International Series on Actuarial Science

Actuarial Mathematics for Life Contingent Risks

David C. M. Dickson, Mary R. Hardy and Howard R. Waters

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Actuarial Mathematics for Life Contingent Risks

How can actuaries best equip themselves for the products and risk structures of the future? In this new textbook, three leaders in actuarial science give a modern perspective on life contingencies.

The book begins traditionally, covering actuarial models and theory, and emphasizing practical applications using computational techniques. The authors then develop a more contemporary outlook, introducing multiple state models, emerging cash flows and embedded options. Using spreadsheet-style software, the book presents large-scale, realistic examples. Over 150 exercises and solutions teach skills in simulation and projection through computational practice.

Balancing rigour with intuition, and emphasizing applications, this textbook is ideal not only for university courses, but also for individuals preparing for professional actuarial examinations and qualified actuaries wishing to renew and update their skills.

International Series on Actuarial Science

Christopher Daykin, Independent Consultant and Actuary
Angus Macdonald, Heriot-Watt University

The *International Series on Actuarial Science*, published by Cambridge University Press in conjunction with the Institute of Actuaries and the Faculty of Actuaries, contains textbooks for students taking courses in or related to actuarial science, as well as more advanced works designed for continuing professional development or for describing and synthesizing research. The series is a vehicle for publishing books that reflect changes and developments in the curriculum, that encourage the introduction of courses on actuarial science in universities, and that show how actuarial science can be used in all areas where there is long-term financial risk.

ACTUARIAL MATHEMATICS FOR LIFE CONTINGENT RISKS

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521118255

First published in print format 2009

ISBN-13 978-0-511-62965-5 eBook (Adobe Reader)

ISBN-13 978-0-521-11825-5 Hardback

To
Carolann,
Vivien
and Phelim

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Preface

Life insurance has undergone enormous change in the last two to three decades. New and innovative products have been developed at the same time as we have seen vast increases in computational power. In addition, the field of finance has experienced a revolution in the development of a mathematical theory of options and financial guarantees, first pioneered in the work of Black, Scholes and Merton, and actuaries have come to realize the importance of that work to risk management in actuarial contexts.

Given the changes occurring in the interconnected worlds of finance and life insurance, we believe that this is a good time to recast the mathematics of life contingent risk to be better adapted to the products, science and technology that are relevant to current and future actuaries.

In this book we have developed the theory to measure and manage risks that are contingent on demographic experience as well as on financial variables. The material is presented with a certain level of mathematical rigour; we intend for readers to understand the principles involved, rather than to memorize methods or formulae. The reason is that a rigorous approach will prove more useful in the long run than a short-term utilitarian outlook, as theory can be adapted to changing products and technology in ways that techniques, without scientific support, cannot.

We start from a traditional approach, and then develop a more contemporary perspective. The first seven chapters set the context for the material, and cover traditional actuarial models and theory of life contingencies, with modern computational techniques integrated throughout, and with an emphasis on the practical context for the survival models and valuation methods presented. Through the focus on realistic contracts and assumptions, we aim to foster a general business awareness in the life insurance context, at the same time as we develop the mathematical tools for risk management in that context.

In Chapter 8 we introduce multiple state models, which generalize the life–death contingency structure of previous chapters. Using multiple state models allows a single framework for a wide range of insurance, including benefits which depend on health status, on cause of death benefits, or on two or more lives.

In Chapter 9 we apply the theory developed in the earlier chapters to problems involving pension benefits. Pension mathematics has some specialized concepts, particularly in funding principles, but in general this chapter is an application of the theory in the preceding chapters.

In Chapter 10 we move to a more sophisticated view of interest rate models and interest rate risk. In this chapter we explore the crucially important difference between diversifiable and non-diversifiable risk. Investment risk represents a source of non-diversifiable risk, and in this chapter we show how we can reduce the risk by matching cash flows from assets and liabilities.

In Chapter 11 we continue the cash flow approach, developing the emerging cash flows for traditional insurance products. One of the liberating aspects of the computer revolution for actuaries is that we are no longer required to summarize complex benefits in a single actuarial value; we can go much further in projecting the cash flows to see how and when surplus will emerge. This is much richer information that the actuary can use to assess profitability and to better manage portfolio assets and liabilities.

In Chapter 12 we repeat the emerging cash flow approach, but here we look at equity-linked contracts, where a financial guarantee is commonly part of the contingent benefit. The real risks for such products can only be assessed taking the random variation in potential outcomes into consideration, and we demonstrate this with Monte Carlo simulation of the emerging cash flows.

The products that are explored in Chapter 12 contain financial guarantees embedded in the life contingent benefits. Option theory is the mathematics of valuation and risk management of financial guarantees. In Chapter 13 we introduce the fundamental assumptions and results of option theory.

In Chapter 14 we apply option theory to the embedded options of financial guarantees in insurance products. The theory can be used for pricing and for determining appropriate reserves, as well as for assessing profitability.

The material in this book is designed for undergraduate and graduate programmes in actuarial science, and for those self-studying for professional actuarial exams. Students should have sufficient background in probability to be able to calculate moments of functions of one or two random variables, and to handle conditional expectations and variances. We also assume familiarity with the binomial, uniform, exponential, normal and lognormal distributions. Some of the more important results are reviewed in Appendix A. We also assume that readers have completed an introductory level course in the mathematics of finance, and are aware of the actuarial notation for annuities-certain.

Throughout, we have opted to use examples that liberally call on spreadsheet-style software. Spreadsheets are ubiquitous tools in actuarial practice, and it is natural to use them throughout, allowing us to use more realistic examples, rather than having to simplify for the sake of mathematical tractability. Other software could be used equally effectively, but spreadsheets represent a fairly universal language that is easily accessible. To keep the computation requirements reasonable, we have ensured that every example and exercise can be completed in Microsoft Excel, without needing any VBA code or macros. Readers who have sufficient familiarity to write their own code may find more efficient solutions than those that we have presented, but our principle was that no reader should need to know more than the basic Excel functions and applications. It will be very useful for anyone working through the material of this book to construct their own spreadsheet tables as they work through the first seven chapters, to generate mortality and actuarial functions for a range of mortality models and interest rates. In the worked examples in the text, we have worked with greater accuracy than we record, so there will be some differences from rounding when working with intermediate figures.

One of the advantages of spreadsheets is the ease of implementation of numerical integration algorithms. We assume that students are aware of the principles of numerical integration, and we give some of the most useful algorithms in Appendix B.

The material in this book is appropriate for two one-semester courses. The first seven chapters form a fairly traditional basis, and would reasonably constitute a first course. Chapters 8–14 introduce more contemporary material. Chapter 13 may be omitted by readers who have studied an introductory course covering pricing and delta hedging in a Black–Scholes–Merton model. Chapter 9, on pension mathematics, is not required for subsequent chapters, and could be omitted if a single focus on life insurance is preferred.

Acknowledgements

Many of our students and colleagues have made valuable comments on earlier drafts of parts of the book. Particular thanks go to Carole Bernard, Phelim Boyle, Johnny Li, Ana Maria Mera, Kok Keng Siaw and Matthew Till.

The authors gratefully acknowledge the contribution of the Departments of Statistics and Actuarial Science, University of Waterloo, and Actuarial Mathematics and Statistics, Heriot-Watt University, in welcoming the non-resident

authors for short visits to work on this book. These visits significantly shortened the time it has taken to write the book (to only one year beyond the original deadline).

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1

Introduction to life insurance

1.1 Summary

Actuaries apply scientific principles and techniques from a range of other disciplines to problems involving risk, uncertainty and finance. In this chapter we set the context for the mathematics of later chapters, by describing some of the background to modern actuarial practice in life insurance, followed by a brief description of the major types of life insurance products that are sold in developed insurance markets. Because pension liabilities are similar in many ways to life insurance liabilities, we also describe some common pension benefits. We give examples of the actuarial questions arising from the risk management of these contracts. How to answer such questions, and solve the resulting problems, is the subject of the following chapters.

1.2 Background

The first actuaries were employed by life insurance companies in the early eighteenth century to provide a scientific basis for managing the companies' assets and liabilities. The liabilities depended on the number of deaths occurring amongst the insured lives each year. The modelling of mortality became a topic of both commercial and general scientific interest, and it attracted many significant scientists and mathematicians to actuarial problems, with the result that much of the early work in the field of probability was closely connected with the development of solutions to actuarial problems.

The earliest life insurance policies provided that the policyholder would pay an amount, called the **premium**, to the insurer. If the named life insured died during the year that the contract was in force, the insurer would pay a predetermined lump sum, the **sum insured**, to the policyholder or his or her estate. So, the first life insurance contracts were annual contracts. Each year the premium would increase as the probability of death increased. If the insured life became very ill at the renewal date, the insurance might not be renewed, in which case no benefit would be paid on the life's subsequent death. Over a large number of contracts, the premium income each year should approximately match the claims outgo. This method of matching income and outgo annually, with no attempt to smooth or balance the premiums over the years, is called **assessmentism**. This method is still used for group life insurance, where an employer purchases life insurance cover for its employees on a year-to-year basis.

The radical development in the later eighteenth century was the level premium contract. The problem with assessmentism was that the annual increases in premiums discouraged policyholders from renewing their contracts. The level premium policy offered the policyholder the option to lock-in a regular premium, payable perhaps weekly, monthly, quarterly or annually, for a number of years. This was much more popular with policyholders, as they would not be priced out of the insurance contract just when it might be most needed. For the insurer, the attraction of the longer contract was a greater likelihood of the policyholder paying premiums for a longer period. However, a problem for the insurer was that the longer contracts were more complex to model, and offered more financial risk. For these contracts then, actuarial techniques had to develop beyond the year-to-year modelling of mortality probabilities. In particular, it became necessary to incorporate financial considerations into the modelling of income and outgo. Over a one-year contract, the time value of money is not a critical aspect. Over, say, a 30-year contract, it becomes a very important part of the modelling and management of risk.

Another development in life insurance in the nineteenth century was the concept of **insurable interest**. This was a requirement in law that the person contracting to pay the life insurance premiums should face a financial loss on the death of the insured life that was no less than the sum insured under the policy. The insurable interest requirement disallowed the use of insurance as a form of gambling on the lives of public figures, but more importantly, removed the incentive for a policyholder to hasten the death of the named insured life. Subsequently, insurance policies tended to be purchased by the insured life, and in the rest of this book we use the convention that the policyholder who pays the premiums is also the life insured, whose survival or death triggers the payment of the sum insured under the conditions of the contract.

The earliest studies of mortality include life tables constructed by John Graunt and Edmund Halley. A life table summarizes a survival model by specifying the proportion of lives that are expected to survive to each age. Using London mortality data from the early seventeenth century, Graunt proposed, for example, that each new life had a probability of 40% of surviving to age 16, and a probability of 1% of surviving to age 76. Edmund Halley, famous for his astronomical calculations, used mortality data from the city of Breslau in the late seventeenth century as the basis for his life table, which, like Graunt's, was constructed by proposing the average ('medium' in Halley's phrase) proportion of survivors to each age from an arbitrary number of births. Halley took the work two steps further. First, he used the table to draw inference about the conditional survival probabilities at intermediate ages. That is, given the probability that a newborn life survives to each subsequent age, it is possible to infer the probability that a life aged, say, 20, will survive to each subsequent age, using the condition that a life aged zero survives to age 20. The second major innovation was that Halley combined the mortality data with an assumption about interest rates to find the value of a whole life annuity at different ages. A whole life annuity is a contract paying a level sum at regular intervals while the named life (the annuitant) is still alive. The calculations in Halley's paper bear a remarkable similarity to some of the work still used by actuaries in pensions and life insurance.

This book continues in the tradition of combining models of mortality with models in finance to develop a framework for pricing and risk management of long-term policies in life insurance. Many of the same techniques are relevant also in pensions mathematics. However, there have been many changes since the first long-term policies of the late eighteenth century.

1.3 Life insurance and annuity contracts

1.3.1 Introduction

The life insurance and annuity contracts that were the object of study of the early actuaries were very similar to the contracts written up to the 1980s in all the developed insurance markets. Recently, however, the design of life insurance products has radically changed, and the techniques needed to manage these more modern contracts are more complex than ever. The reasons for the changes include:

- Increased interest by the insurers in offering combined savings and insurance products. The original life insurance products offered a payment to indemnify (or offset) the hardship caused by the death of the policyholder. Many modern contracts combine the indemnity concept with an opportunity to invest.
- More powerful computational facilities allow more complex products to be modelled.
- Policyholders have become more sophisticated investors, and require more options in their contracts, allowing them to vary premiums or sums insured, for example.
- More competition has led to insurers creating increasingly complex products in order to attract more business.
- The risk management techniques in financial products have also become increasingly complex, and insurers have offered some benefits, particularly

financial guarantees, that require sophisticated techniques from financial engineering to measure and manage the risk.

In the remainder of this section we describe some of the most important modern insurance contracts, which will later be used as examples in the book. Different countries have different names and types of contracts; we have tried to cover the major contract types in North America, the United Kingdom and Australia.

The basic transaction of life insurance is an exchange; the policyholder pays premiums in return for a later payment from the insurer which is life contingent, by which we mean that it depends on the death or survival or possibly the state of health of the policyholder. We usually use the term '**insurance**' when the benefit is paid as a single lump sum, either on the death of the policyholder or on survival to a predetermined **maturity date**. (In the UK it is common to use the term 'assurance' for insurance contracts involving lives, and insurance for contracts involving property.) An **annuity** is a benefit in the form of a regular series of payments, usually conditional on the survival of the policyholder.

1.3.2 Traditional insurance contracts

Term, whole life and endowment insurance are the traditional products, providing cash benefits on death or maturity, usually with predetermined premium and benefit amounts. We describe each in a little more detail here.

Term insurance pays a lump sum benefit on the death of the policyholder, provided death occurs before the end of a specified term. Term insurance allows a policyholder to provide a fixed sum for his or her dependents in the event of the policyholder's death.

Level term insurance indicates a level sum insured and regular, level premiums.

Decreasing term insurance indicates that the sum insured and (usually) premiums decrease over the term of the contract. Decreasing term insurance is popular in the UK where it is used in conjunction with a home mortgage; if the policyholder dies, the remaining mortgage is paid from the term insurance proceeds.

Renewable term insurance offers the policyholder the option of renewing the policy at the end of the original term, without further evidence of the policyholder's health status. In North America, Yearly Renewable Term (YRT) insurance is common, under which insurability is guaranteed for some fixed period, though the contract is written only for one year at a time.

Convertible term insurance offers the policyholder the option to convert to a whole life or endowment insurance at the end of the original term, without further evidence of the policyholder's health status.

Whole life insurance pays a lump sum benefit on the death of the policyholder whenever it occurs. For regular premium contracts, the premium is often payable only up to some maximum age, such as 80. This avoids the problem that older lives may be less able to pay the premiums.

Endowment insurance offers a lump sum benefit paid either on the death of the policyholder or at the end of a specified term, whichever occurs first. This is a mixture of a term insurance benefit and a savings element. If the policyholder dies, the sum insured is paid just as under term insurance; if the policyholder survives, the sum insured is treated as a maturing investment. Endowment insurance is obsolete in many jurisdictions. Traditional endowment insurance policies are not currently sold in the UK, but there are large portfolios of policies on the books of UK insurers, because until the late 1990s, endowment insurance policies were often used to repay home mortgages. The policyholder (who is the home owner) paid interest on the mortgage loan, and the principal was paid from the proceeds on the endowment insurance, either on the death of the policyholder or at the final mortgage repayment date.

Endowment insurance policies are becoming popular in developing nations, particularly for 'micro-insurance' where the amounts involved are small. It is hard for small investors to achieve good rates of return on investments, because of heavy expense charges. By pooling the death and survival benefits under the endowment contract, the policyholder gains on the investment side from the resulting economies of scale, and from the investment expertise of the insurer.

With-profit insurance

Also part of the traditional design of insurance is the division of business into 'with-profit' (also known, especially in North America, as 'participating', or 'par' business), and 'without profit' (also known as 'non-participating' or 'non-par'). Under with-profit arrangements, the profits earned on the invested premiums are shared with the policyholders. In North America, the with-profit arrangement often takes the form of cash dividends or reduced premiums. In the UK and in Australia the traditional approach is to use the profits to increase the sum insured, through bonuses called '**reversionary bonuses**' and '**terminal bonuses**'. Reversionary bonuses are awarded during the term of the contract; once a reversionary bonus is awarded it is guaranteed. Terminal bonuses are awarded when the policy matures, either through the death of the insured, or when an endowment policy reaches the end of the term. Reversionary bonuses

Table 1.1.

Year	Bonus on original sum insured	Bonus on bonus	Total bonus
1	2%	5%	2000.00
2	2.5%	6%	4620.00
3	2.5%	6%	7397.20
⋮	⋮	⋮	⋮

may be expressed as a percentage of the total of the previous sum insured plus bonus, or as a percentage of the original sum insured plus a different percentage of the previously declared bonuses. Reversionary and terminal bonuses are determined by the insurer based on the investment performance of the invested premiums.

For example, suppose an insurance is issued with sum insured \$100 000. At the end of the first year of the contract a bonus of 2% on the sum insured and 5% on previous bonuses is declared; in the following two years, the rates are 2.5% and 6%. Then the total guaranteed sum insured increases each year as shown in Table 1.1.

If the policyholder dies, the total death benefit payable would be the original sum insured plus reversionary bonuses already declared, increased by a terminal bonus if the investment returns earned on the premiums have been sufficient.

With-profits contracts may be used to offer policyholders a savings element with their life insurance. However, the traditional with-profit contract is designed primarily for the life insurance cover, with the savings aspect a secondary feature.

1.3.3 Modern insurance contracts

In recent years insurers have provided more flexible products that combine the death benefit coverage with a significant investment element, as a way of competing for policyholders' savings with other institutions, for example, banks or open-ended investment companies (e.g. mutual funds in North America, or unit trusts in the UK). Additional flexibility also allows policyholders to purchase less insurance when their finances are tight, and then increase the insurance coverage when they have more money available.

In this section we describe some examples of modern, flexible insurance contracts.

Universal life insurance combines investment and life insurance. The policyholder determines a premium and a level of life insurance cover. Some of the premium is used to fund the life insurance; the remainder is paid into an investment fund. Premiums are flexible, as long as they are sufficient to pay for the designated sum insured under the term insurance part of the contract. Under variable universal life, there is a range of funds available for the policyholder to select from. Universal life is a common insurance contract in North America.

Unitized with-profit is a UK insurance contract; it is an evolution from the conventional with-profit policy, designed to be more transparent than the original. Premiums are used to purchase units (shares) of an investment fund, called the with-profit fund. As the fund earns investment return, the shares increase in value (or more shares are issued), increasing the benefit entitlement as reversionary bonus. The shares will not decrease in value. On death or maturity, a further terminal bonus may be payable depending on the performance of the with-profit fund.

After some poor publicity surrounding with-profit business, and, by association, unitized with-profit business, these product designs were withdrawn from the UK and Australian markets by the early 2000s. However, they will remain important for many years as many companies carry very large portfolios of with-profit (traditional and unitized) policies issued during the second half of the twentieth century.

Equity-linked insurance has a benefit linked to the performance of an investment fund. There are two different forms. The first is where the policyholder's premiums are invested in an open-ended investment company style account; at maturity, the benefit is the accumulated value of the premiums. There is a guaranteed minimum death benefit payable if the policyholder dies before the contract matures. In some cases, there is also a guaranteed minimum maturity benefit payable. In the UK and most of Europe, these are called **unit-linked** policies, and they rarely carry a guaranteed maturity benefit. In Canada they are known as **segregated fund** policies and always carry a maturity guarantee. In the USA these contracts are called **variable annuity** contracts; maturity guarantees are increasingly common for these policies. (The use of the term 'annuity' for these contracts is very misleading. The benefits are designed with a single lump sum payout, though there may be an option to convert the lump sum to an annuity.)

The second form of equity-linked insurance is the **Equity-Indexed Annuity** (EIA) in the USA. Under an EIA the policyholder is guaranteed a minimum return on their premium (minus an initial expense charge). At maturity, the policyholder receives a proportion of the return on a specified stock index, if that is greater than the guaranteed minimum return.

EIAs are generally rather shorter in term than unit-linked products, with seven-year policies being typical; variable annuity contracts commonly have terms of twenty years or more. EIAs are much less popular with consumers than variable annuities.

1.3.4 Distribution methods

Most people find insurance dauntingly complex. Brokers who connect individuals to an appropriate insurance product have, since the earliest times, played an important role in the market. There is an old saying amongst actuaries that 'insurance is sold, not bought', which means that the role of an intermediary in persuading potential policyholders to take out an insurance policy is crucial in maintaining an adequate volume of new business.

Brokers, or other financial advisors, are often remunerated through a **commission system**. The commission would be specified as a percentage of the premium paid. Typically, there is a higher percentage paid on the first premium than on subsequent premiums. This is referred to as a **front-end load**. Some advisors may be remunerated on a fixed fee basis, or may be employed by one or more insurance companies on a salary basis.

An alternative to the broker method of selling insurance is **direct marketing**. Insurers may use television advertising or other telemarketing methods to sell direct to the public. The nature of the business sold by direct marketing methods tends to differ from the broker sold business. For example, often the sum insured is smaller. The policy may be aimed at a niche market, such as older lives concerned with insurance to cover their own funeral expenses (called pre-need insurance in the USA). Another mass marketed insurance contract is loan or credit insurance, where an insurer might cover loan or credit card payments in the event of the borrower's death, disability or unemployment.

1.3.5 Underwriting

It is important in modelling life insurance liabilities to consider what happens when a life insurance policy is purchased. Selling life insurance policies is a competitive business and life insurance companies (also known as life offices) are constantly considering ways in which to change their procedures so that they can improve the service to their customers and gain a commercial advantage over their competitors. The account given below of how policies are sold covers some essential points but is necessarily a simplified version of what actually happens.

For a given type of policy, say a 10-year term insurance, the life office will have a schedule of premium rates. These rates will depend on the size of the policy and some other factors known as **rating factors**. An applicant's risk level is assessed by asking them to complete a **proposal form** giving information on relevant rating factors, generally including their age, gender, smoking habits, occupation, any dangerous hobbies, and personal and family health history. The life insurer may ask for permission to contact the applicant's doctor to enquire about their medical history. In some cases, particularly for very large sums insured, the life insurer may require that the applicant's health be checked by a doctor employed by the insurer.

The process of collecting and evaluating this information is called **underwriting**. The purpose of underwriting is, first, to classify potential policyholders into broadly homogeneous risk categories, and secondly to assess what additional premium would be appropriate for applicants whose risk factors indicate that standard premium rates would be too low.

On the basis of the application and supporting medical information, potential life insurance policyholders will generally be categorized into one of the following groups:

- **Preferred lives** have very low mortality risk based on the standard information. The preferred applicant would have no recent record of smoking; no evidence of drug or alcohol abuse; no high-risk hobbies or occupations; no family history of disease known to have a strong genetic component; no adverse medical indicators such as high blood pressure or cholesterol level or body mass index.

  The preferred life category is common in North America, but has not yet caught on elsewhere. In other areas there is no separation of preferred and normal lives.
- **Normal lives** may have some higher rated risk factors than preferred lives (where this category exists), but are still insurable at standard rates. Most applicants fall into this category.
- **Rated lives** have one or more risk factors at raised levels and so are not acceptable at standard premium rates. However, they can be insured for a higher premium. An example might be someone having a family history of heart disease. These lives might be individually assessed for the appropriate additional premium to be charged. This category would also include lives with hazardous jobs or hobbies which put them at increased risk.
- **Uninsurable lives** have such significant risk that the insurer will not enter an insurance contract at any price.

Within the first three groups, applicants would be further categorized according to the relative values of the various risk factors, with the most fundamental being age, gender and smoking status. Most applicants (around 95% for traditional life insurance) will be accepted at preferred or standard rates for the relevant risk category. Another 2–3% may be accepted at non-standard rates because of an impairment, or a dangerous occupation, leaving around 2–3% who will be refused insurance.

The rigour of the underwriting process will depend on the type of insurance being purchased, on the sum insured and on the distribution process of the insurance company. Term insurance is generally more strictly underwritten than whole life insurance, as the risk taken by the insurer is greater. Under whole life insurance, the payment of the sum insured is certain, the uncertainty is in the timing. Under, say, 10-year term insurance, it is assumed that the majority of contracts will expire with no death benefit paid. If the underwriting is not strict there is a risk of **adverse selection** by policyholders – that is, that very high-risk individuals will buy insurance in disproportionate numbers, leading to excessive losses. Since high sum insured contracts carry more risk than low sum insured, high sums insured would generally trigger more rigorous underwriting.

The marketing method also affects the level of underwriting. Often, direct marketed contracts are sold with relatively low benefit levels, and with the attraction that no medical evidence will be sought beyond a standard questionnaire. The insurer may assume relatively heavy mortality for these lives to compensate for potential adverse selection. By keeping the underwriting relatively light, the expenses of writing new business can be kept low, which is an attraction for high-volume, low sum insured contracts.

It is interesting to note that with no third party medical evidence the insurer is placing a lot of weight on the veracity of the policyholder. Insurers have a phrase for this – that both insurer and policyholder may assume 'utmost good faith' or '*uberrima fides*' on the part of the other side of the contract. In practice, in the event of the death of the insured life, the insurer may investigate whether any pertinent information was withheld from the application. If it appears that the policyholder held back information, or submitted false or misleading information, the insurer may not pay the full sum insured.

1.3.6 Premiums

A life insurance policy may involve a single premium, payable at the outset of the contract, or a regular series of premiums payable provided the policyholder survives, perhaps with a fixed end date. In traditional contracts the regular premium is generally a level amount throughout the term of the contract; in more modern contracts the premium might be variable, at the policyholder's discretion for investment products such as equity-linked insurance, or at the insurer's discretion for certain types of term insurance.

Regular premiums may be paid annually, semi-annually, quarterly, monthly or weekly. Monthly premiums are common as it is convenient for policyholders to have their outgoings payable with approximately the same frequency as their income.

An important feature of all premiums is that they are paid at the start of each period. Suppose a policyholder contracts to pay annual premiums for a 10-year insurance contract. The premiums will be paid at the start of the contract, and then at the start of each subsequent year provided the policyholder is alive. So, if we count time in years from $t = 0$ at the start of the contract, the first premium is paid at $t = 0$, the second is paid at $t = 1$, and so on, to the tenth premium paid at $t = 9$. Similarly, if the premiums are monthly, then the first monthly instalment will be paid at $t = 0$, and the final premium will be paid at the start of the final month at $t = 9\frac{11}{12}$ years. (Throughout this book we assume that all months are equal in length, at $\frac{1}{12}$ years.)

1.3.7 Life annuities

Annuity contracts offer a regular series of payments. When an annuity depends on the survival of the recipient, it is called a 'life annuity'. The recipient is called an annuitant. If the annuity continues until the death of the annuitant, it is called a **whole life annuity**. If the annuity is paid for some maximum period, provided the annuitant survives that period, it is called a **term life annuity**.

Annuities are often purchased by older lives to provide income in retirement. Buying a whole life annuity guarantees that the income will not run out before the annuitant dies.

Single Premium Deferred Annuity (SPDA) Under an SPDA contract, the policyholder pays a single premium in return for an annuity which commences payment at some future, specified date. The annuity is 'life contingent', by which we mean the annuity is paid only if the policyholder survives to the payment dates. If the policyholder dies before the annuity commences, there may be a death benefit due. If the policyholder dies soon after the annuity commences, there may be some minimum payment period, called the guarantee period, and the balance would be paid to the policyholder's estate.

Single Premium Immediate Annuity (SPIA) This contract is the same as the SPDA, except that the annuity commences as soon as the contract is effected. This might, for example, be used to convert a lump sum retirement benefit into a life annuity to supplement a pension. As with the SPDA, there may be a guarantee period applying in the event of the early death of the annuitant.

Regular Premium Deferred Annuity (RPDA) The RPDA offers a deferred life annuity with premiums paid through the deferred period. It is otherwise the same as the SPDA.

Joint life annuity A joint life annuity is issued on two lives, typically a married couple. The annuity (which may be single premium or regular premium, immediate or deferred) continues while both lives survive, and ceases on the first death of the couple.

Last survivor annuity A last survivor annuity is similar to the joint life annuity, except that payment continues while at least one of the lives survives, and ceases on the second death of the couple.

Reversionary annuity A reversionary annuity is contingent on two lives, usually a couple. One is designated as the annuitant, and one the insured. No annuity benefit is paid while the insured life survives. On the death of the insured life, if the annuitant is still alive, the annuitant receives an annuity for the remainder of his or her life.

1.4 Other insurance contracts

The insurance and annuity contracts described above are all contingent on death or survival. There are other life contingent risks, in particular involving short-term or long-term disability. These are known as morbidity risks.

Income protection insurance When a person becomes sick and cannot work, their income will, eventually, be affected. For someone in regular employment, the employer may cover salary for a period, but if the sickness continues the salary will be decreased, and ultimately will stop being paid at all. For someone who is self-employed, the effects of sickness on income will be immediate. Income protection policies replace at least some income during periods of sickness. They usually cease at retirement age.

Critical illness insurance Some serious illnesses can cause significant expense at the onset of the illness. The patient may have to leave employment, or alter their home, or incur severe medical expenses. Critical illness insurance pays a benefit on diagnosis of one of a number of severe conditions, such as certain cancers or heart disease. The benefit is usually in the form of a lump sum.

Long-term care insurance This is purchased to cover the costs of care in old age, when the insured life is unable to continue living independently. The benefit would be in the form of the long-term care costs, so is an annuity benefit.

1.5 Pension benefits

Many actuaries work in the area of pension plan design, valuation and risk management. The pension plan is usually sponsored by an employer. Pension plans typically offer employees (also called pension plan members) either lump sums or annuity benefits or both on retirement, or deferred lump sum or annuity benefits (or both) on earlier withdrawal. Some offer a lump sum benefit if the employee dies while still employed. The benefits therefore depend on the survival and employment status of the member, and are quite similar in nature to life insurance benefits – that is, they involve investment of contributions long into the future to pay for future life contingent benefits.

1.5.1 Defined benefit and defined contribution pensions

Defined Benefit (DB) pensions offer retirement income based on service and salary with an employer, using a defined formula to determine the pension. For example, suppose an employee reaches retirement age with n years of service (i.e. membership of the pension plan), and with pensionable salary averaging S in, say, the final three years of employment. A typical **final salary** plan might offer an annual pension at retirement of $B = Sn\alpha$, where α is called the **accrual rate**, and is usually around 1%–2%. The formula may be interpreted as a pension benefit of, say, 2% of the final average salary for each year of service.

The defined benefit is funded by contributions paid by the employer and (usually) the employee over the working lifetime of the employee. The contributions are invested, and the accumulated contributions must be enough, on average, to pay the pensions when they become due.

Defined Contribution (DC) pensions work more like a bank account. The employee and employer pay a predetermined contribution (usually a fixed percentage of salary) into a fund, and the fund earns interest. When the employee leaves or retires, the proceeds are available to provide income throughout retirement. In the UK most of the proceeds must be converted to an annuity. In the USA and Canada there are more options – the pensioner may draw funds to live on without necessarily purchasing an annuity from an insurance company.

1.5.2 Defined benefit pension design

The **age retirement pension** described in the section above defines the pension payable from retirement in a standard final salary plan. **Career average salary** plans are also common in some jurisdictions, where the benefit formula is the same as the final salary formula above, except that the average salary over the employee's entire career is used in place of the final salary.

Many employees leave their jobs before they retire. A typical **withdrawal benefit** would be a pension based on the same formula as the age retirement benefit, but with the start date deferred until the employee reaches the normal retirement age. Employees may have the option of taking a lump sum with the same value as the deferred pension, which can be invested in the pension plan of the new employer.

Some pension plans also offer **death-in-service** benefits, for employees who die during their period of employment. Such benefits might include a lump sum, often based on salary and sometimes service, as well as a pension for the employee's spouse.

1.6 Mutual and proprietary insurers

A **mutual** insurance company is one that has no shareholders. The insurer is owned by the with-profit policyholders. All profits are distributed to the with-profit policyholders through dividends or bonuses.

A **proprietary** insurance company has shareholders, and usually has with-profit policyholders as well. The participating policyholders are not owners, but have a specified right to some of the profits. Thus, in a proprietary insurer, the profits must be shared in some predetermined proportion, between the shareholders and the with-profit policyholders.

Many early life insurance companies were formed as mutual companies. More recently, in the UK, Canada and the USA, there has been a trend towards demutualization, which means the transition of a mutual company to a proprietary company, through issuing shares (or cash) to the with-profit policyholders. Although it would appear that a mutual insurer would have marketing advantages, as participating policyholders receive all the profits and other benefits of ownership, the advantages cited by companies who have demutualized include increased ability to raise capital, clearer corporate structure and improved efficiency.

1.7 Typical problems

We are concerned in this book with developing the mathematical models and techniques used by actuaries working in life insurance and pensions. The primary responsibility of the life insurance actuary is to maintain the solvency and profitability of the insurer. Premiums must be sufficient to pay benefits; the assets held must be sufficient to pay the contingent liabilities; bonuses to policyholders should be fair.

Consider, for example, a whole life insurance contract issued to a life aged 50. The sum insured may not be paid for 30 years or more. The premiums paid over the period will be invested by the insurer to earn significant interest; the accumulated premiums must be sufficient to pay the benefits, on average. To ensure this, the actuary needs to model the survival probabilities of the policyholder, the investment returns likely to be earned and the expenses likely to be incurred in maintaining the policy. The actuary may take into consideration the probability that the policyholder decides to terminate the contract early. The actuary may also consider the profitability requirements for the contract. Then, when all of these factors have been modelled, they must be combined to set a premium.

Each year or so, the actuary must determine how much money the insurer or pension plan should hold to ensure that future liabilities will be covered with adequately high probability. This is called the valuation process. For with-profit insurance, the actuary must determine a suitable level of bonus.

The problems are rather more complex if the insurance also covers morbidity risk, or involves several lives. All of these topics are covered in the following chapters.

The actuary may also be involved in decisions about how the premiums are invested. It is vitally important that the insurer remains solvent, as the contracts are very long-term and insurers are responsible for protecting the financial security of the general public. The way the underlying investments are selected can increase or mitigate the risk of insolvency. The precise selection of investments to manage the risk is particularly important where the contracts involve financial guarantees.

The pensions actuary working with defined benefit pensions must determine appropriate contribution rates to meet the benefits promised, using models that allow for the working patterns of the employees. Sometimes, the employer may want to change the benefit structure, and the actuary is responsible for assessing the cost and impact. When one company with a pension plan takes over another, the actuary must assist with determining the best way to allocate the assets from the two plans, and perhaps how to merge the benefits.

1.8 Notes and further reading

A number of essays describing actuarial practice can be found in Renn (ed.) (1998). This book also provides both historical and more contemporary contexts for life contingencies.

The original papers of Graunt and Halley are available online (and any search engine will find them). Anyone interested in the history of probability and actuarial science will find these interesting, and remarkably modern.

1.9 Exercises

Exercise 1.1 Why do insurers generally require evidence of health from a person applying for life insurance but not for an annuity?

Exercise 1.2 Explain why an insurer might demand more rigorous evidence of a prospective policyholder's health status for a term insurance than for a whole life insurance.

Exercise 1.3 Explain why premiums are payable in advance, so that the first premium is due now rather than in one year's time.

Exercise 1.4 Lenders offering mortgages to home owners may require the borrower to purchase life insurance to cover the outstanding loan on the death of the borrower, even though the mortgaged property is the loan collateral.

(a) Explain why the lender might require term insurance in this circumstance.
(b) Describe how this term insurance might differ from the standard term insurance described in Section 1.3.2.
(c) Can you see any problems with lenders demanding term insurance from borrowers?

Exercise 1.5 Describe the difference between a cash bonus and a reversionary bonus for with-profit whole life insurance. What are the advantages and disadvantages of each for (a) the insurer and (b) the policyholder?

Exercise 1.6 It is common for insurers to design whole life contracts with premiums payable only up to age 80. Why?

Exercise 1.7 Andrew is retired. He has no pension, but has capital of \$500 000. He is considering the following options for using the money:

(a) Purchase an annuity from an insurance company that will pay a level amount for the rest of his life.
(b) Purchase an annuity from an insurance company that will pay an amount that increases with the cost of living for the rest of his life.
(c) Purchase a 20-year annuity certain.
(d) Invest the capital and live on the interest income.
(e) Invest the capital and draw \$40 000 per year to live on.

What are the advantages and disadvantages of each option?

2

Survival models

2.1 Summary

In this chapter we represent the future lifetime of an individual as a random variable, and show how probabilities of death or survival can be calculated under this framework. We then define an important quantity known as the force of mortality, introduce some actuarial notation, and discuss some properties of the distribution of future lifetime. We introduce the curtate future lifetime random variable. This is a function of the future lifetime random variable which represents the number of complete years of future life. We explain why this function is useful and derive its probability function.

2.2 The future lifetime random variable

In Chapter 1 we saw that many insurance policies provide a benefit on the death of the policyholder. When an insurance company issues such a policy, the policyholder's date of death is unknown, so the insurer does not know exactly when the death benefit will be payable. In order to estimate the time at which a death benefit is payable, the insurer needs a model of human mortality, from which probabilities of death at particular ages can be calculated, and this is the topic of this chapter.

We start with some notation. Let (x) denote a life aged x, where $x \geq 0$. The death of (x) can occur at any age greater than x, and we model the future lifetime of (x) by a continuous random variable which we denote by T_x. This means that $x + T_x$ represents the age-at-death random variable for (x).

Let F_x be the distribution function of T_x, so that

$$F_x(t) = \Pr[T_x \leq t].$$

Then $F_x(t)$ represents the probability that (x) does not survive beyond age $x + t$, and we refer to F_x as the **lifetime distribution** from age x. In many life insurance problems we are interested in the probability of survival rather than death, and so we define S_x as

$$S_x(t) = 1 - F_x(t) = \Pr[T_x > t].$$

Thus, $S_x(t)$ represents the probability that (x) survives for at least t years, and S_x is known as the **survival function**.

Given our interpretation of the collection of random variables $\{T_x\}_{x\geq 0}$ as the future lifetimes of individuals, we need a connection between any pair of them. To see this, consider T_0 and T_x for a particular individual who is now aged x. The random variable T_0 represented the future lifetime at birth for this individual, so that, at birth, the individual's age at death would have been represented by T_0. This individual could have died before reaching age x – the probability of this was $\Pr[T_0 < x]$ – but has survived. Now that the individual has survived to age x, so that $T_0 > x$, his or her future lifetime is represented by T_x and the age at death is now $x + T_x$. If the individual dies within t years from now, then $T_x \leq t$ and $T_0 \leq x + t$. Loosely speaking, we require the events $[T_x \leq t]$ and $[T_0 \leq x + t]$ to be equivalent, given that the individual survives to age x. We achieve this by making the following assumption for all $x \geq 0$ and for all $t > 0$

$$\boxed{\Pr[T_x \leq t] = \Pr[T_0 \leq x + t | T_0 > x].} \tag{2.1}$$

This is an important relationship.

Now, recall from probability theory that for two events A and B

$$\Pr[A|B] = \frac{\Pr[A \text{ and } B]}{\Pr[B]},$$

so, interpreting $[T_0 \leq x + t]$ as event A, and $[T_0 > x]$ as event B, we can rearrange the right-hand side of (2.1) to give

$$\Pr[T_x \leq t] = \frac{\Pr[x < T_0 \leq x + t]}{\Pr[T_0 > x]},$$

that is,

$$F_x(t) = \frac{F_0(x+t) - F_0(x)}{S_0(x)}. \tag{2.2}$$

Also, using $S_x(t) = 1 - F_x(t)$,

$$\boxed{S_x(t) = \frac{S_0(x+t)}{S_0(x)},} \tag{2.3}$$

which can be written as

$$\boxed{S_0(x+t) = S_0(x)\, S_x(t).} \tag{2.4}$$

This is a very important result. It shows that we can interpret the probability of survival from age x to age $x+t$ as the product of

(1) the probability of survival to age x from birth, and
(2) the probability, having survived to age x, of further surviving to age $x+t$.

Note that $S_x(t)$ can be thought of as the probability that (0) survives to at least age $x+t$ given that (0) survives to age x, so this result can be derived from the standard probability relationship

$$\Pr[A \text{ and } B] = \Pr[A|B]\Pr[B]$$

where the events here are $A = [T_0 > x+t]$ and $B = [T_0 > x]$, so that

$$\Pr[A|B] = \Pr[T_0 > x+t | T_0 > x],$$

which we know from (2.1) is equal to $\Pr[T_x > t]$.

Similarly, any survival probability for (x), for, say, $t+u$ years can be split into the probability of surviving the first t years, and then, given survival to age $x+t$, subsequently surviving another u years. That is,

$$\begin{aligned}
S_x(t+u) &= \frac{S_0(x+t+u)}{S_0(x)} \\
\Rightarrow S_x(t+u) &= \frac{S_0(x+t)}{S_0(x)}\,\frac{S_0(x+t+u)}{S_0(x+t)} \\
\Rightarrow S_x(t+u) &= S_x(t)S_{x+t}(u).
\end{aligned} \tag{2.5}$$

We have already seen that if we know survival probabilities from birth, then, using formula (2.4), we also know survival probabilities for our individual from any future age x. Formula (2.5) takes this a stage further. It shows that if we know survival probabilities from any age x (≥ 0), then we also know survival probabilities from any future age $x+t$ ($\geq x$).

Any survival function for a lifetime distribution must satisfy the following conditions to be valid.

Condition 1. $S_x(0) = 1$; that is, the probability that a life currently aged x survives 0 years is 1.

Condition 2. $\lim_{t\to\infty} S_x(t) = 0$; that is, all lives eventually die.

Condition 3. The survival function must be a non-increasing function of t; it cannot be more likely that (x) survives, say 10.5 years than 10 years, because in order to survive 10.5 years, (x) must first survive 10 years.

These conditions are both necessary and sufficient, so that any function S_x which satisfies these three conditions as a function of t (≥ 0), for a fixed x (≥ 0), defines a lifetime distribution from age x, and, using formula (2.5), for all ages greater than x.

For all the distributions used in this book, we make three additional assumptions:

Assumption 1. $S_x(t)$ is differentiable for all $t > 0$. Note that together with Condition 3 above, this means that $\frac{d}{dt}S_x(t) \leq 0$ for all $t > 0$.

Assumption 2. $\lim_{t\to\infty} t\, S_x(t) = 0$.

Assumption 3. $\lim_{t\to\infty} t^2\, S_x(t) = 0$.

These last two assumptions ensure that the mean and variance of the distribution of T_x exist. These are not particularly restrictive constraints – we do not need to worry about distributions with infinite mean or variance in the context of individuals' future lifetimes. These three extra assumptions are valid for all distributions that are feasible for human lifetime modelling.

Example 2.1 Let

$$F_0(t) = 1 - (1 - t/120)^{1/6} \quad \text{for } 0 \leq t \leq 120.$$

Calculate the probability that

(a) a newborn life survives beyond age 30,
(b) a life aged 30 dies before age 50, and
(c) a life aged 40 survives beyond age 65.

Solution 2.1 (a) The required probability is

$$S_0(30) = 1 - F_0(30) = (1 - 30/120)^{1/6} = 0.9532.$$

(b) From formula (2.2), the required probability is

$$F_{30}(20) = \frac{F_0(50) - F_0(30)}{1 - F_0(30)} = 0.0410.$$

(c) From formula (2.3), the required probability is

$$S_{40}(25) = \frac{S_0(65)}{S_0(40)} = 0.9395.$$

□

We remark that in the above example, $S_0(120) = 0$, which means that under this model, survival beyond age 120 is not possible. In this case we refer to 120 as the limiting age of the model. In general, if there is a limiting age, we use the Greek letter ω to denote it. In models where there is no limiting age, it is often practical to introduce a limiting age in calculations, as we will see later in this chapter.

2.3 The force of mortality

The force of mortality is an important and fundamental concept in modelling future lifetime. We denote the force of mortality at age x by μ_x and define it as

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} \Pr[T_0 \le x + dx \mid T_0 > x]. \tag{2.6}$$

From equation (2.1) we see that an equivalent way of defining μ_x is

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} \Pr[T_x \le dx],$$

which can be written in tems of the survival function S_x as

$$\mu_x = \lim_{dx \to 0^+} \frac{1}{dx} \left(1 - S_x(dx)\right). \tag{2.7}$$

Note that the force of mortality depends, numerically, on the unit of time; if we are measuring time in years, then μ_x is measured per year.

The force of mortality is best understood by noting that for very small dx, formula (2.6) gives the approximation

$$\mu_x \, dx \approx \Pr[T_0 \le x + dx \mid T_0 > x]. \tag{2.8}$$

Thus, for very small dx, we can interpret $\mu_x dx$ as the probability that a life who has attained age x dies before attaining age $x + dx$. For example, suppose we have a male aged exactly 50 and that the force of mortality at age 50 is 0.0044 per year. A small value of dx might be a single day, or 0.00274 years. Then the approximate probability that (50) dies on his birthday is $0.0044 \times 0.00274 = 1.2 \times 10^{-5}$.

We can relate the force of mortality to the survival function from birth, S_0. As

$$S_x(dx) = \frac{S_0(x + dx)}{S_0(x)},$$

formula (2.7) gives

$$\mu_x = \frac{1}{S_0(x)} \lim_{dx\to 0^+} \frac{S_0(x) - S_0(x+dx)}{dx}$$
$$= \frac{1}{S_0(x)} \left(-\frac{d}{dx} S_0(x) \right).$$

Thus,

$$\boxed{\mu_x = \frac{-1}{S_0(x)} \frac{d}{dx} S_0(x).} \tag{2.9}$$

From standard results in probability theory, we know that the probability density function for the random variable T_x, which we denote f_x, is related to the distribution function F_x and the survival function S_x by

$$f_x(t) = \frac{d}{dt} F_x(t) = -\frac{d}{dt} S_x(t).$$

So, it follows from equation (2.9) that

$$\boxed{\mu_x = \frac{f_0(x)}{S_0(x)}.}$$

We can also relate the force of mortality function at any age $x + t$, $t > 0$, to the lifetime distribution of T_x. Assume x is fixed and t is variable. Then $d(x+t) = dt$ and so

$$\begin{aligned}
\mu_{x+t} &= -\frac{1}{S_0(x+t)} \frac{d}{d(x+t)} S_0(x+t) \\
&= -\frac{1}{S_0(x+t)} \frac{d}{dt} S_0(x+t) \\
&= -\frac{1}{S_0(x+t)} \frac{d}{dt} (S_0(x) S_x(t)) \\
&= -\frac{S_0(x)}{S_0(x+t)} \frac{d}{dt} S_x(t) \\
&= \frac{-1}{S_x(t)} \frac{d}{dt} S_x(t).
\end{aligned}$$

Hence

$$\mu_{x+t} = \frac{f_x(t)}{S_x(t)}. \tag{2.10}$$

This relationship gives a way of finding μ_{x+t} given $S_x(t)$. We can also use equation (2.9) to develop a formula for $S_x(t)$ in terms of the force of mortality function. We use the fact that for a function h whose derivative exists,

$$\frac{d}{dx}\log h(x) = \frac{1}{h(x)}\frac{d}{dx}h(x),$$

so from equation (2.9) we have

$$\mu_x = -\frac{d}{dx}\log S_0(x),$$

and integrating this identity over $(0, y)$ yields

$$\int_0^y \mu_x dx = -\left(\log S_0(y) - \log S_0(0)\right).$$

As $\log S_0(0) = \log \Pr[T_0 > 0] = \log 1 = 0$, we obtain

$$S_0(y) = \exp\left\{-\int_0^y \mu_x dx\right\},$$

from which it follows that

$$\boxed{S_x(t) = \frac{S_0(x+t)}{S_0(x)} = \exp\left\{-\int_x^{x+t} \mu_r dr\right\} = \exp\left\{-\int_0^t \mu_{x+s} ds\right\}.} \quad (2.11)$$

This means that if we know μ_x for all $x \geq 0$, then we can calculate all the survival probabilities $S_x(t)$, for any x and t. In other words, the force of mortality function fully describes the lifetime distribution, just as the function S_0 does. In fact, it is often more convenient to describe the lifetime distribution using the force of mortality function than the survival function.

Example 2.2 As in Example 2.1, let

$$F_0(x) = 1 - (1 - x/120)^{1/6}$$

for $0 \leq x \leq 120$. Derive an expression for μ_x.

Solution 2.2 As $S_0(x) = (1 - x/120)^{1/6}$, it follows that

$$\frac{d}{dx}S_0(x) = \tfrac{1}{6}(1 - x/120)^{-5/6}\left(-\tfrac{1}{120}\right),$$

and so

$$\mu_x = \frac{-1}{S_0(x)}\frac{d}{dx}S_0(x) = \tfrac{1}{720}(1 - x/120)^{-1} = \frac{1}{720 - 6x}.$$

As an alternative, we could use the relationship

$$\mu_x = -\frac{d}{dx}\log S_0(x) = -\frac{d}{dx}\left(\frac{1}{6}\log(1 - x/120)\right) = \frac{1}{720(1 - x/120)}$$
$$= \frac{1}{720 - 6x}.$$

□

Example 2.3 Let $\mu_x = Bc^x$, $x > 0$, where B and c are constants such that $0 < B < 1$ and $c > 1$. This model is called **Gompertz' law of mortality**. Derive an expression for $S_x(t)$.

Solution 2.3 From equation (2.11),

$$S_x(t) = \exp\left\{-\int_x^{x+t} Bc^r\,dr\right\}.$$

Writing c^r as $\exp\{r\log c\}$,

$$\begin{aligned}\int_x^{x+t} Bc^r\,dr &= B\int_x^{x+t}\exp\{r\log c\}dr\\ &= \frac{B}{\log c}\exp\{r\log c\}\Big|_x^{x+t}\\ &= \frac{B}{\log c}\left(c^{x+t} - c^x\right),\end{aligned}$$

giving

$$S_x(t) = \exp\left\{\frac{-B}{\log c}c^x(c^t - 1)\right\}.$$

□

The force of mortality under Gompertz' law increases exponentially with age. At first sight this seems reasonable, but as we will see in the next chapter, the force of mortality for most populations is not an increasing function of age over the entire age range. Nevertheless, the Gompertz model does provide a fairly good fit to mortality data over some age ranges, particularly from middle age to early old age.

Example 2.4 Calculate the survival function and probability density function for T_x using Gompertz' law of mortality, with $B = 0.0003$ and $c = 1.07$, for $x = 20$, $x = 50$ and $x = 80$. Plot the results and comment on the features of the graphs.

Solution 2.4 For $x = 20$, the force of mortality is $\mu_{20+t} = Bc^{20+t}$ and the survival function is

$$S_{20}(t) = \exp\left\{\frac{-B}{\log c} c^{20}(c^t - 1)\right\}.$$

The probability density function is found from (2.10):

$$\mu_{20+t} = \frac{f_{20}(t)}{S_{20}(t)} \Rightarrow f_{20}(t) = \mu_{20+t}\, S_{20}(t) = Bc^{20+t} \exp\left\{\frac{-B}{\log c} c^{20}(c^t - 1)\right\}.$$

Figure 2.1 shows the survival functions for ages 20, 50 and 80, and Figure 2.2 shows the corresponding probability density functions. These figures illustrate some general points about lifetime distributions.

First, we see an effective limiting age, even though, in principle there is no age to which the survival probability is exactly zero. Looking at Figure 2.1, we see that although $S_x(t) > 0$ for all combinations of x and t, survival beyond age 120 is very unlikely.

Second, we note that the survival functions are ordered according to age, with the probability of survival for any given value of t being highest for age 20 and lowest for age 80. For survival functions that give a more realistic representation of human mortality, this ordering can be violated, but it usually holds at ages of interest to insurers. An example of the violation of this ordering is that $S_0(1)$ may be smaller than $S_x(1)$ for $x \geq 1$, as a result of perinatal mortality.

Looking at Figure 2.2, we see that the densities for ages 20 and 50 have similar shapes, but the density for age 80 has a quite different shape. For ages 20 and 50, the densities have their respective maximums at (approximately)

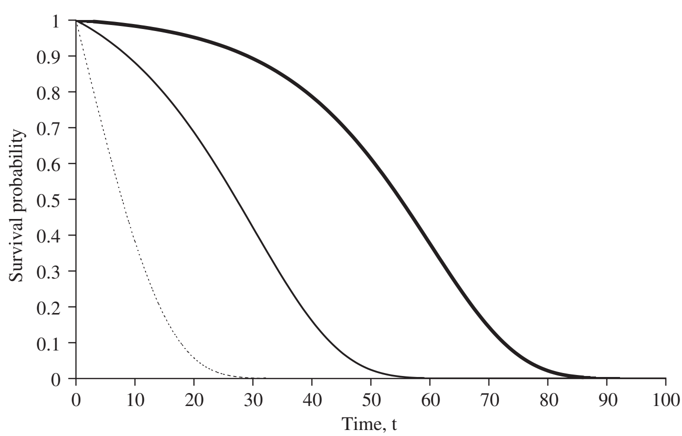

Figure 2.1 $S_x(t)$ for $x = 20$ (bold), 50 (solid) and 80 (dotted).

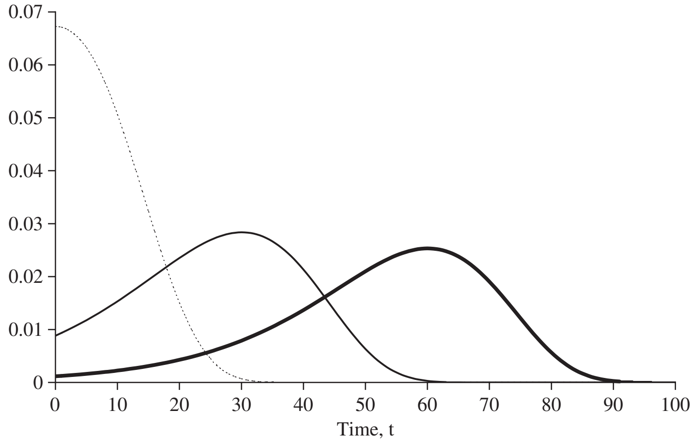

Figure 2.2 $f_x(t)$ for $x = 20$ (bold), 50 (solid) and 80 (dotted).

$t = 60$ and $t = 30$, indicating that death is most likely to occur around age 80. The decreasing form of the density for age 80 also indicates that death is more likely to occur at age 80 than at any other age for a life now aged 80. A further point to note about these density functions is that although each density function is defined on $(0, \infty)$, the spread of values of $f_x(t)$ is much greater for $x = 20$ than for $x = 50$, which, as we will see in Table 2.1, results in a greater variance of future lifetime for $x = 20$ than for $x = 50$. □

2.4 Actuarial notation

The notation used in the previous sections, $S_x(t)$, $F_x(t)$ and $f_x(t)$, is standard in statistics. Actuarial science has developed its own notation, International Actuarial Notation, that encapsulates the probabilities and functions of greatest interest and usefulness to actuaries. The force of mortality notation, μ_x, comes from International Actuarial Notation. We summarize the relevant actuarial notation in this section, and rewrite the important results developed so far in this chapter in terms of actuarial functions. The actuarial notation for survival and mortality probabilities is

$$\boxed{{}_tp_x = \Pr[T_x > t] = S_x(t),} \tag{2.12}$$

$$\boxed{{}_tq_x = \Pr[T_x \le t] = 1 - S_x(t) = F_x(t),} \tag{2.13}$$

$$\boxed{{}_{u|t}q_x = \Pr[u < T_x \le u + t] = S_x(u) - S_x(u + t).} \tag{2.14}$$

So

- ${}_tp_x$ is the probability that (x) survives to at least age $x + t$,
- ${}_tq_x$ is the probability that (x) dies before age $x + t$,
- ${}_{u|t}q_x$ is the probability that (x) survives u years, and then dies in the subsequent t years, that is, between ages $x + u$ and $x + u + t$. This is called a **deferred mortality probability**, because it is the probability that death occurs in some interval following a deferred period.

We may drop the subscript t if its value is 1, so that p_x represents the probability that (x) survives to at least age $x + 1$. Similarly, q_x is the probability that (x) dies before age $x + 1$. In actuarial terminology q_x is called the **mortality rate** at age x.

The relationships below follow immediately from the definitions above and the previous results in this chapter:

$$
\begin{aligned}
{}_tp_x + {}_tq_x &= 1, \\
{}_{u|t}q_x &= {}_up_x - {}_{u+t}p_x, \\
{}_{t+u}p_x &= {}_tp_x \, {}_up_{x+t} \quad \text{from (2.5)}, && (2.15) \\
\mu_x &= -\frac{1}{{}_xp_0} \frac{d}{dx} {}_xp_0 \quad \text{from (2.9)}. && (2.16)
\end{aligned}
$$

Similarly,

$$
\mu_{x+t} = -\frac{1}{{}_tp_x} \frac{d}{dt} {}_tp_x \Rightarrow \frac{d}{dt} {}_tp_x = -{}_tp_x \, \mu_{x+t}, \tag{2.17}
$$

$$
\mu_{x+t} = \frac{f_x(t)}{S_x(t)} \Rightarrow f_x(t) = {}_tp_x \, \mu_{x+t} \quad \text{from (2.10)}, \tag{2.18}
$$

$$
{}_tp_x = \exp\left\{ -\int_0^t \mu_{x+s} ds \right\} \quad \text{from (2.11)}. \tag{2.19}
$$

As F_x is a distribution function and f_x is its density function, it follows that

$$
F_x(t) = \int_0^t f_x(s) ds,
$$

which can be written in actuarial notation as

$$
{}_tq_x = \int_0^t {}_sp_x \, \mu_{x+s} ds. \tag{2.20}
$$

This is an important formula, which can be interpreted as follows. Consider time s, where $0 \leq s < t$. The probability that (x) is alive at time s is ${}_sp_x$,

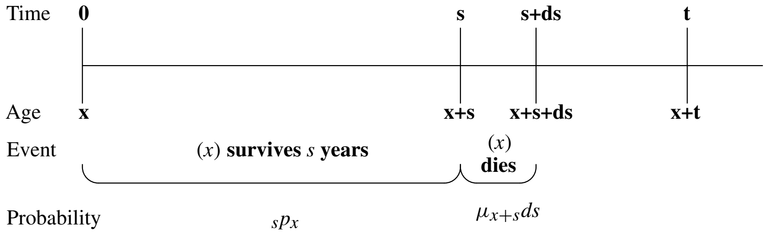

Figure 2.3 Time-line diagram for $_tq_x$

and the probability that (x) dies between ages $x + s$ and $x + s + ds$, having survived to age $x + s$, is (loosely) $\mu_{x+s}ds$, provided that ds is very small. Thus $_sp_x\, \mu_{x+s}ds$ can be interpreted as the probability that (x) dies between ages $x + s$ and $x + s + ds$. Now, we can sum over all the possible death intervals s to $s + ds$ – which requires integrating because these are infinitesimal intervals – to obtain the probability of death before age $x + t$.

We can illustrate this event sequence using the time-line diagram shown in Figure 2.3.

This type of interpretation is important as it can be applied to more complicated situations, and we will employ the time-line again in later chapters.

In the special case when $t = 1$, formula (2.20) becomes

$$q_x = \int_0^1 {}_sp_x\, \mu_{x+s}ds.$$

When q_x is small, it follows that p_x is close to 1, and hence $_sp_x$ is close to 1 for $0 \le s < 1$. Thus

$$q_x \approx \int_0^1 \mu_{x+s}ds \approx \mu_{x+1/2},$$

where the second relationship follows by the mid-point rule for numerical integration.

Example 2.5 As in Example 2.1, let

$$F_0(x) = 1 - (1 - x/120)^{1/6}$$

for $0 \le x \le 120$. Calculate both q_x and $\mu_{x+1/2}$ for $x = 20$ and for $x = 110$, and comment on these values.

Solution 2.5 We have

$$p_x = \frac{S_0(x+1)}{S_0(x)} = \left(1 - \frac{1}{120-x}\right)^{1/6},$$

giving $q_{20} = 0.00167$ and $q_{110} = 0.01741$, and from the solution to Example 2.2, $\mu_{20\frac{1}{2}} = 0.00168$ and $\mu_{110\frac{1}{2}} = 0.01754$. We see that $\mu_{x+1/2}$ is a good approximation to q_x when the mortality rate is small, but is not such a good approximation, at least in absolute terms, when the mortality rate is not close to 0. □

2.5 Mean and standard deviation of T_x

Next, we consider the expected future lifetime of (x), $\mathrm{E}[T_x]$, denoted in actuarial notation by $\overset{\circ}{e}_x$. We also call this the **complete expectation of life**. In order to evaluate $\overset{\circ}{e}_x$, we note from formulae (2.17) and (2.18) that

$$f_x(t) = {}_tp_x\,\mu_{x+t} = -\frac{d}{dt}\,{}_tp_x. \tag{2.21}$$

From the definition of an expected value, we have

$$\begin{aligned}\overset{\circ}{e}_x &= \int_0^\infty t\,f_x(t)dt \\ &= \int_0^\infty t\,{}_tp_x\,\mu_{x+t}dt.\end{aligned}$$

We can now use (2.21) to evaluate this integral using integration by parts as

$$\begin{aligned}\overset{\circ}{e}_x &= -\int_0^\infty t\left(\frac{d}{dt}\,{}_tp_x\right)dt \\ &= -\left(t\,{}_tp_x\Big|_0^\infty - \int_0^\infty {}_tp_x dt\right).\end{aligned}$$

In Section 2.2 we stated the assumption that $\lim_{t\to\infty} t\,{}_tp_x = 0$, which gives

$$\boxed{\overset{\circ}{e}_x = \int_0^\infty {}_tp_x dt.} \tag{2.22}$$

Similarly, for $\mathrm{E}[T_x^2]$, we have

$$\begin{aligned}
\mathrm{E}[T_x^2] &= \int_0^\infty t^2{}_t p_x\, \mu_{x+t} dt \\
&= -\int_0^\infty t^2 \left(\frac{d}{dt}{}_t p_x\right) dt \\
&= -\left(t^2{}_t p_x \Big|_0^\infty - \int_0^\infty {}_t p_x\, 2t\, dt \right) \\
&= 2\int_0^\infty t\, {}_t p_x\, dt. \qquad (2.23)
\end{aligned}$$

So we have integral expressions for $\mathrm{E}[T_x]$ and $\mathrm{E}[T_x^2]$. For some lifetime distributions we are able to integrate directly. In other cases we have to use numerical integration techniques to evaluate the integrals in (2.22) and (2.23). The variance of T_x can then be calculated as

$$\mathrm{V}[T_x] = \mathrm{E}\left[T_x^2\right] - \left(\mathring{e}_x\right)^2.$$

Example 2.6 As in Example 2.1, let

$$F_0(x) = 1 - (1 - x/120)^{1/6}$$

for $0 \le x \le 120$. Calculate $\mathring{e}_x$ and $\mathrm{V}[T_x]$ for (a) $x = 30$ and (b) $x = 80$.

Solution 2.6 As $S_0(x) = (1 - x/120)^{1/6}$, we have

$${}_t p_x = \frac{S_0(x+t)}{S_0(x)} = \left(1 - \frac{t}{120 - x}\right)^{1/6}.$$

Now recall that this formula is valid for $0 \le t \le 120 - x$, since under this model survival beyond age 120 is impossible. Technically, we have

$${}_t p_x = \begin{cases} \left(1 - \frac{t}{120-x}\right)^{1/6} & \text{for } x + t \le 120, \\ 0 & \text{for } x + t > 120. \end{cases}$$

So the upper limit of integration in equation (2.22) is $120 - x$, and

$$\mathring{e}_x = \int_0^{120-x} \left(1 - \frac{t}{120 - x}\right)^{1/6} dt.$$

We make the substitution $y = 1 - t/(120 - x)$, so that $t = (120 - x)(1 - y)$, giving

$$\begin{aligned}\mathring{e}_x &= (120 - x) \int_0^1 y^{1/6} dy \\ &= \tfrac{6}{7}(120 - x).\end{aligned}$$

Then $\mathring{e}_{30} = 77.143$ and $\mathring{e}_{80} = 34.286$.

Under this model the expectation of life at any age x is 6/7 of the time to age 120.

For the variance we require $\mathrm{E}[T_x^2]$. Using equation (2.23) we have

$$\begin{aligned}E\left[T_x^2\right] &= 2 \int_0^{120-x} t \; {}_t p_x dt \\ &= 2 \int_0^{120-x} t \left(1 - \frac{t}{120 - x}\right)^{1/6} dt.\end{aligned}$$

Again, we substitute $y = 1 - t/(120 - x)$ giving

$$\begin{aligned}E\left[T_x^2\right] &= 2(120 - x)^2 \int_0^1 (y^{1/6} - y^{7/6})\, dy \\ &= 2(120 - x)^2 \left(\frac{6}{7} - \frac{6}{13}\right).\end{aligned}$$

Then

$$\begin{aligned}\mathrm{V}[T_x] = \mathrm{E}[T_x^2] - \left(\mathring{e}_x\right)^2 &= (120 - x)^2 \left(2(6/7 - 6/13) - (6/7)^2\right) \\ &= (120 - x)^2\, (0.056515) = \left((120 - x)\, (0.23773)\right)^2.\end{aligned}$$

So $\mathrm{V}[T_{30}] = 21.396^2$ and $\mathrm{V}[T_{80}] = 9.509^2$.

Since we know under this model that all lives will die before age 120, it makes sense that the uncertainty in the future lifetime should be greater for younger lives than for older lives. □

A feature of the model used in Example 2.6 is that we can obtain formulae for quantities of interest such as $\mathring{e}_x$, but for many models this is not possible. For example, when we model mortality using Gompertz' law, there is no explicit formula for $\mathring{e}_x$ and we must use numerical integration to calculate moments of T_x. In Appendix B we describe in detail how to do this.

Table 2.1. *Values of* $\overset{\circ}{e}_x$, *SD*$[T_x]$ *and expected age at death for the Gompertz model with* $B = 0.0003$ *and* $c = 1.07$.

x	$\overset{\circ}{e}_x$	SD$[T_x]$	$x + \overset{\circ}{e}_x$
0	71.938	18.074	71.938
10	62.223	17.579	72.223
20	52.703	16.857	72.703
30	43.492	15.841	73.492
40	34.252	14.477	74.752
50	26.691	12.746	76.691
60	19.550	10.693	79.550
70	13.555	8.449	83.555
80	8.848	6.224	88.848
90	5.433	4.246	95.433
100	3.152	2.682	103.152

Table 2.1 shows values of $\overset{\circ}{e}_x$ and the standard deviation of T_x (denoted SD$[T_x]$) for a range of values of x using Gompertz' law, $\mu_x = Bc^x$, where $B = 0.0003$ and $c = 1.07$. For this survival model, ${}_{130}p_0 = 1.9 \times 10^{-13}$, so that using 130 as the maximum attainable age in our numerical integration is accurate enough for practical purposes.

We see that $\overset{\circ}{e}_x$ is a decreasing function of x, as it was in Example 2.6. In that example $\overset{\circ}{e}_x$ was a linear function of x, but we see that this is not true in Table 2.1.

2.6 Curtate future lifetime

2.6.1 K_x and e_x

In many insurance applications we are interested not only in the future lifetime of an individual, but also in what is known as the individual's curtate future lifetime. The curtate future lifetime random variable is defined as the integer part of future lifetime, and is denoted by K_x for a life aged x. If we let $\lfloor\ \rfloor$ denote the floor function, we have

$$K_x = \lfloor T_x \rfloor.$$

We can think of the curtate future lifetime as the number of whole years lived in the future by an individual. As an illustration of the importance of curtate future lifetime, consider the situation where a life aged x at time 0 is entitled to payments of 1 at times $1, 2, 3, \ldots$ provided that (x) is alive at these times. Then