2\hat{a}_j^\dagger G_{jk} \hat{a}_k + \hat{a}_j^\dagger F_{jk}^* \hat{a}_k^\dagger\right], \end{aligned} \tag{1.173}$$

where

$$F_{jk} = \sum_i A_{ij} B_{ik}^* \tag{1.174}$$

and

$$G_{jk} = \frac{1}{2}\sum_i \left(A_{ij} A_{ik}^* + B_{ij}^* B_{ik}\right) + \frac{\delta_{jk}}{2} = \sum_i A_{ij} A_{ik}^*. \tag{1.175}$$

The last equation follows from Eq. (1.166). The Hamiltonian in Eq. (1.173) is the most general quadratic Hamiltonian. It governs the dynamics of free fields, where the mixing of modes is given by the Bogoliubov transformations. These transformations can be implemented with beam splitters and phase shifters, as well as single-mode squeezers.

At this point we should mention a possible source of confusion in the nomenclature. The Bogoliubov transformation in Eq. (1.42) is *linear*, since it transforms a mode operator into a linear combination of other mode operators. As we have seen, these linear transformations are induced by generators (Hamiltonians) that are *quadratic* in the mode operators. In particular, both (generalized) beam-splitter and squeezing transformations are generated by quadratic Hamiltonians. On the other hand, when people mention linear optics, they often refer specifically to optical elements that are described by generalized beam-splitters, and not squeezers. The reason is that beam-splitters and phase-shifters are implemented with linear dielectric media, whereas squeezing requires a *nonlinear* dielectric medium. When using the term *linear*, we must therefore always give its context, namely either the mode transformations of optical elements or the physical implementation.

1.5 Quantum states of the electromagnetic field

We have seen how the creation and annihilation operators produce and destroy photons in their respective optical modes. In this section we study how linear and quadratic functions of the mode operators can be used to define two important classes of states of the electromagnetic field. Linear Hamiltonians give rise to ‘coherent states’, while quadratic Hamiltonians produce ‘squeezed states’.