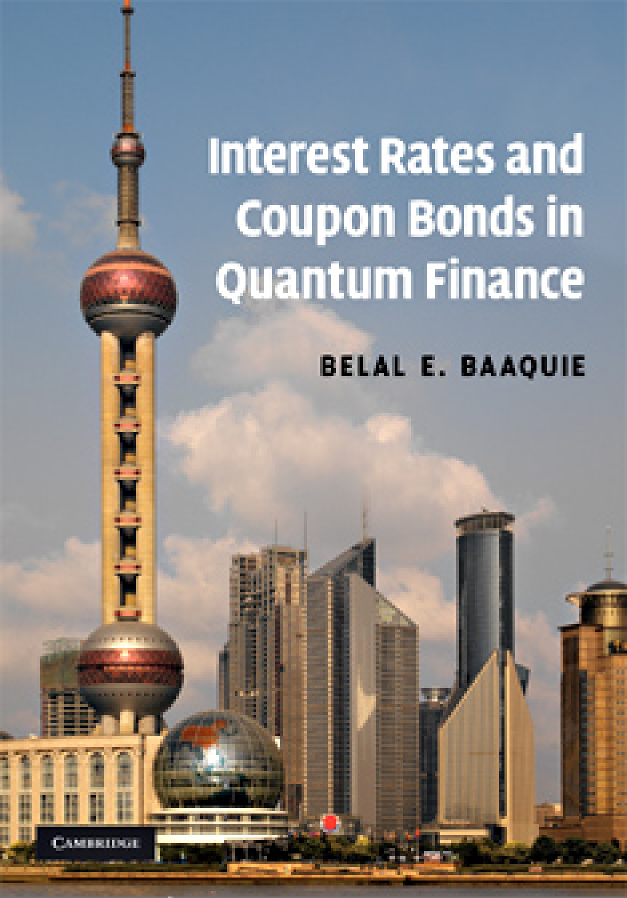
Interest Rates and Coupon Bonds in Quantum Finance

BELAL E. BAAQUIE



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The economic crisis of 2008 has shown that the capital markets need new theoretical and mathematical concepts to describe and price financial instruments.

Focusing on interest rates and coupon bonds, this book does not employ stochastic calculus – the bedrock of the present day mathematical finance – for any of the derivations. Instead, it analyzes interest rates and coupon bonds using quantum finance. The Heath–Jarrow–Morton model and the Libor Market Model are generalized by realizing the forward and Libor interest rates as an imperfectly correlated quantum field. Theoretical models have been calibrated and tested using bond and interest rates market data.

Building on the principles formulated in the author's previous book (*Quantum Finance*, Cambridge University Press, 2004), this ground-breaking book brings together a diverse collection of theoretical and mathematical interest rate models. It will interest physicists and mathematicians researching in finance, and professionals working in the finance industry.

BELAL E. BAAQUIE is Professor of Physics in the Department of Physics at the National University of Singapore. He obtained his B.S. from Caltech and his Ph.D. from Cornell University. His specialization is in quantum field theory, and he has spent the last ten years applying quantum mathematics, and quantum field theory in particular, to quantitative finance. Professor Baaquie is an affiliated researcher with the Risk Management Institute, Singapore, and is a founding Editor of the *International Journal of Theoretical and Applied Finance.* His pioneering book *Quantum Finance* has created a new branch of research in theoretical and applied finance.

Cover illustrations: Shanghai skyline and the Bund.

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BELAL E. BAAQUIE
National University of Singapore

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This book is dedicated to my wife Najma Sultana Baaquie,
my son Arzish Falaqul Baaquie, and my daughter Tazkiah Faizaan Baaquie.
Their precious love, affection, support, and optimism
have made this book possible.

Contents

Prologue

The 2008 economic crisis has shown that the capital markets need new and fresh theoretical and mathematical concepts for designing and pricing financial instruments. Focusing on interest rates and coupon bonds, this book does not employ stochastic calculus – the bedrock of the present-day mathematical finance – for any of the derivations. Interest rates and coupon bonds are studied in the self-contained framework of quantum finance that is independent of stochastic calculus. Quantum finance provides solutions and results that go beyond the formalism of stochastic calculus.

It is five years since *Quantum Finance* [12] was published in 2004 and it is indeed gratifying to see how well it has been received. No attempt has been made to re-work the principles of finance. Rather, the main thrust of this book is to employ the methods of theoretical physics in addressing the subject of finance. Theoretical physics has accumulated a vast and rich repertoire of mathematical concepts and techniques; it is only natural that this treasure house of quantitative tools be employed to analyze the field of finance, and the debt market in particular.

The term 'quantum' in *Quantum Finance* refers to the use of *quantum mathematics*, namely the mathematics and theoretical concepts of quantum mechanics and quantum field theory, in analyzing and studying finance. Finance is an entirely classical subject and there is no $\hbar$ – Planck's constant, the *sine qua non* of quantum phenomena – in quantum finance: the term 'quantum' is a *metaphor*. Consider the case of classical phase transformations that result from the random fluctuations of classical fields; critical exponents, which characterize phase transitions, are computed using the mathematics of nonlinear quantum field theories [95]. Similar to the case of phase transitions, quantum mathematics provides powerful theoretical and mathematical tools for studying the underlying random processes that drive modern finance.

The principles of quantum finance provide a comprehensive and self-contained theoretical platform for modeling all forms of financial instruments. This book, in particular, is focused on studying interest rates and coupon bonds. A detailed analytical, computational, and empirical study of debt instruments constitutes the main content of this book.

The Libor Market Model and the Heath–Jarrow–Morton model, which are the industry standards for modeling interest rates and coupon bonds, are both based on exactly correlated Libor and forward interest rates. The book makes a quantum finance generalization of these models to *imperfectly correlated* interest rates by modeling the forward interest rates as a quantum field. Empirical studies provide strong evidence supporting the imperfect correlation of interest rates. Many ground-breaking results are obtained for debt instruments. In particular, it is shown that quantum field theory provides a generalization of Ito calculus that is required for studying imperfectly correlated interest rates.

In the capital markets, interest rates determine the returns on cash deposits. Coupon bonds, on the other hand, are loans that are disbursed – with the objective of earning interest – against promissory notes. In principle, the interest paid on cash deposits and the interest earned on loans are equivalent. However, all interest rates are only defined for a *finite* time interval – of which the minimum is overnight (24 hours). In particular, all interest rate *derivatives* are based on benchmark interest rates for cash deposits of a duration of 90 days. The bond (derivatives) markets, in contrast, have no such minimum duration. The existence of a finite duration for the (benchmark) interest rates creates *two distinct sectors* of the debt derivatives market, namely derivatives of interest rates and derivatives of coupon bonds – with a nonlinear transformation connecting the two sectors.

Numerous and exhaustive calculations are carried out for diverse forms of interest rate and coupon bond options. Complicated concepts and calculations that are typical for debt instruments are introduced and motivated, in some cases by first discussing analogous and simpler equity instruments. It is my view that only by actually working out the various steps required in a calculation can a reader grasp the principles and techniques of what is still a subject in its infancy. Almost all the intermediate steps in the various calculations are included so as to clear the way for the interested reader. A few key ideas are repeated in the various chapters so that each chapter can be read more or less independently.

The material covered in the book is primarily meant for physicists and mathematicians engaged with research in the field of finance, as well as professional theorists working in the finance industry. Specialists working in the field of debt instruments will hopefully find that the theoretical tools and mathematical ideas developed in this book broaden their repertoire of quantitative approaches to finance. The material could also be of interest to physicists, probabilists, applied mathematicians, and statisticians – as well as graduates students in science and engineering – who are thinking of pursuing research in the field of finance.

One of the aims of this book is to be self-contained and comprehensive. All derivations and concepts are introduced from first principles, and all important results are derived *ab initio*. Given the diverse nature of the potential audience, fundamental concepts of finance have been reviewed for readers who are new to this field. Appendix A reviews the essential mathematical background required for following the various derivations and is meant to introduce specialists working in finance to the concepts of quantum mathematics.

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1

Synopsis

The book consists of *three major themes*. Any one of the three components can be read without many gaps in the analysis.

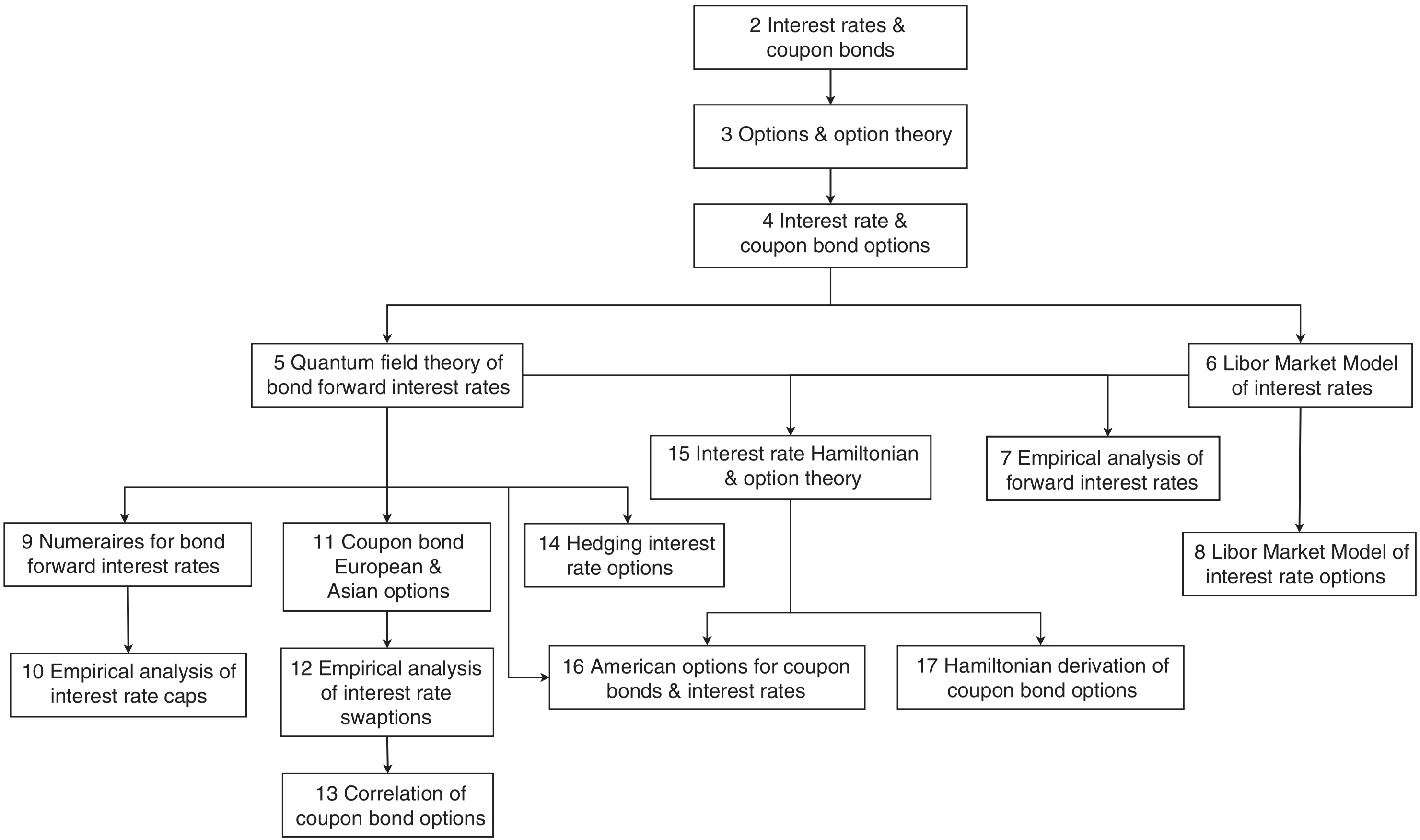
1 The introductory chapters are primarily intended for readers who are unfamiliar with the *fundamental concepts of finance*. The principles and mathematical expressions for debt instruments, which are analyzed in later chapters, are reviewed in Chapter 2,3, and 4. Options are briefly discussed and the Black–Scholes option theory is given a path integral formulation.

2 A major subject matter of the book is the theory of coupon bonds. A quantum field theory of the *bond forward interest rates* $f(t, x)$ is developed in Chapter 5 and forms a core chapter. It provides a model for the study of coupon and zero coupon bonds. Many of the derivations in later chapters are based on the quantum finance model of bond forward interest rates.

3 The quantum finance formulation of Libor interest rates is another major topic. The Libor Market Model is formulated in Chapter 6; the nonlinear Libor forward interest rates $f_L(t, x)$ that it is based upon are transformed into *logarithmic Libor interest rates* $\phi(t, x)$. In Chapter 7 some empirical properties of the Libor Market Model are studied and in Chapter 8 the prices of Libor options are obtained by using techniques of quantum field theory. A derivation of the Libor Market Model's nonlinear drift term is given in Chapter 15, based on the Libor Hamiltonian and state space of $\phi(t, x)$.

The inter-connection of the various chapters is shown in the flowchart given overleaf.

Chapter dependency flowchart



2

Interest rates and coupon bonds

Interest rates, coupon bonds, and their derivatives are the main instruments of the debt markets, which constitute well over 60% of the entire capital markets. A brief discussion locates the debt markets in the general framework of finance and points to the growing importance of the debt markets in the global economy. Interest rates are a measure of the returns on cash deposits, whereas coupon bonds are a measure of the present value of future cash flows. From this intuitive and apparently simple idea flow all the various definitions of interest rates and coupon bonds. The fundamental concept of forward interest rates that describe the bond market is introduced. The interest rate markets are driven by Libor and Euribor; these two instruments are defined and a few of their important features are discussed.

2.1 Introduction

Finance is the discipline that studies the borrowing, lending, and investing of money capital. The main form of money capital is *paper* issued by various governments and private organizations, which includes corporations and individuals. The three pillars of finance are equity, debt, and foreign exchange and are the basis of all financial instruments. Financial markets, collectively known as the capital markets, trade in these instruments [31].

Capital in economics represents the collection of productive assets required for carrying out economic activities. Financial 'paper' is not merely ordinary paper, but, rather, the preferred form of money capital that is used for *representing* value: a value based not on how it has been generated but, rather, on its present day and future value in the capital markets – and in economic activity in general. Money capital carries an intrinsic risk since expectations of what can be realized in the present and future are always subject to uncertainties inherent in any form of forecasting. Unlike traditional economies – where finance is a passive force and auxiliary to the real economy – the capital markets today are one of the most powerful and dynamic components of the modern global economy and a potent force for economic growth and expansion. The capital markets are expected to become increasingly important with the increasing inter-connections of the global economy. However, there is a downside to the increasing importance of finance. Due to the inherently uncertain and random nature of money capital, the capital markets have an uncontrollable and unpredictable component that can wreak havoc on the real economy. Advanced theories of money capital are required for creating financial instruments that can be used for managing risk and reducing instabilities of the capital markets – and thus help to *tame* the destabilizing spikes, bubbles, meltdowns, and crashes of the financial markets.

Money capital comes in many forms with the main three forms being stocks and shares of companies, debt instruments, and cash of various currencies. Money, or more precisely money capital that is seeking returns from the economy, is a dynamic quantity – with opportunities for money to yield profit constantly changing with time. Interest rates reflect the relation of the value of money with time and quantify the time-dependent and dynamic aspect of money.

Debtors pay a return – the amount depending on the interest rate – to the providers of credit. Debt and surplus capital are two sides of the same coin, since debt for one party is the complement of the credit that the other party has provided. The world's debt market is an expression of the net savings that the world economy has generated.

One needs sophisticated and effective models of interest rates to manage and expand the net global savings so as to maximize its returns. It is from this perspective – of optimizing the management of the international debt markets – that quantum finance models of interest rates and coupon bonds have been developed and form the main content of this book.

Optimizing the management of international liquidity will result in better allocation and returns on investments as well as create conditions for the prosperity of society at large. In particular, managing flows of international capital to developing and other higher risk economies, using customized financial instruments, would result in a larger fraction of mankind having access to investment capital – leading to the betterment of people's lives and wealth.

2.2 Expanding global money capital

The nature of finance has undergone a radical change in the last 30 years, with the financial sector of the economy becoming increasingly more important. There are many indicators that point to this fundamental change in the financial superstructure of economically advanced countries.

In 2006, the world economy generated about US$65 trillion worth of goods and services, of which raw materials (taken directly from nature) constituted about two-thirds (US$43 trillion) of the total value. The remaining one-third (US$22 trillion) was the value added by human labor. For example, in 2007 – based on daily production of 85 million barrels (about 31 billion barrels a year) – the sale of petroleum at around US$100 per barrel generated a revenue worth about US$3.1 trillion, with a large part of this revenue being invested in the capital markets.

In general, a substantial fraction of the net profit generated by the world economy as well as the savings and net accumulated surplus capital of many individuals, organizations, and countries is held in the form of money capital. In particular, cash rich oil and gas producers as well as East Asian economies (with substantial national reserves) have created 'sovereign funds' for investing their surplus in the capital markets. Money capital is bound to be increasingly important; due to the enormous scale of the global economy and the net savings it generates, there is not enough gold or other precious commodities that can hold this value. Paper seems the only way to represent and store the generated global surplus value.

Risk management, based on models that quantify the degree of risk, allows many institutional investors to convert net savings into money capital. Better risk management instruments have drawn risk-averse investors, such as insurance companies and pension and sovereign funds amongst others, to place their assets in the capital markets, contributing to the current explosion of the money capital.

IMF estimates that in 2005 the total value of the stocks, bonds, and bank loans worldwide was about US$165 trillion. The global bond (debt) market's share was close to US$104 trillion – by far the largest component of the global capital markets – accounting for over 63% of the total; banking credit in 2008 amounted to about US$23 trillion. In 2005, cross-border money flows (stocks, bonds, real estate, and so on) amounted to about US$6 trillion. The foreign exchange markets have also undergone a phenomenal increase, with about US$3 trillion being traded daily in 2007.

In 2007, global stocks were worth about US$56 trillion – about 35% of the capital markets – with the US and Eurozone each accounting for US$18 trillion and the rest of world accounting for US$20 trillion. The US capital markets had a total worth of US$42 trillion of which US$24 trillion was in the bond (debt) market and US$18 trillion in stocks (equity).

In 2006 global debt issuance rose to a record US$6.9 trillion with the global syndicated loan volume exceeding US$3.2 trillion. During the period of 2000–2005 nonfinancial companies worldwide issued $19.3 trillion worth of debt, in the form of corporate medium-term notes (MTNs), with the biggest issuers being the automotive industry, issuing 70 MTNs worth US$4.54 trillion followed by insurance companies issuing 26 MTNs worth US$4.49 trillion.

Market liquidity and risk management – two of the current lynch pins of the financial system – require the participation of speculators. A speculator, who can be an individual, a corporation, or a financial institution, makes an estimate of the future and if right profits and if wrong loses. Speculating on the capital markets usually means taking high risks since the future is always uncertain. Speculative positions create market liquidity as well as provide a mechanism for sharing risk, which, for example, a (manufacturing) business, not having in-house expertise in risk management, may want to dispense with.

Although the term 'speculator', to some, carries a negative connotation, the market needs both informed and uninformed, traders. Speculators are not inside traders but, instead, should be called uninformed traders, in contrast to informed traders who buy or sell a specific instrument. If only informed traders were market players, any move to buy or sell would lead to slippage in the offered prices, leading to the informed traders being held to ransom by the market. Uninformed traders provide the 'veil', a background of 'noise', that allows informed traders to enter the market without causing major slippages in prices. One needs both the informed and uninformed traders for the market to function efficiently.

2.2.1 Securitization

Another reason for the expansion of the capital markets is that financial engineering has created instruments that allow diverse forms of future cash flows to be used for issuing vast amounts of securitized debt. Securitization is the consolidation and structuring of cash-flow producing financial instruments, called asset-backed securities, that can then be traded in the capital markets. For example, the securitization of cash flows, such as mortgage payments and rentals, has allowed these to be traded in the capital markets – adding to the depth and liquidity of the capital markets.

Securitization is a relatively new concept in finance, having gained acceptance only over the last 20 years. Securitized debt has grown in the issuance of new loans and covers such diverse sectors as residential mortgages, commercial real estate, corporate loans, auto loans, student loans, and so on. In 1990, just 10% of mortgages in the United States were securitized, compared to 70% in 2007. It is estimated that by the middle of 2008 there were asset-backed securities worth US$10.2 trillion in the US and US$2.3 trillion in Europe. In 2007, new issues of asset-backed securities amounted to US$3.5 trillion in the US and US$650 billion in Europe. Securitization has had a major setback due to the 2008 US economic crisis, with the issuance of new mortgage-backed securities dropping by almost 85% in the first half of 2008 compared to the same period in 2007. The 2008 subprime crises in US home mortgages is claimed by some critics to be a negative example of securitization; this is not entirely correct and is discussed in Appendix B.2.

The lack of securitization can be a formidable barrier to economic development. It has been argued by Soto [91] that the securitization of third world developing countries' real estate, and of property in general, into tradable financial instruments could release vast amounts of capital. It was estimated that, in 1997, capital worth about US$9.3 trillion was locked up due to lack of securitization, an amount twice of the then total US public debt [91]. This 'dead capital', if securitized, could play a major role in the economic growth of the developing countries. Mortgages are fungible (a commodity that is freely interchangeable with another in satisfying an obligation) only in countries where the rule of law is well established and the legal system guarantees ownership. To securitize real estate assets in third world countries, hence, requires a stable political system that is accountable and relatively free from corruption. For these reasons, third world countries will have to overcome many major hurdles before they are in a position to create mortgage and other asset-backed securities, which would in turn release presently inert capital.

2.2.2 Profitability of the financial sector

At present, the rate of return of the financial sector and services in general is about 20% for the advanced economies of the US, Europe, and Japan – much higher than the 8–10% returns from manufacturing.[1] For example, from 2002 to 2006 five leading US investment banks – Goldman Sachs, Merrill Lynch, Morgan Stanley, Lehman Brothers, and Bears Stern – had an average return on equity of about 22%, amounting to US$30 billion – rivaling returns for such profitable industrial sectors as pharmaceuticals and energy.

The increasing volume of financial money capital reflects the overall expansion of the world economy, with vast amounts of surplus finding its way to the capital markets. The high rates of return from finance capital is one of the reasons for the immense infusion of savings and other assets into the global capital markets. The higher rate of return is thought to be due to the finance industry not being as mature as manufacturing and is taken to indicate a shift of the global economy to a new regime. There is, however, a contrarian view that the high returns from finance are primarily the result of the formation of an asset bubble – and hence intrinsically unstable and not sustainable.

The September–October 2008 global financial meltdown seems to provide strong evidence in support of the contrarian view. By the end of the September–October

[1] The rate of return on manufacturing is thought to be low due to the increasingly large capital investment required for setting up and upgrading modern industries.

2008 US financial meltdown, all the five US investment banks had ceased to exist – with Goldman Sachs and Morgan Stanley having converted themselves into bank holding companies. The consistently high returns of 22% from 2002 to 2006 shown by the five investment banks, with hind sight, is seen to completely coincide with the formation and expansion of the US subprime mortgage loans' financial bubble and may have simply been a result of this bubble.

Finance may still give a return higher than manufacturing due to the creation of new financial instruments, but in the current climate of financial turmoil and contraction it will be a while before such innovations find acceptance in the capital markets.

2.3 New centers of global finance

The United States (US) capital markets, since 1945, have been the most important component of the global capital markets, playing a central role in shaping and developing the international financial system. In Appendix B, the structure of the US debt markets is briefly reviewed.

The US is losing its pre-eminent position in the global capital markets due to the following reasons: (a) massive financial losses caused by the 2008 economic meltdown – in the US stock market, for example stocks on the Dow Jones lost 34% of their value in 2008 (the largest drop since 1931), and in the bankruptcy of major US financial institutions; (b) the rise of other capital markets and centers of wealth. The year 2007 saw a sea change in the distribution of global wealth. Largely due to the rise of China and India and investments by oil and gas producing countries, for the first time since the Second World War (1945) London displaced New York to become the center of the global capital markets. Over 40% of the world's foreign equities were traded in London, more than New York. Over 30% of the world's foreign currency trading took place in London, being larger than New York and Tokyo combined.

The US capital markets, in 2007, were worth US\$42 trillion of which US\$7.3 trillion was owned by foreigners, namely 17%, who also held 44% of the US national debt. In contrast to both New York and Tokyo, which depend largely on their domestic and East Asian markets, 80% of London's business comes from international sources, spread widely over many regions and countries.

The shift away from a US-centered global financial system can also be seen in the emergence of the Euro as an international reserve currency, as can be seen from Table 2.1. The Euro was introduced in 1999 and by 2008 had appreciated over 50% against the US Dollar. International reserves are now held in both the US Dollar and the Euro, with estimates that by 2010 about 34% will be held in Euro and 54% in the US Dollar, in contrast to 2000 when 71% of world reserves were held in

Table 2.1 *International reserve in Euros and US Dollars, and the projected currency distribution of these reserves by the year 2010.*

	Currency of international reserve		
	2000	2007	2010 (projected)
US Dollar $	71%	63%	54%
Euro €	18%	26%	34%

US Dollars. Some economists have predicted that, by as early as 2015, the Euro may overtake the US Dollar as the main international reserve currency provided two conditions hold: (a) more countries, including the UK, join the Eurozone countries and (b) the 2008 US economic crisis causes a deterioration in the value of the US Dollar.

With the increasing pace of globalization, one can expect the emergence of new international centers of finance in Shanghai, Hong Kong, Singapore, Mumbai, Dubai, Sao Paolo, and so on.

2.4 Interest rates

Interest rates, in essence, represent the interplay of time with economic activity, money capital, and real (tangible) assets.

The money form of capital represents real productive assets of society that can erode over time; furthermore, other factors like inflation, currency devaluations, new technologies, and so on make the value represented by financial assets a variable quantity that responds to changing circumstances. Financial assets represent the ability to initiate or facilitate economic activities, opportunities for which are tied to many social factors. For these and many other reasons, the effective value of money is strongly dependent on time.

How does one estimate the time value of money? From economic theory, the sum total of all the endogenous and exogenous factors that affect the time value of money are contained in the interest rates that one earns on cash deposits or on Treasury Bonds. Money invested in other financial instruments is more complicated to value as risk premiums are involved, perceptions of which differ between investors. Ultimately, the time value of money involves discounting expected future cash flows from bonds to obtain its present-day value; or, inversely, compounding present-day cash deposits for obtaining its expected future value.

Interest rates fix the cost of borrowing capital, the ‘cost’ of money, and are determined by both, the supply and demand for money – which depend on the prevailing interest rates – and by the macroeconomic policies of central banks.

Central banks would, ideally, like to hold down inflation while at the same time engendering economic growth; central banks balance inflation against the rate of economic growth by regulating the supply of money. One of the major tools for influencing the supply and demand for money is by setting interest rates.

Market forces of supply and demand and central banks' setting of interest rates are in a state of constant tension. Market forces sometimes force the central banks to change the interest rates so as to bring them in line with the market; at other times, central banks intervene by changing the interest rates and thus affecting the market's demand for money.

The concepts of *discounting* and *compounding* are fundamental to finance. However, contrary to what one intuitively expects, the relation turns out to be far more complex than discounting and compounding simply being the inverse of each other. The different forms of compounding (discounting) present (future) cash flows provide different ways by which interest rates are defined.

Consider the future value of a fixed deposit that is rolled over continuously; a constant interest rate leads to an exponential compounding of the value of the initial fixed deposit. Discounting, on the other hand, is the procedure that yields the present-day value of a pre-fixed future cash flow and is exponentially smaller for constant interest rates. In essence, all measures of interest rates arise by either discounting expected future cash flows to obtain their present-day value or by compounding the present-day value of fixed deposits to obtain the value of future cash flows.

2.5 Three definitions of interest rates

The following procedures for defining interest rates are widely used in the financial markets, with an interest rate 'yield curve' for each case.

- Returns on cash deposits using simple interest rates. This is the basis of defining Libor and Euribor, the two fundamental market determined interest rates.
- Discrete compounding of cash deposits and discrete discounting of bonds. This procedure is the basis for the definition of the zero coupon yield curve (ZCYC), which is fundamental to the interest rates and bond markets.
- Instantaneous compounding and discounting future cash flows. This definition leads to the concept of instantaneous forward interest rates, the main theoretical construct of the bond market.

To simplify the discussion of the central concepts, all interest rates for now are taken to be constant. The more complex generalizations of these concepts are discussed in the later sections.

2.5.1 Simple interest rates

Consider a principal sum of amount M, kept in a bank fixed deposit at time t and earning a simple interest at the rate of L per year. After a period of say T years, the initial amount M increases to $M[1 + (T - t)L]$. Conversely, if one is to receive a pre-fixed amount B at time T in the future, the value of that amount at time t is given by $B/[1 + (T - t)L]$. In summary

$$M \text{ at time } t = M[1 + (T - t)L] \text{ at time } T$$
$$\frac{B}{[1 + (T - t)L]} \text{ at time } t = B \text{ at time } T \tag{2.1}$$

2.5.2 Discrete compounding and discounting: yield to maturity

Consider a fixed deposit made at time t; the principal earns a *yield to maturity* z, a dimensionless quantity that is a measure of simple interest for a period, usually taken to be one year. At the end of one year, the interest earned is compounded – namely, the interest earned is *added* to the principal sum. At the end of the first year $M(1 + z)$ is the amount in the fixed deposit; at the end of the second year the amount in the fixed deposit is $M(1 + z)^2$, and so on. For a deposit of duration $T - t$ years, there are $[T - t] = (T - t)/1$ number of compounding.[2]

Hence, at time T, the discretely compounded amount for a fixed deposit made at time t is given by

$$M \text{ at time } t = M(1 + z)^{[T-t]} \text{ at time } T$$
$$\frac{B}{(1 + z)^{[T-t]}} \text{ at time } t = B \text{ at time } T \tag{2.2}$$

where the last equation gives the discretely discounted value at time t of a pre-fixed payment B at time T.

2.5.3 Continuous compounding and discounting

Consider the case of discrete compounding, but now let ϵ be an infinitesimal period of discrete compounding. Consider the limit of $\epsilon \to 0$; simple interest payments are now given by $z = \epsilon r$; r is the instantaneous spot interest rate and has the dimension of 1/time. The interest generated in the time interval t to $t + \epsilon$, is $M\epsilon r$ and the fixed deposit is compounded to yield $M(1 + \epsilon r)$. For the time interval $T - t$, the

[2] Note $[T - t]$ is always an *integer*.

number of times the principal is compounded is $(T-t)/\epsilon$; hence the value of the continuously compounded fixed deposit at time T is given by

$$\lim_{\epsilon \to 0} M(1+\epsilon r)^{(T-t)/\epsilon} = Me^{r(T-t)}$$

In summary, for continuously compounded interest rates

$$\begin{aligned} M \text{ at time } t &= Me^{r(T-t)} \text{ at time } T \\ \frac{B}{e^{r(T-t)}} \text{ at time } t &= B \text{ at time } T \end{aligned} \tag{2.3}$$

All the different ways of defining interest rates are of course consistent. Any inconsistency or incompatibility in the different definitions of interest rates leads to arbitrage opportunities in the prices of debt instruments.[3] This in turn would lead to trades that remove any pricing inconsistency.

2.6 Coupon and zero coupon bonds

Cash represents present-day value, whereas bonds represent future cash flows.

Bonds are fundamental instruments of debt; the seller of a bond issues a promissory note to the buyer that states the seller's (legal) obligation to make a future payment of a certain pre-determined amount. The amount includes a component that is the return on the bond and reflects the interest rate paid by the issuer of the bond.

One of the primary financial instruments of the national and international debt markets are government and corporate bonds. Interest rates can be derived from the market prices of bonds. Given the vast diversity of the bond market, only those aspects of bonds are discussed that are of direct relevance to the material covered in this book. The readers are referred to the extensive literature on bonds [73].

A *zero coupon bond* is a financial instrument that gives a single pre-determined payoff, of say €1, called the principal amount, when it matures at some fixed future time T; its price at earlier time $t < T$ is given by $B(t, T)$. Note that for a zero coupon bond there are no coupon payments and hence the name.

At time t there are, in principle, infinitely many zero coupon bonds with varying maturities; that is, bonds $B(t, T)$, in principle, exist for all $T \in [t, t+\infty]$ years. In practice, in the capital markets, the zero coupon bonds are usually issued with maturity from one day to about 30 years in the future and hence $T \in [t, t+30]$ years. The collection of the prices of all zero coupon bonds $B(t, T)$, with maturity from present time t to a maximum time T is called the zero coupon bond term structure.

[3] Arbitrage opportunities means that one can make risk-free profit that is higher than the (risk-free) rate of return on fixed deposits. See Section 3.5.

Consider a *coupon bond*, denoted by $\mathcal{B}(t)$, that pays a principal of L when it matures at time T, and pays fixed dividends (coupons) a_i at times $T_i, i = 1, 2, \ldots, N$. The value of the coupon bond at time $t < T_i$ can be shown [63, 65] to be equivalent to a portfolio of zero coupon bonds with maturities coinciding with the payment dates of the coupons. Quantitatively

$$\mathcal{B}(t) = \sum_{i=1}^{N} a_i B(t, T_i) + LB(t, T) = \sum_{i=1}^{N} c_i B(t, T_i) \tag{2.4}$$

For simplicity of notation, the time of maturity of the coupon bond is taken to be the date of the last coupon payment, that is $T = T_N$. The final payment is included in the sum by setting $c_i = a_i;\ c_N = a_N + L$.

Intuitively, the reason that a portfolio of zero coupon bonds is equal to a coupon bond is because the two instruments have the same cash flow. Every coupon payment for the coupon bond is equivalent to a zero coupon bond maturing at the time of the payment. A fundamental theorem of finance states that any two financial instruments that have the same cash flow are identical [63]. The proof follows from the fact that, otherwise, arbitrage opportunities would exist for the prices – which is ruled out in an efficient market.

2.6.1 Coupon bond yield-to-maturity y

Given the wide variety of coupon bonds, with different face values L, different amounts and number of coupon payments a_i and N respectively, it is difficult to compare the *rates of return* of two different coupon bonds. For this reason, a generalization of the zero coupon bond yield-to-maturity z, given in Eq. (2.23), is defined for coupon bonds and denoted by y.

Coupon bond yield-to-maturity y is the annual yield such that, at time t, the present values of the future cash flows, discretely discounted yearly by y, equal the face value of the coupon bond. For coupon bonds with N number of (annual) payments, the yield-to-maturity is defined as follows

$$\mathcal{B}(t) = \sum_{i=1}^{N} \frac{a_i}{(1+y)^i} + \frac{L}{(1+y)^N}$$

Given the values of $\mathcal{B}(t), a_i$, and N, it is in general a nonlinear problem to evaluate y, and is usually done numerically. Once the y value of a coupon bond is determined, one can accurately compare it with other coupon bonds with very different cash flows. One can readily generalize the definition of the coupon bond yield-to-maturity y for coupons that are paid out c times a year and so on.

From Eq. (2.4) one can conclude that the zero coupon bonds are the fundamental instruments of the bond market. If one can model the behavior of the zero coupon bonds, one automatically has, in principle, a model for the coupon bonds. However, as is to be expected, the coupon bond is a much more complex instrument than the zero coupon bond.

All bonds have a credit risk, which is the likelihood of default, due by the possible inability of the issuer to pay either the coupons or the principal amount. Credit risk arises from various sources and the financial consequences of default are taken into account in the pricing models of such defaultable, or risky, bonds; in particular, the higher the possibility of default, the higher the interest rate that has to be paid out by the issuer of the bond.

An important class of both coupon and zero coupon bonds are those that carry no risk of default; such bonds are called *Treasury Bonds*. In practice, bonds issued by the US federal government are taken to be risk-free Treasury Bonds and consequently have the lowest interest rates in the debt market. Almost all the discussions on bonds, in the later chapters, are confined to the study of risk-free Treasury Bonds.

Since bonds generate pre-fixed (series of) cash flows, they belong to the larger class of financial instruments called *fixed-income securities*. The ownership of a fixed-income security is often, erroneously, considered to be less risky than the ownership of equity since – short of the issuer going bankrupt – the owner of a fixed-income security is guaranteed a return. However, due to interest rate risk, credit risk, and currency risk (for the bonds that are issued in a foreign currency), a bond portfolio before maturity can lose as much value, or even more, than a portfolio of equities.

2.7 Continuous compounding: forward interest rates

The present-day value of a bond is obtained by discounting future cash flow(s) using various methods, with each method providing a definition of interest rates.

Consider the simplest case of an economy that has a constant interest rate r. As discussed in Eq. (2.3) a continuously compounded fixed cash deposit of €1 made at time t will yield, at time T in the future, a cash of amount $\exp\{(T-t)r\}$. Hence a zero coupon bond yielding €1 at time T has a present value of

$$B(t,T) = e^{-(T-t)r}$$

In general, a real economy never has an interest rate that is constant over future time. Instead, for each future time T, there is a separate effective interest rate, denoted by $r(t,T)$, called the *term structure of interest rates* and also known as the *interest yield curve*. The zero coupon bond is given by

$$B(t,T) = e^{-(T-t)r(t,T)}$$
$$\Rightarrow r(t,T) = -\frac{1}{T-t}\ln B(t,T) \tag{2.5}$$

The interest yield curve can also be used for determining the future value of a fixed deposit that is continuously compounded; for €1 deposited at time t and continuously compounded, its future value at time T is locked in at time t to be equal to $\exp\{(T-t)r(t,T)\}$.

Forward interest rates, denoted by $f(t;T_1,T_2)$, are continuous rates that are available in the debt market such that one can lock-in, at time t, the interest rate for a deposit from future time T_1 to T_2, with $T_2 > T_1$.

To understand the relation of $f(t;T_1,T_2)$ to zero coupon bonds, consider two zero coupon bonds $B(t,T_1)$ and $B(t,T_2)$, with $T_2 > T_1$. The definition of bonds in terms of the interest yield curve given in Eq. (2.5) yields

$$B(t,T_1) = e^{-(T_2-T_1)f(t;T_1,T_2)}B(t,T_2)$$
$$\Rightarrow f(t;T_1,T_2) = -\frac{1}{T_2-T_1}\ln\left[\frac{B(t,T_1)}{B(t,T_2)}\right] \tag{2.6}$$

Discounting of bonds, from future to present time, is shown in Figure 2.1.

For a deposit made at time t, the future value at times T_1 and T_2 are $\exp\{(T_1-t)r(t,T_1)\}$ and $\exp\{(T_2-t)r(t,T_2)\}$, respectively. However, the value of the two deposits are related, as shown in Figure 6.6, since one can take the cash obtained at time T_1 and lock-in the interest at time t, for the duration from T_1 to T_2 using $f(t;T_1,T_2)$. The principle of no-arbitrage yields

$$e^{(T_2-t)r(t,T_2)} = e^{(T_1-t)r(t,T_1)}e^{(T_2-T_1)f(t;T_1,T_2)} \tag{2.7}$$

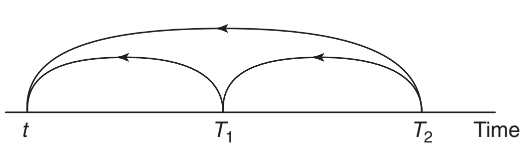


Figure 2.1 The discounting of bond payoff directly from time T_2 to time t and via an intermediate time T_1.

2.8 Instantaneous forward interest rates

Forward interest rates play a central role in the study of interest rates and coupon bonds. The forward interest rates provide a representation of zero coupon term structure that is analytically and conceptually very useful in the study of the bond market.

To derive the instantaneous forward interest rates from the term structure of the zero coupon bonds, consider two bonds that are mature at infinitesimally separated future times. More precisely, in Eq. (2.6) let $T_2 = T_1 + \epsilon$; hence one obtains the following[4]

$$B(t, T + \epsilon) = e^{-\epsilon f(t,T,T+\epsilon)} B(t, T) \tag{2.8}$$

The limit of forward interest rates

$$f(t, T) \equiv \lim_{\epsilon \to 0} f(t, T, T + \epsilon) \tag{2.9}$$

defines the instantaneous *forward interest rates*, namely $f(t, T)$. Instantaneous forward interest rates $f(t, T)$ are the rate, fixed at time t, for instantaneous loans at future time $T > t$; as expected $f(t, T)$ has the dimensions of 1/time. All forward interest rates are always positive and hence

$$f(t, T) > 0 \quad \text{for all } t,\ T \tag{2.10}$$

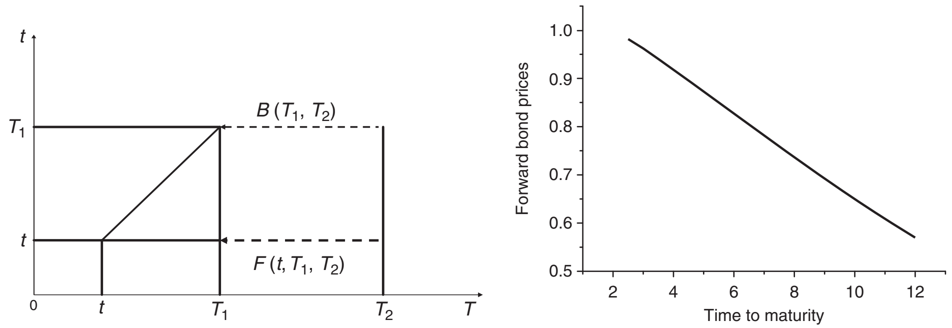
The spot interest rate $r(t)$ is the instantaneous interest rate at time t; the definition of the instantaneous forward rates yields

$$r(t) = f(t, t)$$

Eq. (2.8) provides a recursion equation. Let maturity time be discretized into a lattice with $T - k\epsilon$ points; then, since $B(t, t) = 1$, Eq. (2.8) yields the following

$$\begin{aligned}
B(t, T) &= \exp\{-\epsilon f(t, T - \epsilon)\} B(t, T - \epsilon) \\
&= \exp\left\{-\epsilon \sum_{k=1}^{(T-t)/\epsilon} f(t, T - \epsilon k)\right\} B(t, t) \\
&\to \exp\left\{-\int_0^{T-t} dy\, f(t, T - y)\right\} \\
\Rightarrow B(t, T) &= \exp\left\{-\int_t^T dx\, f(t, x)\right\}; \quad x = T - y
\end{aligned} \tag{2.11}$$

[4] In practice, one takes $\epsilon = 1$ day $= 1/360$ year.



(a) (b)

Figure 2.2 (a) The forward interest rates, indicated by the dashed lines, that define a zero coupon bond $B(T_1, T_2)$ and its forward price $F(t, T_1, T_2)$. (b) The forward bond price $F = F(t, T_1, T_2)$ for zero coupon bonds maturing at different times T_2, with $T_1 - t = 2$ years in the future. The forward interest rates $f(t, T)$ were obtained from the US$ zero coupon yield curve for $t = 29$ January 2003.

Figure 2.2(a) graphically represents the forward interest rates that define a zero coupon bond $B(T_1, T_2)$.

It is worth noting that one can directly obtain the current value of the bond $B(t, T)$ by discounting the €1 payoff taking infinitesimal backward time steps ϵ from maturity T to present time t, which yields[5]

$$B(t, T) = e^{-\epsilon f(t,t+\epsilon)} e^{-\epsilon f(t,t+2\epsilon)} \ldots e^{-\epsilon f(t,x)} \ldots e^{-\epsilon f(t,T)}$$

$$\Rightarrow B(t, T) = \exp\left\{-\int_t^T dx f(t, x)\right\} \tag{2.12}$$

$$\frac{\partial B(t, T)}{\partial T} = -f(t, T)B(t, T) \tag{2.13}$$

In fact, the result given above, using the concept of discounting, is obtained more formally in Eq. (2.11), using the recursion equation.

Eq. (2.12) shows that $f(t, x)$ is a set of variables equivalent to the zero coupon bonds. From the definition of the instantaneous forward interest rates given in Eq. (2.12), the forward interest rate and the interest yield curve are given by the following

$$f(t : T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} dx f(t, x)$$

$$r(t, T) = \frac{1}{T - t} \int_t^T dx f(t, x)$$

[5] The fixed payoff €1 is assumed and is not written out explicitly.

Suppose a zero coupon bond $B(T_1, T_2)$ is going to be issued at some future time $T_1 > t$, with expiry at time T_2; the *forward price* of the zero coupon bond is the price that one pays at time t to lock-in the delivery of the bond when it is issued at future time T_1. Hence, the *forward bond price* is given by

$$\begin{aligned} F(t, T_1, T_2) &= \exp\left\{-\int_{T_1}^{T_2} dx f(t, x)\right\} \\ &= \frac{B(t, T_2)}{B(t, T_1)} : \quad \text{forward bond price} \end{aligned} \tag{2.14}$$

In terms of the forward interest rates the forward bond price is given by

$$F(t, T_1, T_2) = \exp\{-(T_2 - T_1) f(t; T_1, T_2)\}$$

Figure 2.2(b) shows the forward bond price $F = F(t, T_1, T_2)$ of the bond $B(T_1, T_2)$. The values of the forward bond price are plotted in Figure 2.2(b), as a function of maturity time. It can be seen that the forward price falls rapidly, as is expected, given the exponential discounting of the bond prices.

At any instant t, the capital markets (implicitly) have instantaneous forward interest rates from present t out to a time T_{FR} in the future; for example, if t refers to present time t_0, then one has forward rates from t_0 till time $t_0 + T_{FR}$ in the future. In the market, T_{FR} is at least about 30 years, and hence we have $T_{FR} > 30$ years. In general, at any time t, all the forward interest rates $f(t, x)$ exist till time $t + T_{FR}$ and, hence, have future time x with $t < x < t + T_{FR}$.

2.9 Libor and Euribor

The two main international currencies are the US Dollar and the Euro, which is the currency of the European Union. As can be seen from Table 2.1, almost 90% of international cash reserves are in the form of US Dollars or Euros. Cash fixed deposits in these currencies account for almost 90% of simple interest rates that are traded in the capital markets. Cash deposits in US Dollars as well as British Pounds earn simple interest at the rate fixed by Libor and deposits in Euros earn interest rates fixed by Euribor.

2.9.1 Libor

The interest rates offered for time deposits are often based on Libor, the *London Interbank Offered Rate* [12]. Libor is one of the main instruments for interest rates in the debt market, and is widely used for multifarious purposes.

Libor was launched on 1 January 1986 by the British Bankers' Association. Libor is a daily quoted rate based on the interest rates at which *commercial banks* are willing to lend funds to other banks in the London interbank money market. The minimum deposit for a Libor has a par value of \$1,000,000. Libor is a simple interest rate for fixed bank deposits and the British Bankers' Association has daily quotes of Libor for loans in the money market of the following duration: overnight; one and two weeks; one, three, four, five, six, nine, and 12 months. Libors of longer duration are obtained from the interest rate swap market and are quoted for future loans of duration from two years to 30 years. A Libor zero coupon yield curve is constructed from the swap market and is quoted by vendors of financial data. The Libor market is active in maturities ranging from a few days to 30 years, with the greatest depth in the 90- and 180-day time deposits.

The three-month Libor is the benchmark rate that forms the basis of the Libor derivatives market. All Libor swaps, futures, caps, floors, swaptions, and so on are based on the three-month deposit. The main focus of this book is Libor derivatives and the term Libor will be taken to be synonymous with the three-month Libor.

In 1999 the open positions on Eurodollar futures had a par value of about US\$750 billion, and has grown tremendously since then. The Chicago Mercantile Exchange (CME) Libor futures represent one-month Libor rates on a \$3 million deposit. In 2008, CME had Eurodollar futures and options on Libor with open interest of over 40 million Libor contracts and an average daily volume of 3.0 million. Libor is amongst the world's most liquid short-term interest rate futures contracts. Interest rate swaps, with Libor taken as the floating rate, currently trade on the interbank market for maturities of up to 50 years.

Market data on Libor futures are given for daily time t in the form of $L(t, T_i - t)$, with fixed dates of maturity T_i (March, June, September, and December) and shown in Figure 2.2(a). The shortest maturity time is $\theta_{\text{min}} = 3$ months, and the spot rate is taken to be $r(t) = f(t, \theta_{\text{min}})$.

2.9.2 Euribor

Euribor (Euro Interbank Offered Rate) is the benchmark rate of the Euro money market, which has emerged since 1999. Euribor is simple interest on fixed deposits in the Euro currency; the duration of the deposits can vary from overnight, weekly, monthly, three monthly out to long duration deposits of ten years and longer. Euribor is sponsored by the Financial Markets Association (ACI) and by the European Banking Federation (FBE), which represents 4,500 banks in the 24 member states of the European Union and in Iceland, Norway, and Switzerland. Euribor is the rate at which Euro interbank term deposits are offered by one prime bank to another.

The choice of banks quoting for Euribor is based on market criteria. These banks are of first-class credit standing. They are selected to ensure that the diversity of the Euro money market is adequately reflected, thereby making Euribor an efficient and

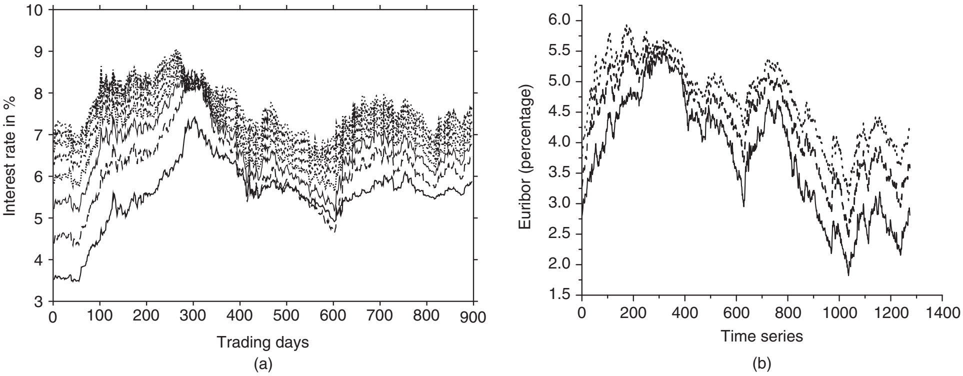


Figure 2.3 (a) Daily Eurodollar futures for Libor rates $L(t, t+7 \text{ years}), \ldots L(t, t+6 \text{ years}), \ldots L(t, t+1 \text{ year})$, and $L(t, t+0.25 \text{ years})$ with $t \in [1996, 1999]$. (b) Euribor maturing one, two, and three years in the future, from 26 May 1999 to 17 May 2004.

representative benchmark. All the features discussed for Libor can also be applied to Euribor.

Euribor was first announced on 30 December 1998 for deposits starting on 4 January 1999. Figure 2.3(b) shows daily values for three Euribor forward interest rates for 90-day deposits one, two, and three years in the future. Since its launch, Euribor has been actively trading on the options markets and is the underlying rate of many derivatives transactions, both over-the-counter and exchange-traded. Euribor is one of the most liquid global interest rate instruments, second only to Libor. The Euribor zero coupon yield curve, based on the rates being contracted in the Euribor swaps market, extends out to 50 years in the future.

2.10 Simple interest rate

Cash deposits can earn simple interest rates for a given period of time. For example, one can lock-in at time t, a simple interest rate, denoted by $L(t; T_1, T_2)$, for a fixed deposit from future time T_1 to T_2. The period of the deposit, namely $T_2 - T_1$, is called the *tenor* of the simple interest rate.

A deposit of €1, made from time T_1 to T_2, will increase, as in Eq. (2.1), to an amount $1 + (T_2 - T_1)L(t; T_1, T_2)$. Similarly, the present-day value of a zero coupon bond $B(t, T)$ is given by

$$B(t, T) = \frac{1}{1 + (T - t)L(t; t, T)}$$

and more generally

$$B(t, T_2) = B(t, T_1)\frac{1}{1 + (T_2 - T_1)L(t; T_1, T_2)}$$

From the definition of zero coupon bonds given in Eq. (2.12) the simple interest rates are given in terms of the instantaneous forward interest rates by the following

$$\frac{1}{1 + (T_2 - T_1)L(t; T_1, T_2)} = \exp\left\{-\int_{T_1}^{T_2} dx f(t, x)\right\}$$
$$\Rightarrow L(t; T_1, T_2) = \frac{1}{T_2 - T_1}\left[\exp\left\{\int_{T_1}^{T_2} dx f(t, x)\right\} - 1\right] \tag{2.15}$$

From Eq. (2.10) one has $f(t, x) > 0$ and this leads to

$$L(t; T_1, T_2) > 0 \tag{2.16}$$

The forward interest rates for returns on fixed cash deposits are the same as $f(t, x)$; these rates are, in principle, identical to the forward interest rates discussed in Section 2.8.

Consider a future time falling within the fixed maturity times, say $\theta = x - t$, with $T_i - t \leq \theta \leq T_{i+1} - t$; to obtain $L(t, \theta)$ with fixed θ, a spline interpolation for the values of the Libor yields the values of $L(t, T)$ for continuous future time T. The spline interpolation is necessary since the Libor data are provided only for discrete maturity times $T_i - t$, whereas for empirically studying interest rates, data are required for constant θ. The daily interpolated data, from 1996 to 1999 for the Libor rates, is plotted in Figure 2.3(a).

A futures contract is an undertaking by participating parties, entered into at time t, to loan or borrow a fixed amount of principal at an interest rate fixed by Libor $L(t, T_1, T_2)$; the contract is executed at a specified future date $T_1 > t$. Consider a futures contract entered into at time t for a 90-day deposit of the principal P, from future time T to $T + \ell$ ($\ell = 90/360$ year). On maturity, an investor who is long on the contract receives P plus simple interest I; hence

$$P + I = P[1 + \ell L(t; T, T + \ell)] \tag{2.17}$$

where $L(t; T, T + \ell)$ is the (annualized) three-month Libor interest rate. For simplicity of notation, a 90-day tenor is written as ℓ.

One can express the principal plus interest based on compounding by instantaneous forward interest rates and obtain

$$P + I = Pe^{\int_T^{T+\ell} dx f(t,x)} \tag{2.18}$$

Define the *benchmark* three-month Libor by

$$L(t;T) \equiv L(t;T,T+\ell) \tag{2.19}$$

The relationship between Libor and forward interest rates, from Eqs. (2.17) and (2.18), is given by

$$1+\ell L(t,T) = e^{\int_T^{T+\ell} dx f(t,x)}; \quad \Rightarrow L(t,T) = \frac{e^{\int_T^{T+\ell} dx f(t,x)} - 1}{\ell} \tag{2.20}$$

Note that the above equation is a special case of the relation between $L(t;T_1,T_2)$ and $f(t,x)$ given in Eq. (2.15) with $T_1 = T$ and $T_2 = T+\ell$.

Forward interest rates $f(t,x)$ can be extracted from Libor futures data. Since Libor is determined on a daily basis, the data for the forward interest rates are given only for discrete calendar time. Future time is also discrete, with the benchmark Libor given at 90-day intervals.

In terms of zero coupon bonds $B(t,T)$, from Eqs. (2.12) and (2.20), Libor has the following representation

$$L(t,T) = \frac{1}{\ell}\frac{B(t,T) - B(t,T+\ell)}{B(t,T+\ell)} \tag{2.21}$$

It is sometimes assumed that the Libor futures prices are approximately equal to the forward interest rates. More precisely, from Eq. (2.20)

$$L(t,T) = \frac{e^{\int_T^{T+\ell} dx f(t,x)} - 1}{\ell} \simeq f(t,T) + O(\ell) \tag{2.22}$$

The errors in setting Libor equal to the forward interest rates are usually negligible, given the other errors that arise in the empirical study; the justification for this assumption is discussed in [12]. In summary, Libor can be identified with the forward interest rates, but sometimes it is more appropriate to use the full expression for $L(t,T)$.

2.11 Discrete discounting: zero coupon yield curve

Recall that, from Section 2.5, the *yield-to-maturity* z of a zero coupon bond is the annual simple interest that is discretely compounded every year. Let T,t be the maturity and issue date of the bond; as before, let $[T-t] = (T-t)/\text{year}$ be an integer equal to the number of years. On maturing, the bond value of €1 will compound to $(1+z)^{[T-t]}$. Since, on maturity, the payoff of the bond is €1, the relation of z to the price of the zero coupon bond at t is given by

$$B(t,T) = \frac{1}{(1+z)^{[T-t]}} \tag{2.23}$$

Note the yield-to-maturity varies for the different bonds; hence, a more precise notation is to have a *term structure* for the yield-to-maturity, called the zero coupon yield curve (ZCYC) and denoted by $Z(t, T)$; similar to z, $Z(t, T)$ is dimensionless. Eq. (2.23), for a ZCYC that is annually compounded, has the following generalization

$$B(t, T) = \frac{1}{(1 + Z(t, T))^{[T-t]}} \tag{2.24}$$

Equation (2.24) states that $Z(t, T)$ is the dimensionless yield-to-maturity, compounded annually, that is earned by the zero coupon bond $B(t, T)$. If the interest is paid out c times a year, then the number of payments is $c[T - t]$ with each payment of interest being $Z(t, T)/c$; hence, for a ZCYC for interest that is compounded c times a year, the bond is given by

$$B(t, T) = \frac{1}{\left(1 + \frac{1}{c} Z(t, T)\right)^{c[T-t]}} \tag{2.25}$$

In the bond market, for semi-annual (six monthly) payments, $c = 2$ and hence

$$B(t, T) = \frac{1}{\left(1 + \frac{1}{2} Z(t, T)\right)^{2[T-t]}} \tag{2.26}$$

Market data for $Z(t, T)$ from the bond market are given in Figure 2.4(a) for fixed future remaining time, that is for $Z(t, t + \theta)$, with future remaining time θ ranging

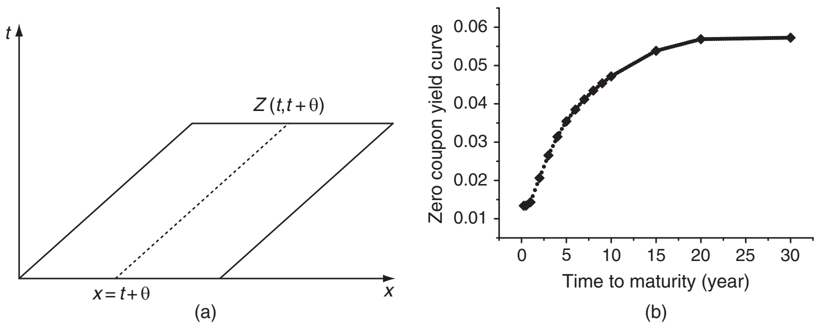


Figure 2.4 (a) The zero coupon yield curve (ZCYC) $Z(t, T)$ is given along the lines of constant $\theta = x - t$; the diagram shows $Z(t, t + \theta)$ for θ = constant. (b) The spline curve fit for the US Treasury Bonds semi-annually compounded zero coupon yield curve (ZCYC) $Z(t, T)$, with market data given by the filled squares. The curve is given for calendar time t = 29 January 2003 out to 30 years into the future. The market values of the ZCYC are given for discrete future remaining time θ equal to 3m, 6m, 1y, 2y, 3y, 4y, 5y, 6y, 7y, 8y, 9y, 10y, 15y, 20y, 30y.

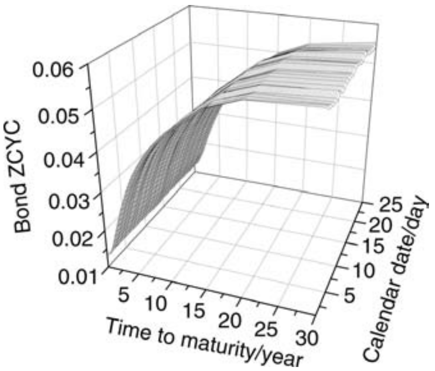


Figure 2.5 Zero coupon yield curve obtained from US Treasury Bonds. The calendar date is from 29 January 2003 to 4 March 2003, with a total of 25 trading days.

from three months out to 30 years. Figure 2.4(b) shows the structure of the ZCYC $Z(t, T)$ for US Treasury Bonds as a function of remaining future time $\theta = T - t$ with calendar time t fixed at 29 January 2003; market data are the discrete points and the interpolation curve is the result of a spline fit.[6]

Figure 2.5 shows the ZCYC obtained from US Treasury Bonds. The ZCYC rises and flattens out, as expected, since forward interest rates for the future are in general higher than near-term loans and spot rates, given that risks accumulate further out into the future. However, there are cases in which the ZCYC may have an inversion in the future reflecting some regulation or other exogenous factors that affect the future behavior of the bond market. There are several theories of interest rates that study the long- and short-term behavior of the ZCYC [73].

ZCYC is given by the capital markets for future remaining time $T - t$ from zero up to 30 years; in other words, every day, due to trading in the bond markets, the value of the $Z(t, T)$, from one day up to 30 years in the future is refreshed and updated by the bond market. The long duration of data for the ZCYC makes it one of the most important interest rate instruments for modeling the long-term behavior of the bond markets. To obtain the values of $Z(t, T)$ for continuous values of future time $\theta = T - t$ one fits the discretely given values of the ZCYC by a smooth spline curve.[7]

[6] For the empirical studies in later chapters, daily Treasury Bond ZCYC data for calendar times from 29 January 2003 to 28 January 2005 were used.

[7] It is assumed that the ZCYC rates are smooth; the assumption is a reasonable one to make as one would intuitively expect that the ZCYC, say three years into the future would not be too different from that of three years and one month into the future. The loss in accuracy due to the spline interpolation is unimportant since the future times at which $Z(t, T)$ is specified, namely values of T, are separated by at least by three months. The market data that are being studied have random errors larger than the errors introduced by the spline interpolation. See Section 2.14.

Both the bond markets as well as the Libor markets provide a ZCYC. For the case of Libor, the British Bankers' Association quotes daily interest rates for overnight (24 hours) deposits up to rates for deposits made one year in the future. Libors for deposits made at future time from one to 30 years are obtained from the interest rate swaps market. All the Libors are combined to produce a single ZCYC that is semi-annually compounded to produce the effective Libor zero coupon bonds. More precisely

$$B_L(t,T) = \frac{1}{\left(1+\frac{1}{2}Z_L(t,T)\right)^{2[T-t]}} \tag{2.27}$$

where the subscript L indicates Libor. In Chapter 6 the Libor forward interest rates that are derived from $B_L(t,T)$ are discussed. In principle, $B(t,T) = B_L(t,T)$, but there are differences related to the risk of default in the Libor market being greater than in the Treasury Bond market. Figure 2.6(a) shows the Libor ZCYC for two days five years apart and Figure 2.6(b) shows the Libor ZCYC $Z_L(t,x)$ for 25 consecutive days until 8 August 2008.

The term structure of the zero coupon bonds $B(t,T)$, for some fixed time t, consists of the prices for all $T \in [t, t+30$ years]. The market usually gives the term structure of the zero coupon bonds $B(t,T)$ in terms of the ZCYC. Figure 2.7(a) shows the term structure of zero coupon bonds as reconstructed from the US Treasury Bonds' ZCYC. Figure 2.7(b) shows the term structure for Libor zero coupon bonds $B_L(t,T)$ for two days five years apart; the shape of the Libor is different from the Treasuries result.

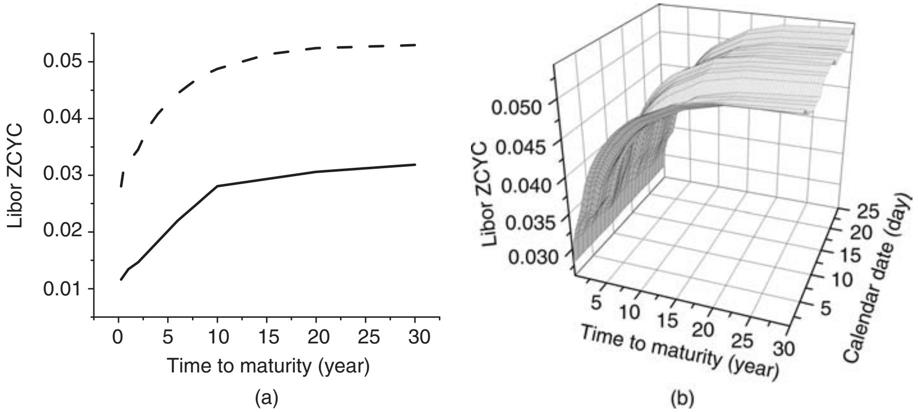


Figure 2.6 (a) A graph of Libor $Z_L(t,T)$. The solid line is for 8 August 2008 and the dashed line is for 28 October 2003. (b) A graph of Libor $Z_L(t,T)$, with future time $T-t$ shown along the x-axis out to 30 years; the daily values for t – for 25 subsequent days until 8 August 2008 – are shown along the y-axis.

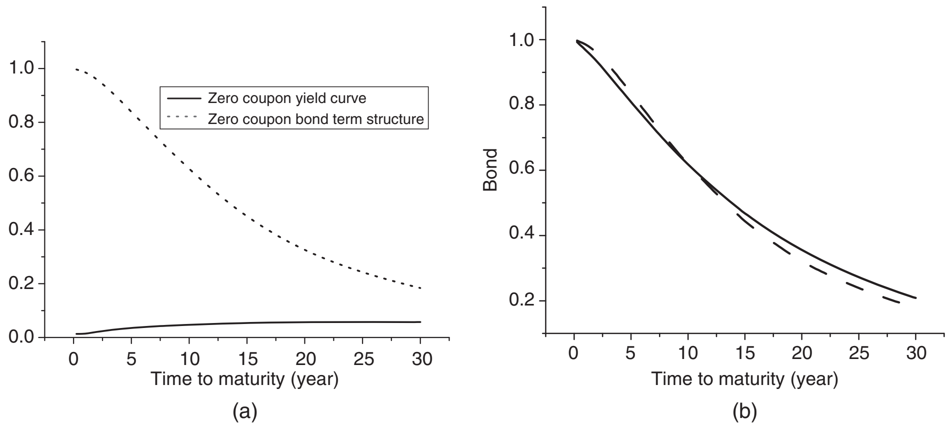


Figure 2.7 The ZCYC data are for $t = 29$ January 2003 and with $T - t$ up to 30 years into the future. (a) The term structure zero coupon bonds $B(t, T)$ (dotted line) obtained from ZCYC $Z(t, T)$ data (unbroken line). Note $B(t, T)$ falls off exponentially due to exponential discounting. (b) The term structure, up to 30 years for Libor zero coupon bonds $B_L(t, T)$ – obtained from the Libor ZCYC. The solid line is for 8 August 2008 and the dashed line is for 28 October 2003.

2.12 Zero coupon yield curve and interest rates

Both the ZCYC and the forward interest rates are descriptions of the same financial instrument, namely the zero coupon bonds; in the case of Libor, the ZCYC does not correspond to any actual traded zero coupon bonds, but, rather, is a compact way of expressing market data on the term structure for all the Libors taken together.

The two different descriptions, namely the ZCYC and the forward interest rates, are useful for representing different aspects of the interest rate and bond markets. Recall, from Section 2.5, that discounting future cash flows provides the following two definitions of the underlying interest rates: (a) the zero coupon yield curve (ZCYC) $Z(t, T)$, defined by the annual or semi-annual discounting and (b) instantaneous forward interest rates $f(t, T)$, defined by instantaneous discounting.

The traded zero coupon bond prices are quoted in the bond markets by specifying the ZCYC. In the case of the interest rate markets, the Libor ZCYC is directly quoted, based on a semi-annual compounding for obtaining the hypothetical Libor zero coupon bonds. Eqs. (2.12) and (2.25) are the key for relating the zero coupon bond price to the underlying interest rates and yield

$$B(t, T) = \frac{1}{\left(1 + \frac{1}{c} Z(t, T)\right)^{c[T-t]}} = \exp\left\{-\int_t^T dx f(t, x)\right\}$$

$$\Rightarrow \int_t^T dx f(t, x) = c[T - t] \ln\left(1 + \frac{1}{c} Z(t, T)\right) \tag{2.28}$$

Eq. (2.28) is dimensionally consistent. The left-hand side is dimensionless; as required, the right-hand side is also dimensionless; c is the dimensionless number of payments per year, $Z(t,T)$ is dimensionless, and, furthermore, the integer $[T-t]$ is also dimensionless.

From Eq. (2.28) one has, by differentiating on future time T, the following

$$f(t,T) = \frac{c}{\epsilon}\ln\left(1+\frac{1}{c}Z(t,T)\right) + \frac{[T-t]}{1+\frac{1}{c}Z(t,T)}\frac{\partial Z(t,T)}{\partial T} \tag{2.29}$$

One can numerically differentiate the ZCYC to extract $f(t,T)$; this procedure does yield an estimate of $f(t,x)$ from Eq. (2.29), but with such large errors that it makes the estimate quite useless for any empirical purpose.

The zero coupon bonds $B(t,T)$ are reconstructed directly from the ZCYC using Eq. (2.25) in Figure 2.8(a) (continuous line) and from forward interest rates $f(t,T)$ (dotted line), which have been extracted from the ZCYC using Eq. (2.29). One can see from Figure 2.8(a) that one gets large and systematic errors by using $f(t,T)$: the longer the time in the future the larger the systematic errors.

Both the interest rate and bond markets directly provide the ZCYC that is the *integral* of the forward interest rates over an interval of future time $[t,T]$. Hence, to minimize errors, all the numerical procedures that employ the ZCYC data should, as far as possible, directly employ the ZCYC data. One needs to avoid numerically differentiating $Z(t,T)$.

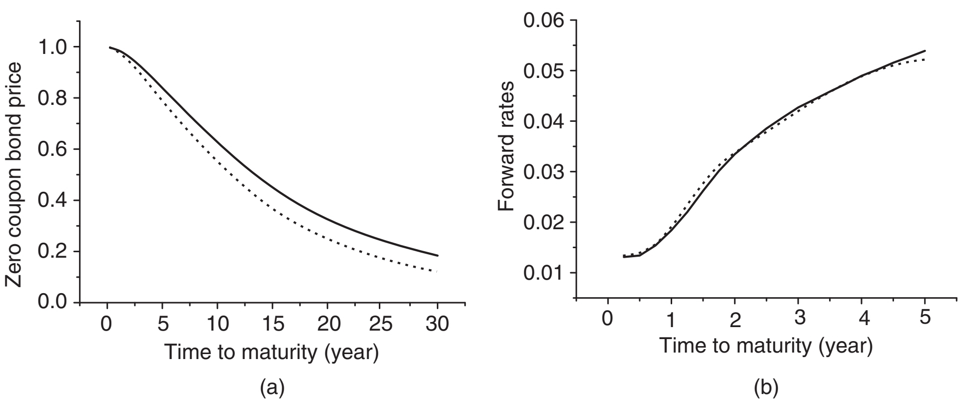


Figure 2.8 The ZCYC data are for $t = 29$ January 2003 and with $T-t$ up to 30 years into the future. (a) Zero coupon term structure $B(t,T)$ constructed from ZCYC (unbroken line) and from the forward interest rates (dotted line). (b) Forward interest rates with maturity time up to five years in the future, constructed from Libor $L(t,T)$ (dotted line) and from ZCYC $Z(t,T)$ (unbroken line).

Figure 2.8(b) shows the zero coupon term structure $B(t, T)$, obtained empirically from the ZCYC and from Libor. It is seen, as expected, that for relatively short remaining future time of $T - t \leq 5$ years the two curves give almost identical results. The ZCYC produce zero coupon term structure for maturities of up to 30 years in the future and one needs to generate the long duration zero coupon bond directly from the Libor ZCYC and not from market Libors, which are usually of a duration of up to ten years.

There is an empirical difference between Libor and Treasury Bonds. Libor has a finite probability of default, whereas Treasuries are risk free; the difference between these two rates is expressed by the TED (Treasury Eurodollar) spread. The difference is the spread between the Libor and forward interest rates derived from the ZCYC and is a measure of the risk of default of financial institutions that lend and borrow at Libor; in most cases, the spread is negligible and is ignored in all of the later discussions.

2.13 Summary

A brief review of finance shows the key role that the debt markets play in the capital markets and in the global economy. The changing nature of global capital markets and, in particular, the shift of the international capital markets to new centers were briefly discussed.

Given the growing importance of financial markets for the global economy, instabilities, such as the 2008 US financial crisis, need to be curtailed so that the financial system provides a stable environment for steady global economic growth. No country or region, no matter how large or 'important', should be allowed to hold the global economy to ransom. International financial instruments and regulations should address the current imbalances in the global capital markets. A fair, efficient and transparent financial system would mobilize currently untapped capital as well as release entrepreneurial energy that would be beneficial to all players – and to the world economy in general.

Some experts have declared that the 2008 financial crisis has sounded the death knell for financial engineering, which is said to have become irrelevant; such pronouncements are far from the truth. The importance of the capital markets, and in particular of the debt market, is indisputable; in particular, one can expect the global debt markets to play an increasingly important role in the international capital markets and in the world economy in general. Far from financial engineering being irrelevant, powerful quantitative financial models will continue to be indispensable in managing risk and maximizing returns on capital. Interest rate models of increasing sophistication will be required for designing and pricing ever-more complex debt instruments as well as for efficiently deploying a vast and expanding mountain of debt capital.

The chief component of the global capital markets is the debt market, which in turn consists mainly of the bond and interest rate markets. Bonds and interest rates are fundamental financial instruments of the debt market and reflect the time value of money. The different ways of defining interest rates and yields of bonds are the result of different ways of either discounting future cash flows or compounding present-day cash deposits or other tradable assets. The two ways of defining the future value of time lead to forward interest rates.

Coupon bond and forward interest rates and their derivatives will be discussed at length in the following chapters. Libor and Euribor were briefly discussed as these are the most important interest rate instruments, having the greatest liquidity and being the most widely traded. The three-month Libor and Euribor are taken to be the benchmark interest rates earned on cash deposits as these are the most relevant for the interest rate derivatives markets.

2.14 Appendix: De-noising financial data

All the values of financial instruments are influenced by background random noise. Consider for example the market value of a 90-day Libor $L(t, T_0)$ that matured at a fixed date of $T_0 = 16$ December 2003. The original data series on Libor is for the period from 14 June 2000 to 16 December 2003. The daily Libor is plotted from 14 June 2000 to 10 June 2002 in Figure 2.9. One can see the value of Libor is jagged (nondifferentiable) on a small time scale and regular on a long time scale.

It is assumed that Libor, and in general the price of all financial instruments, is composed of its true value, denoted by $s(t)$ and superimposed on it is noise, denoted by $w(t)$. In other words, one has [43]

$$L(t, T_0) = s(t) + w(t)$$

It is assumed that $w(t)$ is *white noise*, specified by the normal random variable given by $N(\mu, \sigma)$; at every instant, the smooth component of the market price, namely $s(t)$, has added to it a noise that is drawn from a normal (Gaussian) random variable. The random noise is assumed to be centered around the market price $s(t)$ and hence it is expected that $\mu = 0$. In other words, the observed random market price for Libor, based on the assumptions discussed, is given by

$$L(t, T_0) = s(t) \pm \sigma \text{ with } 66\% \text{ likelihood}$$

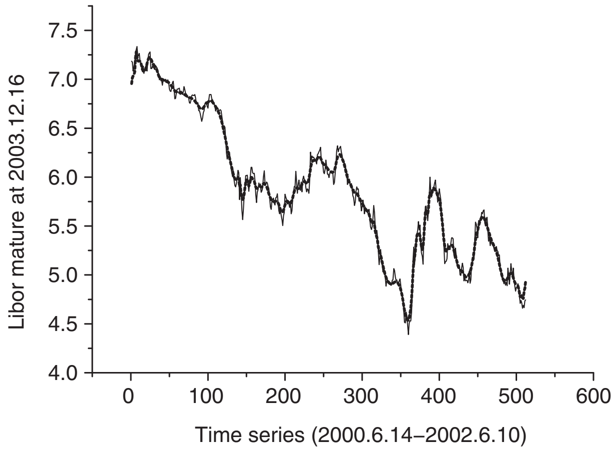


Figure 2.9 Original and de-noised Libor $L(t, T_0)$ maturing at *fixed time* in the future given by $T_0 = 16$ December 2003 and for the time period $t \in$ [14 June 2000, 10 June 2002].

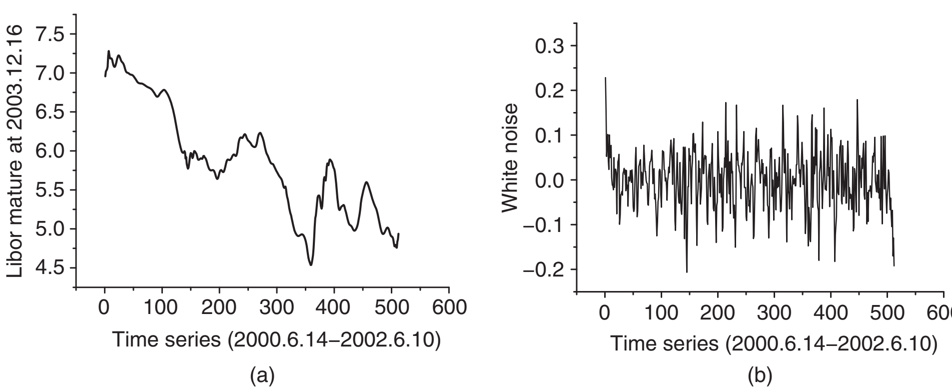


Figure 2.10 (a) The smooth component $s(t)$ of Libor $L(t, T_0)$. (b) Gaussian white noise $w(t) = N(\mu, \sigma)$ inherent in the market value of Libor $L(t, T_0)$, with $\mu = -1.4 \times 10^{-6}\%$ per year and $\sigma = 0.0629\%$ per year.

The behavior of the smooth portion $s(t)$ of Libor can have complicated dynamics and, in particular, is expected to be mean reverting.

De-noising consists of *subtracting*, at *each* instant, the white noise component from the market value of $L(t, T_0)$ and thus obtaining its smooth component, namely $s(t)$. For many purposes, it is the smooth component of the market value of a financial instrument that is required. One of the most efficient procedures is to use wavelet analysis to filter out white noise. There are many different 'basis' states that one can use to transform the market price to its smooth component and the Debauche wavelets D8 were used.

Figure 2.9 shows the original (jagged) swaption market price and the de-noised smooth curve as well. Figure 2.10(a) shows the de-noised swaption price $s(t)$ and Figure 2.10(b) shows the noise component $w(t)$. The distribution of white noise $w(t)$ is given by $N(-1.4 \times 10^{-6}, 0.0629)$, where the units for the parameters of the normal distribution are % per year.

Note that the typical value of Libor, as given in Figure 2.9, is of the order of 5%; the noise component is given by $\sigma = 0.06\%$, which is small – about 1% of the market price. This is what one expects since noise is supposed to be a small background component of the market price. Furthermore, $\mu = -1.4 \times 10^{-6}\%$ per year, which is completely neglible compared to the price of the daily value of Libor, hence confirming the assumption that the random noise is symmetrically distributed about the smooth curve $s(t)$.