Lecture Notes in Logic

LOGIC COLLOQUIUM 2006

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Logic Colloquium 2006

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LECTURE NOTES IN LOGIC

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Logic Colloquium 2006

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ASSOCIATION FOR SYMBOLIC LOGIC

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore,
São Paulo, Delhi, Dubai, Tokyo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521110815

First published in print format 2009

ISBN-13 978-0-511-60164-4 eBook (Adobe Reader)

ISBN-13 978-0-521-11081-5 Hardback

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INTRODUCTION

These are the proceedings of the Logic Colloquium 2006, which was held July 27–August 2 at the Radboud University of Nijmegen in the Netherlands.

The Logic Colloquium is the annual European conference on logic, organized under the auspices of the Association for Symbolic Logic (ASL). The program of LC2006 consisted of a mixture of tutorials, invited plenary talks, special sessions, and contributed talks. Finally, there was a plenary discussion on Gödel's legacy, on the occasion of the 100th birthday of the great logician Kurt Gödel, moderated by William Tait. The program gave a good overview of the recent research developments in logic.

The tutorial speakers were Downey, Moerdijk, and Veličković. The invited plenary speakers were Abramsky, Arslanov, Friedman, Goldstern, Hrushovski, Koenigsmann, Lewis, Montalbán, Palmgren, Pohlers, Schimmerling, Steel, Tait, and Wagner. The five special sessions were devoted to computability theory, computer science logic, model theory, proof theory and type theory, and set theory.

For these proceedings we have invited the tutorial and plenary invited speakers—as well as one invited speaker from each of the special sessions—to submit a paper. All papers have been reviewed by independent referees. This has given rise to these proceedings, which give a good overview of the content and breadth of the Logic Colloquium 2006 and of the state of the art in logic at present.

The Editors
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DEFINABILITY AND ELEMENTARY EQUIVALENCE IN THE ERSHOV DIFFERENCE HIERARCHY

M. M. ARSLANOV

Abstract. In this paper we investigate questions of definability and elementary equivalence in the Ershov difference hierarchy. We give a survey of recent results in this area and discuss a number of related open questions. Finally, properties of reducibilities which are intermediate between Turing and truth table reducibilities and which are connected with infinite levels of the Ershov hierarchy are studied.

§1. Introduction. In this paper we consider the current status of a number of open questions concerning the structural organization of classes of Turing degrees below $\boldsymbol{0}'$, the degree of the Halting Problem. We denote the set of all such degrees by $\mathcal{D}(\leq \boldsymbol{0}')$.

The Ershov hierarchy arranges these degrees into different levels which are determined by a quantitative characteristic of the complexity of algorithmic recognition of the sets composing these degrees.

The finite level n, $n \geq 1$, of the Ershov hierarchy constitutes n-c.e. sets which can be presented in a canonical form as

$$A = \bigcup_{i=0}^{[\frac{n-1}{2}]} \{(R_{2i+1} - R_{2i}) \cup (R_{2i} - R_{2i+1})\}$$

for some c.e. sets $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots \subseteq R_{n-1}$. (Here if n is an odd number then $R_n = \emptyset$.)

A (Turing) degree $\boldsymbol{a}$ is called an n-c.e. degree if it contains an n- c.e. set, and it is called a *properly n-c.e. degree* if it contains an n-c.e. set but no $(n-1)$-c.e. sets. We denote by $\mathcal{D}_n$ the set of all n- c.e. degrees. $\mathcal{R}$ denotes the set of c.e. degrees.

Degrees containing sets from different levels of the Ershov hierarchy, in particular the c.e. degrees, are the most important representatives of $\mathcal{D}(\leq \boldsymbol{0}')$. Investigations of these degree structures pursued in last two-three decades

The author is supported by RFBR Grant 05-01-00830.

Logic Colloquium '06
Edited by Barry Cooper, Herman Geuvers, Anand Pillay, and Jouko Väänänen
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show that the c.e. degrees and the degrees from finite levels of the Ershov hierarchy have similar properties in many respects.

The following theorem, which states that the classes of c.e. and n-c.e. degrees for $1 < n < \omega$ are indistinguishable from the point of view of their ability to compute fixed-point free functions, is a remarkable confirmation of this observation.

THEOREM 1. (*Arslanov* [1] *for* $n = 1$; *Jockusch, Lerman, Soare, Solovay* [17] *for* $n > 1$). *Suppose that* A *is a set which is* n*-c.e. for some* $n \geq 1$. *Then* A *has degree* $\boldsymbol{0}'$ *if and only if there is a function* f *computable in* A *with no fixed point, i.e.* $(\forall e)(W_{f(e)} \neq W_e)$.

Nevertheless, the elementary theories of the c.e. and the n-c.e. degrees for every $n > 1$ are different even at the Σ^0_2-level. This was shown by Downey [13] (the diamond lattice is embeddable in 2-c.e. degrees preserving $\boldsymbol{0}$ and $\boldsymbol{0}'$) and later by Cooper, Harrington, Lachlan, Lempp and Soare [10] (there is a 2-c.e. degree $\boldsymbol{d} < \boldsymbol{0}'$ which is maximal in the partial ordering of all n-c.e. degrees for all $n > 1$). Previous to this, a difference between the elementary theories of these degree structures at the Σ^0_3-level was exhibited in Arslanov [2, 3] (every n-c.e. degree for any $n > 1$ can be cupped to $\boldsymbol{0}'$ by a 2-c.e. degree $< \boldsymbol{0}'$). (Since any Σ^0_1-sentence satisfies $\mathcal{R}$ and $\mathcal{D}_n$ for any $n > 1$ if and only if it is consistent with the theory of partial orderings, $\mathcal{R}$ and $\mathcal{D}_n$ are indistinguishable at the Σ^0_1-level.)

These results initiated an intensive study of the properties of the n-c.e. degree structures for several $n > 1$. For the main part these investigations have concentrated on the following questions:

- Is the relation "$\boldsymbol{x}$ is c.e." definable in $\mathcal{D}_n$ for each (some) $n \geq 2$? Are there nontrivial definable in $\mathcal{D}_n, n \geq 2$, sets of c.e. degrees?
- Is the relation "$\boldsymbol{x}$ is m-c.e." definable in $\mathcal{D}_n$ for each (some) pair $n > m \geq 2$?
- Are $\{\mathcal{D}_2, \leq\}$ and $\{\mathcal{D}_3, \leq\}$ elementarily equivalent? (The famous *Downey's conjecture* states that they are elementarily equivalent.)
- Is $\{\mathcal{D}_2, \leq\}$ an elementary substructure of $\{\mathcal{D}_3, \leq\}$?
- Are $\{\mathcal{D}_m, \leq\}$ and $\{\mathcal{D}_n, \leq\}$ elementarily equivalent for each (some) $n \neq m, m, n \geq 2$?
- Is $\{\mathcal{D}_m, \leq\}$ an elementary substructure of $\{\mathcal{D}_n, \leq\}$ for $1 \leq m < n$?
- Is $\{\mathcal{D}_m, \leq\}$ a Σ_k-substructure of $\{\mathcal{D}_n, \leq\}$ for some $1 \leq m < n$ and some $k \geq 1$?
- Is the elementary theory of $\{\mathcal{D}_n, \leq\}$ undecidable for each (some) $n \geq 2$?

Known results on the definability of the relation "$\boldsymbol{x}$ is c.e." in $\{\mathcal{D}_n, \leq\}$, $n > 1$:

The following definition is from Cooper and Li [11].

DEFINITION 1. A Turing approximation to the class of c.e. degrees $\mathcal{R}$ in the n-c.e. degrees is a Turing definable class $\mathcal{S}_n$ of n-c.e. degrees such that

- either $\mathcal{R} \subseteq \mathcal{S}_n$ (in this case we say that $\mathcal{S}_n$ is an approximation to $\mathcal{R}$ from above), or
- $\mathcal{S}_n \subseteq \mathcal{R}$ ($\mathcal{S}_n$ is an approximation to $\mathcal{R}$ from below).

Obviously, $\mathcal{R}$ is definable in the n-c.e. degrees if and only if there is a Turing definable class $\mathcal{S}_n$ of n-c.e. degrees which is a Turing approximation to the class $\mathcal{R}$ in the n-c.e. degrees simultaneously from above and from below.

Non-trivial approximations from above:

First consider the following set of n-c.e. degrees: $\mathcal{S}_n = \{\boldsymbol{x} \in \mathcal{D}_n | (\forall \boldsymbol{z} > \boldsymbol{x})(\exists \boldsymbol{y})(\boldsymbol{x} < \boldsymbol{y} < \boldsymbol{z})\}$. The following two theorems show that $\mathcal{S}_n$ contains all c.e. degrees but does not coincide with the set of all n-c.e. degrees. Therefore, for any $n > 1$, $\mathcal{S}_n \neq \mathcal{D}_n$, and $\mathcal{S}_n$ is a *nontrivial approximation from above* of $\mathcal{R}$ in n-c.e. degrees.

THEOREM 2. (*Cooper, Yi* [12] *for* $n = 2$; *Arslanov, LaForte and Slaman* [6] *for* $n > 2$) *For any c.e. degree* $\boldsymbol{x}$ *and* n*-c.e. degree* $\boldsymbol{y}$, *if* $\boldsymbol{x} < \boldsymbol{y}$ *then* $\boldsymbol{x} < \boldsymbol{z} < \boldsymbol{y}$ *for some* n*-c.e. degree* $\boldsymbol{z}$.

THEOREM 3. [10] *There is a* 2*-c.e. degree which is maximal in each* $(\mathcal{D}_n, \leq)$, $n > 1$, *therefore* $\mathcal{S}_n \neq \mathcal{D}_n$ *for each* $n > 1$.

Using the Robinson Splitting Theorem technique (see Soare [20]) it is not difficult to construct a properly n-c.e. degree belonging to $\mathcal{S}_n$. Therefore $\mathcal{S}_n$ does not coincide with the class of all c.e. degrees and is not a Turing definition for the class $\mathcal{R}$.

Recently the following refinement of this approximation for the class $\mathcal{D}_2$ was obtained by Cooper and Li [11].

THEOREM 4. *For every c.e. degree* $\boldsymbol{a} < \boldsymbol{0}'$ *every* 2*-c.e. degree* $\boldsymbol{b} > \boldsymbol{a}$ *is splittable in the* 2*-c.e. degrees above* $\boldsymbol{a}$.

And again the set of c.e. degrees with this property does not coincide with the set of all c.e. degrees. Namely, using Cooper's [9] strategy of splitting of d-c.e. degrees over $\boldsymbol{0}$, M. Jamaleev [16] constructed a properly d-c.e. degree $\boldsymbol{a}$ such that any d-c.e. degree $\boldsymbol{d} > \boldsymbol{a}$ is splittable in d-c.e. degrees over $\boldsymbol{a}$.

So far there were no non-trivial approximations from below to the class of c.e. degrees $\mathcal{R}$ in the n-c.e. degrees, $n \geq 2$. Later in this paper the first such example will be presented.

Known results on elementary differences among $\{\mathcal{D}_n, \leq\}$ *for different* $n \geq 1$:

It was already mentioned that the elementary theories of the c.e. and the n-c.e. degrees for every $n > 1$ are different at the Σ^0_2-level, and that for any Σ_1-sentence φ, $\mathcal{D}_n \models \varphi$ if and only if φ is consistent with the theory of partial orderings. Therefore,

– $\mathcal{D}_n \equiv_{\Sigma_1} \mathcal{D}_m \equiv_{\Sigma_1} \mathcal{D}(\leq 0')$ for all $m \neq n, 1 \leq m, n < \omega$,

but

– for any $n \geq 1$, $\mathcal{D}_n$ is not a Σ_1-substructure of $\mathcal{D}(\leq 0')$ (Slaman, unpublished), and
– $\mathcal{R}$ is not a Σ_1-substructure of $\mathcal{D}_2$ (Y. Yang and L. Yu [21])

§2. Questions of definability and elementary equivalence. Investigation into the problems listed above is motivated by a desire for better understanding of the level of structural similarity of classes of c.e. and n-c.e. degrees for different $n > 1$, as well as better understanding of the level of homogeneity for the notion of c.e. with respect to n-c.e. degrees in the sense of the level of similarity of orderings of c.e. degrees and of n-c.e. degrees which are c.e. in a c.e. degree $\boldsymbol{d}$ and $\geq \boldsymbol{d}$. We consider the following two questions as corresponding examples for these two approaches.

Question 1. Let $\boldsymbol{0} < \boldsymbol{d} < \boldsymbol{e}$ be c.e. degrees. There is a c.e. degree $\boldsymbol{c}$ such that $\boldsymbol{c} < \boldsymbol{e}$ and $\boldsymbol{c}|\boldsymbol{d}$. Does this property of the c.e. degrees also hold in the n-c.e. degrees? That is, given 2-c.e. degrees $\boldsymbol{d}$ and $\boldsymbol{e}$ such that $\boldsymbol{0} < \boldsymbol{d} < \boldsymbol{e}$, is always there a 2-c.e. degree $\boldsymbol{c}$ such that $\boldsymbol{c} < \boldsymbol{e}$ and $\boldsymbol{c}|\boldsymbol{d}$?

Question 2. A relativization of the above stated property of the c.e. degrees to a c.e. degree $\boldsymbol{x}$ allows to obtain (having c.e. in and above $\boldsymbol{x}$ degrees $\boldsymbol{d} < \boldsymbol{e}$) a c.e. in $\boldsymbol{x}$ degree $\boldsymbol{c} > \boldsymbol{x}$ such that $\boldsymbol{c} < \boldsymbol{e}$ and $\boldsymbol{c}|\boldsymbol{d}$. Does this property hold also in the realm of 2-c.e. degrees in the following sense: let $\boldsymbol{0} < \boldsymbol{x} < \boldsymbol{d} < \boldsymbol{e}$ be such degrees that $\boldsymbol{x}$ is c.e., $\boldsymbol{d}$ and $\boldsymbol{e}$ are 2-c.e. degrees such that both of them are c.e. in $\boldsymbol{x}$. Is there a c.e. in $\boldsymbol{x}$ 2-c.e. degree $\boldsymbol{c}$ such that $\boldsymbol{c} < \boldsymbol{e}$ and $\boldsymbol{c}|\boldsymbol{d}$?

The following two results show that we have a negative answer to the first question and an affirmative answer to the second question.

Theorem 5. [5] *There are 2-c.e. degrees* $\mathbf{d}$ *and* $\mathbf{e}$ *such that* $\mathbf{0} < \mathbf{d} < \mathbf{e}$ *and for any 2-c.e. degree* $\mathbf{u} < \mathbf{e}$ *either* $\mathbf{u} \leq \mathbf{d}$ *or* $\mathbf{d} \leq \mathbf{u}$.

Theorem 6. [5] *For every c.e. degree* $\mathbf{x}$ *and all 2-c.e. degrees* $\mathbf{d}$ *and* $\mathbf{e}$ *such that* $\mathbf{d}, \mathbf{e}$ *are both c.e. in* $\mathbf{x}$ *and* $\mathbf{0} < \mathbf{x} < \mathbf{d} < \mathbf{e}$, *there is a c.e. in* $\mathbf{x}$ *2-c.e. degree* $\mathbf{u}$ *such that* $\mathbf{x} < \mathbf{u} < \mathbf{e}$ *and* $\mathbf{d}|\mathbf{u}$.

The following theorem is a refinement of Theorem 5.

Theorem 7. a) *In Theorem 5 the degree* $\mathbf{d}$ *is necessarily c.e. and*
b) *for each 2-c.e. degree* $\mathbf{e}$ *there is at most one c.e. degree* $\mathbf{d} < \mathbf{e}$ *with this property.*

Proof. Every 2-c.e. degree $\boldsymbol{u} > \boldsymbol{0}$ is c.e. in a c.e. degree $\boldsymbol{u}' < \boldsymbol{u}$ (this is the so-called Lachlan's Proposition). Therefore $\boldsymbol{e}$ is c.e. in a c.e. degree $\boldsymbol{e}' < \boldsymbol{e}$. If $\boldsymbol{e}' > \boldsymbol{d}$, then by Sacks Splitting Theorem we split $\boldsymbol{e}'$ into two c.e. degrees $\boldsymbol{e}_0$ and $\boldsymbol{e}_1$ avoiding the upper cone of $\boldsymbol{d}$ (avoiding $\boldsymbol{d}$, for short). At least one of these degrees must be incomparable with $\boldsymbol{d}$, a contradiction.

If $\boldsymbol{e}' < \boldsymbol{d}$, then consider the c.e. degree $\boldsymbol{c} = \boldsymbol{e}' \cup \boldsymbol{d}'$, where $\boldsymbol{d}' < \boldsymbol{d}$ is a c.e. degree such that $\boldsymbol{d}$ is c.e. in $\boldsymbol{d}'$. Obviously, $\boldsymbol{c} \leq \boldsymbol{d}$. If $\boldsymbol{c} < \boldsymbol{d}$ then we obtain a contradiction with Theorem 6, since both of the 2-c.e. degrees $\boldsymbol{e}$ and $\boldsymbol{d}$ are c.e. in $\boldsymbol{c}$. Therefore, $\boldsymbol{d} = \boldsymbol{c}$. Similar arguments prove also the second part of the theorem. ⊣

Now consider the following set of c.e. degrees:

$$\mathcal{S}_2 = \{0\} \bigcup \Big\{ \boldsymbol{x} > \boldsymbol{0} \mid (\exists \boldsymbol{y} > \boldsymbol{x})(\forall \boldsymbol{z})(\boldsymbol{z} \leq \boldsymbol{y} \rightarrow \boldsymbol{z} \leq \boldsymbol{x} \vee \boldsymbol{x} \leq \boldsymbol{z}) \Big\}.$$

It follows from Theorems 5 and 7 that

Corollary 1. *$\mathcal{S}_2 \subseteq \mathcal{R}$ and $\mathcal{S}_2 \neq \{0\}$.*

Therefore, $\mathcal{S}_2$ is a nontrivial approximation from below to the class of c.e., degrees $\mathcal{R}$ in the class of 2-c.e. degrees. A small additional construction in Theorem 5 allows to achieve that $\mathcal{S}_2$ contains infinitely many c.e. degrees.

It is a natural question to ask whether $\mathcal{S}_2$ coincides with the class of all c.e. degrees (and, therefore establishes definability of the c.e. degrees in $\mathcal{D}_2$) or not? To give a negative answer to this question we consider the isolated 2-c.e. degrees, which were introduced by Cooper and Yi [12]:

Definition 2. A c.e. degree $\mathbf{d}$ is *isolated* by an isolating 2-c.e. degree $\mathbf{e}$ if $\boldsymbol{d} < \boldsymbol{e}$ and for any c.e. degree $\mathbf{c}$, if $\boldsymbol{c} \leq \boldsymbol{e}$ then $\boldsymbol{c} \leq \boldsymbol{d}$.

A c.e. degree is *non-isolated* if it is not isolated.

Obviously, each non-computable c.e. degree $\boldsymbol{x}$ from $\mathcal{S}_2$ is isolated by a 2-c.e. degree $\boldsymbol{y}$. In Arslanov, Lempp, Shore [7] we proved that the non-isolated degrees are downward dense in the c.e. degrees and that they occur in any jump class. Therefore, $\mathcal{S}_2 \neq \mathcal{R}$.

Open Question. Whether for every pair of c.e. degrees $\boldsymbol{a} < \boldsymbol{b}$ there is a degree $\boldsymbol{c} \in \mathcal{S}_2$ such that $\boldsymbol{a} < \boldsymbol{c} < \boldsymbol{b}$ (i.e. $\mathcal{S}_2$ is dense in $\mathcal{R}$)?

An affirmative answer to this question implies definability of $\mathcal{R}$ in $\mathcal{D}_2$ as follows: given a c.e. degree $\boldsymbol{a} > \boldsymbol{0}$ we first split $\boldsymbol{a}$ into two incomparable c.e. degrees $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$, then using density of $\mathcal{S}_2$ in $\mathcal{R}$ find between $\boldsymbol{a}$ and $\boldsymbol{a}_i, i \leq 1$, a c.e. degree $\boldsymbol{c}_i, i \leq 1$, obtaining $\boldsymbol{a} = \boldsymbol{c}_0 \cup \boldsymbol{c}_1$. This shows that in this case a nonzero 2-c.e. degree is c.e. if and only if it is a least upper bound of two incomparable 2-c.e. degrees from $\mathcal{S}_2$.

Conjecture 1. Each c.e. degree $\boldsymbol{a} > \boldsymbol{0}$ is the least upper bound of two incomparable degrees from $\mathcal{S}_2$ and, therefore, the c.e. degrees are definable in $\mathcal{D}_2$.

Corollary 2 (From Theorem 5). *There are no 2-c.e. degrees $\boldsymbol{f} > \boldsymbol{e} > \boldsymbol{d} > \boldsymbol{0}$ such that for any u,*

(i) *if $\boldsymbol{u} \leq \boldsymbol{f}$ then either $\boldsymbol{e} \leq \boldsymbol{u}$ or $\boldsymbol{u} \leq \boldsymbol{e}$, and*
(ii) *if $\boldsymbol{u} \leq \boldsymbol{e}$ then either $\boldsymbol{d} \leq \boldsymbol{u}$ or $\boldsymbol{u} \leq \boldsymbol{d}$.*

PROOF. If there are such degrees $\boldsymbol{f} > \boldsymbol{e} > \boldsymbol{d} > \boldsymbol{0}$ then by Theorem 7a the degree $\boldsymbol{e}$ is c.e. and by the Sacks Splitting Theorem is splittable avoiding $\boldsymbol{d}$, which is a contradiction. ⊣

OPEN QUESTION. Are there 3-c.e. degrees $\boldsymbol{f} > \boldsymbol{e} > \boldsymbol{d} > \boldsymbol{0}$ with this property?

Obviously, an affirmative answer to this question refutes Downey's Conjecture on the elementarily equivalency of $\mathcal{D}_2$ and $\mathcal{D}_3$.

Though this question still remains open, we can weaken a little this property of degrees $(\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f})$ to carry out the mission imposed to these degrees to refute Downey's Conjecture. We consider triples of non-computable n-c.e. degrees $\{(\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}) \mid \boldsymbol{0} < \boldsymbol{d} < \boldsymbol{e} < \boldsymbol{f}\}$ with the following (weaker) property: for any n-c.e. degree u,

(i) if $\boldsymbol{u} \leq \boldsymbol{f}$ then either $\boldsymbol{u} \leq \boldsymbol{e}$ or $\boldsymbol{e} \leq \boldsymbol{d} \cup \boldsymbol{u}$, and
(ii) if $\boldsymbol{u} \leq \boldsymbol{e}$ then either $\boldsymbol{d} \leq \boldsymbol{u}$ or $\boldsymbol{u} \leq \boldsymbol{d}$.

(In the first line the former condition $\boldsymbol{e} \leq \boldsymbol{u}$ is changed to the weaker condition $\boldsymbol{e} \leq \boldsymbol{d} \cup \boldsymbol{u}$.)

We still have the following corollary from Theorems 5 and 6:

COROLLARY 3. *There are no* 2*-c.e. degrees* $\boldsymbol{f} > \boldsymbol{e} > \boldsymbol{d} > \boldsymbol{0}$ *such that for any* 2*-c.e. degree* u,

(i) *if* $\boldsymbol{u} \leq \boldsymbol{f}$ *then either* $\boldsymbol{u} \leq \boldsymbol{e}$ *or* $\boldsymbol{e} \leq \boldsymbol{d} \cup \boldsymbol{u}$, *and*
(ii) *if* $\boldsymbol{u} \leq \boldsymbol{e}$ *then either* $\boldsymbol{d} \leq \boldsymbol{u}$ *or* $\boldsymbol{u} \leq \boldsymbol{d}$.

PROOF. Suppose that there are such degrees $\boldsymbol{f} > \boldsymbol{e} > \boldsymbol{d} > \boldsymbol{0}$. Let $\boldsymbol{f}' \leq \boldsymbol{f}$ and $\boldsymbol{e}' \leq \boldsymbol{e}$ be c.e. degrees such that $\boldsymbol{f}$ and $\boldsymbol{e}$ are c.e. in $\boldsymbol{f}'$ and $\boldsymbol{e}'$, accordingly. Consider the degree $\boldsymbol{x} = \boldsymbol{d} \cup \boldsymbol{e}' \cup \boldsymbol{f}'$. Obviously, $\boldsymbol{d} \leq \boldsymbol{x} \leq \boldsymbol{f}$.

Since $\boldsymbol{x}$ is c.e., we have $\boldsymbol{e} \not< \boldsymbol{x}$, otherwise $\boldsymbol{x}$ is splittable in c.e. degrees avoiding $\boldsymbol{e}$, which is a contradiction. Also $\boldsymbol{x} \neq \boldsymbol{e}$, since in this case we can split $\boldsymbol{x}$ avoiding $\boldsymbol{d}$, which is again a contradiction. At last, if $\boldsymbol{x} \not\leq \boldsymbol{e}$ then it follows from condition (i) that $\boldsymbol{e} \leq \boldsymbol{d} \cup \boldsymbol{x} = \boldsymbol{x}$, a contradiction. Therefore, $\boldsymbol{x} < \boldsymbol{e}$. Since $\boldsymbol{f}$ and $\boldsymbol{e}$ are both c.e. in $\boldsymbol{x}$, it follows now from Theorem 6 that there is a 2-c.e. degree $\boldsymbol{u}$ such that $\boldsymbol{x} < \boldsymbol{u} < \boldsymbol{f}$ and $\boldsymbol{u} | \boldsymbol{e}$, a contradiction. ⊣

THEOREM 8. [5] *There exists a c.e. degree* $\boldsymbol{d} > \boldsymbol{0}$, *a* 2*-c.e. degree* $\boldsymbol{e} > \boldsymbol{d}$, *and a* 3*-c.e. degree* $\boldsymbol{f} > \boldsymbol{e}$ *such that for any* 3*-c.e. degree* u,

(i) *if* $\boldsymbol{u} \leq \boldsymbol{f}$ *then either* $\boldsymbol{u} \leq \boldsymbol{e}$ *or* $\boldsymbol{e} \leq \boldsymbol{d} \cup \boldsymbol{u}$, *and*
(ii) *if* $\boldsymbol{u} \leq \boldsymbol{e}$ *then either* $\boldsymbol{d} \leq \boldsymbol{u}$ *or* $\boldsymbol{u} \leq \boldsymbol{d}$.

COROLLARY 4. $\mathcal{D}_2 \not\equiv \mathcal{D}_3$ *at the* Σ_2*- level.*

In Theorem 8 we have a c.e. degree $\boldsymbol{d} > \boldsymbol{0}$ and a 2-c.e. degree $\boldsymbol{e} > \boldsymbol{d}$ such that every 3-c.e. degree $\boldsymbol{u} \leq \boldsymbol{e}$ is comparable with $\boldsymbol{d}$. Can this condition be strengthened in the following sense: there exists a c.e. degree $\boldsymbol{d} > \boldsymbol{0}$ and a 2-c.e. degree $\boldsymbol{e} > \boldsymbol{d}$ such that every n-c.e. degree $\leq \boldsymbol{e}$ for every $n < \omega$ is comparable with $\boldsymbol{d}$?

Open Question. Does there exist a c.e. degree $\boldsymbol{d} > \boldsymbol{0}$ and a 2-c.e. degree $\boldsymbol{e} > \boldsymbol{d}$ such that for any $n < \omega$ and any n-c.e. degree $\boldsymbol{u} \leq \boldsymbol{e}$ either $\boldsymbol{u} \leq \boldsymbol{d}$ or $\boldsymbol{d} \leq \boldsymbol{u}$?

An affirmative answer to this question would reveal an interesting property of the finite levels of the Ershov difference hierarchy with far-reaching prospects. From other side, if the question has a negative answer, then let $\boldsymbol{d} > \boldsymbol{0}$ and $\boldsymbol{e} > \boldsymbol{d}$ be accordingly c.e. and 2-c.e. degrees and $n \geq 3$ be the greatest natural number such that every n-c.e. degree $\boldsymbol{u} \leq \boldsymbol{e}$ is comparable with $\boldsymbol{d}$ and there is a $(n+1)$-c.e. degree $\boldsymbol{v} \leq \boldsymbol{e}$ which is incomparable with $\boldsymbol{d}$. Now consider the following Σ_1-formula:

$$\varphi(x, y, z) \equiv \exists u(x < y < z \,\&\, u \leq z \,\&\, u \not\leq y \,\&\, y \not\leq u).$$

Let $\boldsymbol{d}$ and $\boldsymbol{e}$ be degrees and n be the integer whose existence is assumed by the negative answer to the previous question. Then we have $\mathcal{D}_{n+1} \models \varphi(\boldsymbol{0}, \boldsymbol{d}, \boldsymbol{e})$, and $\mathcal{D}_n \models \neg\varphi(\boldsymbol{0}, \boldsymbol{d}, \boldsymbol{e})$, which means that in this case $\mathcal{D}_n$ is not a Σ_1-substructure of $\mathcal{D}_{n+1}$, thus answering a well-known open question.

We see that an answer to this question in any direction leads to very interesting consequences.

Theorems 5 and 8 raise a whole series of questions whose study could lead to a better understanding of the inner structure of the ordering of the n-c.e. degrees. Below we consider some of these questions.

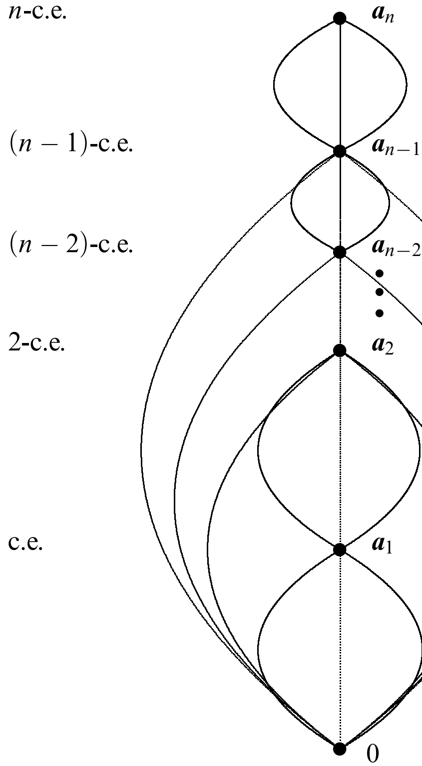
Definition 3. Let $n > 1$. An $(n+1)$-tuple of degrees $\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{n-1}, \boldsymbol{a}_n$ forms an *n-bubble* in $\mathcal{D}_m$ for some $m \geq 1$, if $\boldsymbol{0} = \boldsymbol{a}_0 < \boldsymbol{a}_1 < \boldsymbol{a}_2 < \cdots < \boldsymbol{a}_{n-1} < \boldsymbol{a}_n$, $\boldsymbol{a}_k$ is k-c.e. for each $k, 1 \leq k \leq n$, and for any m-c.e. degree $\boldsymbol{u}$, if $\boldsymbol{u} \leq \boldsymbol{a}_k$ then either $\boldsymbol{u} \leq \boldsymbol{a}_{k-1}$ or $\boldsymbol{a}_{k-1} \leq \boldsymbol{u}$.

An $(n+1)$-tuple of degrees $\boldsymbol{a}_0, \boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_{n-1}, \boldsymbol{a}_n$ forms a *weak n-bubble* in $\mathcal{D}_m$ for some $m \geq 1$, if $\boldsymbol{0} = \boldsymbol{a}_0 < \boldsymbol{a}_1 < \boldsymbol{a}_2 < \cdots < \boldsymbol{a}_{n-1} < \boldsymbol{a}_n$, $\boldsymbol{a}_k$ is k-c.e. for each $k, 1 \leq k \leq n$, and for any m-c.e. degree $\boldsymbol{u}$, if $\boldsymbol{u} \leq \boldsymbol{a}_k$ then either $\boldsymbol{u} \leq \boldsymbol{a}_{k-1}$ or $\boldsymbol{a}_{k-1} \leq \boldsymbol{u} \cup \boldsymbol{a}_{k-2}$.

Obviously, every n-bubble is also an n-weak bubble for every $n > 1$, but we don't know if the reverse holds. Theorem 5 and Corollary 2 state that in the 2-c.e. degrees there are 2-bubbles, but in the 2-c.e. degrees there are no n-bubbles (and even n-weak bubbles) for every $n > 2$. Theorem 8 states that in the 3-c.e. degrees there are 3-weak bubbles. The existence of n-bubbles (and even n-week bubbles) in the n-c.e. degrees for $n > 3$, and the existence of n-bubbles in m-c.e. degrees for $2 < m < n$ are open questions.

Conjecture 2. For every $n, 1 < n < \omega$, $\mathcal{D}_n$ contains an n-bubble, but does not contain m-bubbles for every $m > n$. (As we saw already this is true for $n = 2$.)

Obviously, if this conjecture holds for some $n > 1$ then this means that $\mathcal{D}_n$ is not elementarily equivalent to $\mathcal{D}_m, m > n$.



n-bubble

All known examples of sentences in the language of partial ordering, which are true in the n-c.e. degrees and false in the $(n+1)$-c.e. degrees for some $n \geq 1$, belong to the level $\forall\exists$ or to the higher levels of the arithmetic hierarchy. This and some other observations allow us to state the following plausible conjecture.

CONJECTURE 3. For all $n \geq 1$, for all $\exists\forall$-sentences φ, $\mathcal{D}_n \models \varphi \Rightarrow \mathcal{D}_{n+1} \models \varphi$. (The $\exists\forall$-theory of the n-c.e. degrees is a sub-theory of the $\exists\forall$-theory of the $(n+1)$-c.e. degrees).

The following question is posed in a number of publications (see, for instance, Y. Yang and L. Yu [21]):

QUESTION 3. Fix integers $n \geq 1$ and $m > n$. Is there a function $f : \mathcal{D}_n \to \mathcal{D}_m$ such that for each Σ_1-formula $\varphi(\bar{\boldsymbol{x}})$ and each n-tuple $\bar{\boldsymbol{a}}$ from $\mathcal{D}_n$, $\mathcal{D}_n \models \varphi(\bar{\boldsymbol{a}})$ if and only if $\mathcal{D}_m \models \varphi(f(\bar{\boldsymbol{a}}))$?

If for some $n < m < \omega$ $\mathcal{D}_n$ is not a Σ_1-substructure of $\mathcal{D}_m$, then φ (if exists) cannot be the identity function.

How many parameters contain functions which are witnesses in the proof that $\mathcal{D}_1$ is not a Σ_1-substructure of $\mathcal{D}(\leq \boldsymbol{0}')$ and $\mathcal{D}_2$?

– Slaman's result ($\mathcal{R} \not\preceq_{\Sigma_1} \mathcal{D}(\boldsymbol{0}')$: 3 parameters;
– Yang and Yu ($\mathcal{R} \not\preceq_{\Sigma_1} \mathcal{D}_2$): 4 parameters.

QUESTION 4. Can be these numbers reduced?

§3. Generalized tabular reducibilities. In Arslanov [4] for each constructive ordinal $\alpha \leq \omega^\omega$ we defined the so-called generalized tabular reducibility $\leq_{gtt(\alpha)}$ such that $\leq_{gtt(\omega)}$ coincides with $\leq_{tt}$, for each $\alpha \leq \omega^\omega$ $gtt(\alpha)$-reducibility is intermediate between tt- and T-reducibilities, and for different α all $\leq_{gtt(\alpha)}$ are different. In Arslanov [4] we also outlined a proof that the $gtt(\alpha)$-reducibility carries out the following property of the tt-reducibility to other infinite levels of the Ershov hierarchy: a Δ^0_2-set is ω-c.e. if and only if it is tt-reducible to the creative set K. In this paragraph we give a corrected proof of this and some other related results, eliminating some inaccuracy in the argument.

The following definitions of infinite levels of the hierarchy are from Ershov [14, 15].

Let $P(x, y)$ be a computable predicate which on ω defines a partial ordering. (If $P(x, y)$ we write $x \leq_P y$.) A uniformly c.e. sequence $\{R_x\}$ of c.e. sets is a $\leq_P$-sequence, if for all x, y, $x \leq_P y$ implies $R_x \subseteq R_y$.

Hereinafter we will use the Kleene system of notation $(\mathcal{O}, <_0)$. For $a \in \mathcal{O}$ we denote by $|a|_0$ the ordinal α, which has $\mathcal{O}$-notation a. Therefore $|a|_0$ has the order type $\langle\{x | x <_0 a\}, <_0\rangle$, and an "$a$-sequence of c.e. sets $\{R_x\}$" for $a \in \mathcal{O}$ is to be understood in the usual way. If α is a constructive ordinal and $a \in \mathcal{O}$ its notation, i.e. $|a|_0 = \alpha$, and $\lambda < \alpha$, then knowing a we can effectively find a notation b for λ, $|b|_0 = \lambda$.

An ordinal is even if it is either 0, or a limit ordinal, or a successor of an odd ordinal. Otherwise the ordinal is odd. Therefore if α is even, then α' (the successor of α) is odd and vise versa.

For the system of notation $\mathcal{O}$, the parity function $e(x)$ is defined as follows. Let $n \in \mathcal{O}$. Then $e(n) = 1$ if ordinal $|n|_0$ is odd, and $e(n) = 0$ if $|n|_0$ is even.

For any $a \in \mathcal{O}$ we first define operations S_a and P_a, which map a-sequences $\{R_x\}_{x<_0 a}$ to subsets of ω, as follows:

$S_a(R) = \{z | \exists x <_0 a(e(x) \neq e(a) \,\&\, z \in R_x \,\&\, \forall y <_0 x(z \notin R_y))\}$.

$P_a(R) = \{z | \exists x <_0 a(e(x) = e(a) \,\&\, z \in R_x \,\&$
$\&\, \forall y <_0 x(z \notin R_y))\} \cup \{\omega - \cup_{x<_0 a} R_x\}$.

It follows from these definitions that $P_a(R) = \overline{S_a(R)}$ for all $a \in \mathcal{O}$ and all a-sequences R.

The class Σ^{-1}_a (Π^{-1}_a) for $a \in \mathcal{O}$ is the class of all sets $S_a(R)$ (accordingly all sets $P_a(R)$), where $R = \{R_x\}_{x<_0 a}$ all a-sequences of c.e. sets, $a \in \mathcal{O}$. Define $\Delta^{-1}_a = \Sigma^{-1}_a \cap \Pi^{-1}_a$.

In particular, a set $A \subseteq \omega$ belongs to level Σ_ω^{-1} of the Ershov hierarchy (A is Σ_ω^{-1}-set), if there is a uniformly c.e. sequence of c.e. sets $\{R_x\}_{x\in\omega}$ such that $R_0 \subseteq R_1 \subseteq \cdots$ (ω-sequence of c.e. sets), and $A = \cup_{n=0}^{\infty}(R_{2n+1} - R_{2n})$.

DEFINITION 4. A set A is ω-c.e. set if there is a computable function g of two variables s and x and a computable function f such that for all x $A(x) = lim_s g(s, x)$, $g(0, x) = 0$, and

$$|\{s | g(s+1, x) \neq g(s, x)\}| \leq f(x).$$

THEOREM 9 (Ershov [14, 15]; Carstens [8], Selivanov [19]). *Let $A \subseteq \omega$. The following are equivalent*:

a) *A is ω-c.e.*;
b) *A is a Δ_ω^{-1}-set*;
c) *A is tt-reducible to a creative set K*;
d) *there is a uniformly c.e. sequence of c.e. sets $\{R_x\}_{x\in\omega}$, such that $\bigcup_{x\in\omega} R_x = \omega$, $R_0 \subseteq R_1 \subseteq \cdots$, and $A = \bigcup_{n=0}^{\infty}(R_{2n+1} - R_{2n})$.*

THEOREM 10 (Ershov [14, 15]). *Let $a, b \in \mathcal{O}$ and $a <_0 b$. Then*

a) $\Sigma_a^{-1} \cup \Pi_a^{-1} \subset \Sigma_b^{-1} \cap \Pi_b^{-1}$ *and, therefore, for any $a \in \mathcal{O}$, $\Sigma_a^{-1} \subset \Delta_2^0$*;
b) $\cup_{a\in\mathcal{O}} \Sigma_a^{-1} = \cup_{a\in\mathcal{O}, |a|_{\mathcal{O}}=\omega^2} \Sigma_a^{-1} = \Delta_2^0$.

For convenience we consider only ordinals $\leq \omega^\omega$, and for simplicity instead of notations we use ordinals meaning their representation in normal form

$$\alpha = \omega^m \cdot n_0 + \cdots + \omega \cdot n_{m-1} + n_m.$$

The material of this paragraph can, however, be extended also to all constructive ordinals (considering, for instance, Kleene system of ordinal notation).

We first define classes of generalized truth-table conditions ($gtt(\alpha)$-conditions) $\mathcal{B}_\alpha$, $\alpha \leq \omega^\omega$.

$\alpha = n > 1$: $\mathcal{B}_\alpha$ consists of all tt-conditions with norm $< n$;

$\alpha = \omega$: $\mathcal{B}_\alpha$ consists of all tt-conditions;

$\alpha = \omega^m \cdot n + \beta$, $\beta < \omega^m$ ($n > 1$; if $n = 1$ then $\beta > 0$): $\mathcal{B}_\alpha$ consists of all tt-conditions of the form

$$\sigma_1 \,\&\, \tau_1 \vee \cdots \vee \sigma_n \,\&\, \tau_n \vee \rho, \text{ or } \neg[\sigma_1 \,\&\, \tau_1 \vee \cdots \vee \sigma_n \,\&\, \tau_n \vee \rho],$$

where $\sigma_i \in \mathcal{B}_\omega$, $\tau_i \in B_{\omega^m}$, $\rho \in \mathcal{B}_\beta$;

$\alpha = \omega^{m+1}$: $\mathcal{B}_\alpha = \bigcup_n \mathcal{B}_{\omega^m \cdot n}$;

$\alpha = \omega^\omega$: $\mathcal{B}_\alpha = \bigcup_n \mathcal{B}_{\omega^n}$.

It follows from these definitions that for each α, $\omega \leq \alpha \leq \omega^\omega$, gtt-conditions from $\mathcal{B}_\alpha$ are usual tt-conditions with a fixed inner structure of these conditions. Using this structure we define by induction on α an enumeration $\{\sigma_n^\alpha\}_{n\in\omega}$ of $gtt(\alpha)$-*formulas* related to gtt-conditions from $\mathcal{B}_\alpha$.

We denote by σ_n^ω the n-th tt-condition (which is a formula of propositional logic constructed from atomic propositions $\langle k \in X\rangle$ for several $k \in \omega$, and the norm of the tt-condition is the number of its atomic propositions).

For $\alpha = \omega^m \cdot n + \beta$, $m \geq 1, n \geq 1, \beta < \omega^m$ ($n > 1$; if $n = 1$ then $\beta > 0$) the $gtt(\alpha)$-formula $\sigma^\alpha_{\langle p,q,r\rangle}$ with index $\langle p, q, r\rangle$ is the formula

$$\left[\sigma^\omega_{\Phi_p(0)} \mathbin{\&} \sigma^\gamma_{\Phi_q(0)} \ \vee \ \cdots \ \vee \ \sigma^\omega_{\Phi_p(n-1)} \mathbin{\&} \sigma^\gamma_{\Phi_q(n-1)} \vee \ \sigma^\beta_r\right],$$

where $\gamma = \omega^m$, $\Phi_p(i)$ is the partial-computable function with index p, defined for all $i \leq n-1$, $\Phi_q(i)$ is the partial-computable function with index q.

Therefore, a $gtt(\alpha)$-formula σ^α_i with index $i = \langle p, q, r\rangle$ is a gtt-condition $\sigma \in \mathcal{B}_\alpha$, $\alpha = \omega^m \cdot n + \delta$, $m \geq 1, n \geq 1, \delta < \omega^m$, if and only if $\Phi_p(x)\downarrow$ for all $x \leq n-1$ and r is an index for some gtt-condition from $\mathcal{B}_\delta$.

For $\alpha = \omega^{m+1}$ and $\alpha = \omega^\omega$ the enumeration of $gtt(\alpha)$-formulas $\{\sigma^\alpha\}$ is defined using a fixed effective enumeration of all gtt-formulas from $\bigcup_{n,i} \sigma_i^{\omega^m \cdot n}$ (accordingly from $\bigcup_{n,i} \sigma_i^{\omega^n}$).

For convenience we add to the integers two additional objects *true* and *false*, for which σ^α_{true} is a tt-condition which is identically truth and σ^α_{false} is an inconsistent tt-condition.

From now on we identify the class $\mathcal{B}_\alpha$ with the class of all $gtt(\alpha)$-formulas.

Definition 5. We say that a gtt-formula σ from $\mathcal{B}_\alpha$ *converges* on a set $A \subseteq \omega$, if

1. $\alpha \leq \omega$, i.e. any tt-condition from $\mathcal{B}_\alpha$, $\alpha \leq \omega$, converges on any set $A \subset \omega$, or
2. σ is equal to $\left[\left(\bigvee_{i\leqslant m} \sigma^\omega_{\Phi_p(i)} \mathbin{\&} \sigma^\gamma_{\Phi_q(i)}\right) \vee \sigma^\beta_j\right]$, and for any $i \leq m$ if A satisfies $\sigma^\omega_{\Phi_p(i)}$ (see the definition of "A satisfies $\sigma^\omega_{\Phi_p(i)}$" below), then $\Phi_q(i)\downarrow$ and $\sigma^\gamma_{\Phi_q(i)}$ converges on A.

Definition 6. A gtt-formula σ from $\mathcal{B}_\alpha$ is *satisfied* by a set $A \subseteq \omega$ (written as $A \models \sigma$), if σ converges on A and

- If $\sigma \in \mathcal{B}_\omega$, then A satisfies to the tt-condition σ,
- If σ is equal to $\left(\bigvee_{i\leqslant m} \sigma_i \mathbin{\&} \tau_i\right) \vee \rho$, then $A \models \rho$ or there is an $i \leq m$ such that $A \models \sigma_i$ and $A \models \tau_i$.

$A \not\models \sigma$ means $A \models \neg\sigma$.

Definition 7. A set A is $gtt(\alpha)$-reducible to a set B (written as $A \leq_{gtt(\alpha)} B$), if there is a computable function f such that for any x

(i) gtt-formula $\sigma^\alpha_{f(x)}$ converges on B, and
(ii) $x \in A \leftrightarrow B \models \sigma^\alpha_{f(x)}$.

REMARK 1. Part (i) in this definition requires us to prove that the $\leq_{gtt(\alpha)}$-reducibility implies Turing reducibility for any $\alpha \leq \omega^\omega$ (see Theorem 11 below). From the other side, the presence of the condition (i) in the definition of the $gtt(\alpha)$-reducibility does not allow us to prove the main property which is incumbent on these reducibilities: a Δ^0_2-set A is α-c.e. if and only if $A \leq_{gtt(\alpha)} K$. (We say that a set A is α-c.e. for some infinite ordinal $\alpha \leq \omega^\omega$ if A belong to the level Δ^{-1}_α of the Ershov hierarchy.) In Theorem 12 below we prove a weaker version of this statement: if $A \leq_{gtt(\alpha)} K$ then A belongs to the level Σ^{-1}_α of the Ershov hierarchy, and if $A \in \Sigma^{-1}_\alpha$ then $A \leq_{gtt^\star(\alpha)} K$. Here the $gtt^\star(\alpha)$-reducibility is obtained from the the $gtt(\alpha)$-reducibility by removing part (i) in Definition 7.

If $\alpha \neq \omega^n$ for some $n \leq \omega$, then in general the reducibility $\leq_{gtt(\alpha)}$ is not transitive. For this reason we formulate the following theorem for ordinals $\omega^n, 1 \leq n \leq \omega$ only.

THEOREM 11. *For $\alpha = \omega, \omega^2, \dots, \omega^\omega$ the reducibilities $\leq_{gtt(\alpha)}$ are reducibilities which are intermediate between the tt- and T-reducibilities, and for different α all $\leq_{gtt(\alpha)}$ are different.*

PROOF. From indexes i and j of σ^α_i, σ^α_j we can effectively compute an index k of the gtt-formula σ^α_k, which is obtained by substitution of σ^α_i into σ^α_j, which means that for $\alpha = \omega, \omega^2, \dots, \omega^\omega$ the set $\mathcal{B}_\alpha$ is effectively closed on substitutions, and the relation $\leqslant_{gtt(\alpha)}$ in this case transitive. It is easy to see also that the relation $\leqslant_{gtt(\alpha)}$ is reflexive. To prove that $A \leqslant_{gtt(\alpha)} B \rightarrow A \leq_T B$ for all sets A, B, suppose that $x \in A$ if and only if $B \models \sigma^\alpha_{f(x)} = \left[\left(\bigvee_{i \leqslant m} \sigma^\omega_{\Phi_{g(x)}(i)} \,\&\, \sigma^\beta_{\Phi_{q(x)}(i)}\right) \vee \sigma^\gamma_{r(x)}\right]$ for some computable functions f, g, q and r. For each x using oracle of B we can list all $i \leq m$ such that $B \models \sigma^\omega_{\Phi_{g(x)}(i)}$. For each such i we have $\Phi_{q(x)}(i)\downarrow$ (condition (i) in the Definition 7), and we also check whether $B \models \sigma^\beta_{\Phi_{q(x)}(i)}$. Similarly for the $gtt(\gamma)$-formula $\sigma^\gamma_{r(x)}$. Now $x \in A$ if and only if for some $i \leq m$ we have $\models \sigma^\omega_{\Phi_{g(x)}(i)}$ and $B \models \sigma^\beta_{\Phi_{q(x)}(i)}$, or there is such i in $gtt(\gamma)$-formula $\sigma^\gamma_{r(x)}$.

It is well-known (see Selivanov [19]) that for all $a <_0 b$ the set of T-degrees of Δ^{-1}_a-sets is a proper subset of the set of Δ^{-1}_b-sets and, therefore, it follows from Theorem 12 below that at least the degrees of creative sets for these reducibilities are different. ⊣

DEFINITION 8. A set A is $gtt^\star(\alpha)$-reducible to a set B (written as $A \leq_{gtt^\star(\alpha)} B$), if there is a computable function f such that for any x

$\star$ $x \in A \leftrightarrow B \models \sigma^\alpha_{f(x)}$.

THEOREM 12. *Let $\omega \le \alpha \le \omega^\omega$. Then for any set $A \subseteq \omega$,*

(i) *If $A \in \Sigma^{-1}_\alpha$ then $A \le_{gtt^\star(\alpha)} K$;*
(ii) *If $A \le_{gtt(\alpha)} K$, then $A \in \Sigma^{-1}_\alpha$.*

PROOF. We will consider the case $\alpha = \omega \cdot 2$. After that it will be clear how to prove the theorem in the general case.

(i) Suppose that $R_0 \subseteq R_1 \subseteq \cdots \subseteq R_\omega \subseteq R_{\omega+1} \subseteq \cdots$ is a $\omega \cdot 2$-sequence such that $A = (\cup_{i=0}^\infty (R_{2i+1} - R_{2i})) \cup (\cup_{i=0}^\infty (R_{\omega+2i+1} - R_{\omega+2i}))$.

The proof of the following lemma is straightforward.

LEMMA 1. *Let A be a Σ^{-1}_ω set and $P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots$ be an ω-sequence such that $A = \bigcup_{i=0}^\infty (P_{2i+1} - P_{2i})$. For any $x \in \omega$, if $x \in P_n$ for some $n < \omega$, then there is a tt-condition σ^ω_m such that $x \in A$ if and only if $K \models \sigma^\omega_m$. The integer m can be found effectively from x and n.*

Given x first define $\sigma^\omega_{\Phi_{p(x)}(0)} = \sigma^\omega_{\Phi_{p(x)}(1)} =$ "true", leave $\Phi_{q(x)}(0)$ and $\Phi_{q(x)}(1)$ undefined, and wait for an $i < \omega$ such that either $x \in R_i$ or $x \in R_{\omega+i}$. Using the previous lemma in the first case find a tt-condition $\sigma^\omega_{x_0}$ such that $x \in A$ if and only if $K \models \sigma^\omega_{x_0}$, similarly in the second case find a tt-condition $\sigma^\omega_{x_1}$ (for the ω-sequence $R_\omega \subseteq R_{\omega+1} \subseteq \cdots$) such that $x \in A$ if and only if $K \models \sigma^\omega_{x_1}$. Now in the first case define $\sigma^\omega_{\Phi_{q(x)}(0)} = \sigma^\omega_{x_0}$, in the second case define $\sigma^\omega_{\Phi_{q(x)}(0)} = \sigma^\omega_{x_1}$.

Now it is clear that $\sigma^\omega_{\Phi_{q(x)}(0)} \,\&\, \sigma^\omega_{\Phi_{q(x)}(0)} \vee \sigma^\omega_{\Phi_{p(x)}(1)} \,\&\, \sigma^\omega_{\Phi_{q(x)}(1)}$ is the required $gtt^\star(\omega \cdot 2)$-formula which $gtt^\star(\omega \cdot 2)$-reduces A to K: if $x \notin (\cup_{i=0}^\infty R_i) \cup (\cup_{i=0}^\infty R_{\omega+i})$, then $x \notin A$ and $\Phi_{q(x)}(0)$ and $\Phi_{q(x)}(1)$ remain undefined. If $x \in (\cup_{i=0}^\infty R_i) \cup (\cup_{i=0}^\infty R_{\omega+i})$, then the claim follows directly from the lemma.

(ii) Now suppose that $A \le_{gtt(\omega\cdot 2)} K$, i.e. there is a computable function f such that $x \in A$ if and only if $K \models \sigma^{\omega\cdot 2}_{f(x)}$. By definition there are computable functions p and q such that for all x, $\sigma^{\omega\cdot 2}_{f(x)} = \sigma^\omega_{\Phi_{p(x)}(0)} \,\&\, \sigma^\omega_{\Phi_{q(x)}(0)} \vee \sigma^\omega_{\Phi_{p(x)}(1)} \,\&\, \sigma^\omega_{\Phi_{q(x)}(1)}$, and for $i \le 1$, if $K \models \sigma^\omega_{\Phi_{p(x)}(i)}$ then $\Phi_{q(x)}(i)\downarrow$.

Since $\sigma^\omega_{\Phi_{p(x)}(0)}$ is a usual truth-table condition, there are a Boolean function $\alpha_{0,x}$ and a finite set $F_0 = \{u_{0,1}, u_{0,2}, \dots, u_{0,n_0(x)}\}$ such that $\sigma^\omega_{\Phi_{p(x)}(0)} = (F, \alpha_{0,x})$ (here $n_0(x)$ and $\alpha_{0,x}$ are computable functions on x). Similarly, if $\Phi_{q(x)}(0)\downarrow$, then let $\sigma^\omega_{\Phi_{q(x)}(0)} = (\{v_{0,1}, v_{0,2}, \dots, v_{0,m_0(x)}\}, \beta_{0,x})$ for some functions $m_0(x)$ and $\beta_{0,x}$.

Similarly we define functions $n_1(x), m_1(x), \alpha_{1,x}$ and $\beta_{1,x}$ for tt-conditions $\sigma^\omega_{\Phi_{p(x)}(1)}$ and $\sigma^\omega_{\Phi_{q(x)}(1)}$ (if $\Phi_{q(x)}(1)\downarrow$).

We have $x \in A \leftrightarrow K \models \sigma^\omega_{\Phi_{p(x)}(0)} \,\&\, \Phi_{q(x)}(0)\downarrow \,\&\, K \models \sigma^\omega_{\Phi_{q(x)}(0)} \vee K \models \sigma^\omega_{\Phi_{p(x)}(1)} \,\&\, \Phi_{q(x)}(1)\downarrow \,\&\, K \models \sigma^\omega_{\Phi_{q(x)}(1)}$.

We construct an $\omega \cdot 2$-sequence $R_0 \subseteq R_1 \subseteq \cdots \subseteq R_\omega \subseteq R_{\omega+1} \subseteq \cdots$ such that $A = (\cup_{i=0}^{\infty}(R_{2i+1} - R_{2i})) \cup (\cup_{i=0}^{\infty}(R_{\omega+2i+1} - R_{\omega+2i}))$ as follows:

Given $x \in \omega$ we wait for a stage s such that either $K_s \models \sigma^{\omega}_{\Phi_{p(x)}(0)} \,\&\, \Phi_{q(x),s}(0)\downarrow \,\&\, K_s \models \sigma^{\omega}_{\Phi_{q(x)}(0)}$ or $K_s \models \sigma^{\omega}_{\Phi_{p(x)}(1)} \,\&\, \Phi_{q(x),s}(1)\downarrow \,\&\, K_s \models \sigma^{\omega}_{\Phi_{q(x)}(1)}$.

Suppose that at a stage s for some $i \leq 1$ we have $K_s \models \sigma^{\omega}_{\Phi_{p(x),s}(i)} \,\&\, \Phi_{q(x),s}(i)\downarrow \,\&\, K_s \models \sigma^{\omega}_{\Phi_{q(x),s}(i)}$.

Then we enumerate x into $R_{\omega+n_i(x)+m_i(x)}$, and wait for a stage $t > s$ when $K_t \not\models \sigma^{\omega}_{\Phi_{p(x),t}(i)} \vee K_t \not\models \sigma^{\omega}_{\Phi_{q(x),t}(i)}$. Then enumerate x into $R_{\omega+n_i(x)+m_i(x)-1}$ etc.

If later at a stage $l > s$ we obtain $\Phi_{q(x),l}(1-i)\downarrow$, then we enumerate x into $R_{r(x)+\varepsilon}$, where $r(x) = n_0(x) + n_1(x) + m_0(x) + m_1(x)$, and ε is defined as follows:

(i) $\varepsilon = 1$, if $K_l \models \sigma^{\omega}_{\Phi_{p(x),l}(0)} \,\&\, K_l \models \sigma^{\omega}_{\Phi_{q(x),l}(0)}$ or $K_l \models \sigma^{\omega}_{\Phi_{p(x),l}(1)} \,\&\, K_l \models \sigma^{\omega}_{\Phi_{q(x),l}(1)}$ and $r(x)$ is an even number, or

if $K_l \not\models \sigma^{\omega}_{\Phi_{p(x),l}(0)} \vee K_l \not\models \sigma^{\omega}_{\Phi_{q(x),l}(0)}$, and $K_l \not\models \sigma^{\omega}_{\Phi_{p(x),l}(1)} \vee K_l \not\models \sigma^{\omega}_{\Phi_{q(x),l}(1)}$ and $r(x)$ is an odd number.

(ii) $\varepsilon = 0$, if $K_l \models \sigma^{\omega}_{\Phi_{p(x),l}(0)} \,\&\, K_l \models \sigma^{\omega}_{\Phi_{q(x),l}(0)}$ or $K_l \models \sigma^{\omega}_{\Phi_{p(x),l}(1)} \,\&\, K_l \models \sigma^{\omega}_{\Phi_{q(x),l}(1)}$ and $r(x)$ is an odd number, or

if $K_l \not\models \sigma^{\omega}_{\Phi_{p(x),l}(0)} \vee K_l \not\models \sigma^{\omega}_{\Phi_{q(x),l}(0)}$, and $K_l \not\models \sigma^{\omega}_{\Phi_{p(x),l}(1)} \vee K_l \not\models \sigma^{\omega}_{\Phi_{q(x),l}(1)}$ and $r(x)$ is an even number.

Then as above we may enumerate x into $R_{r(x)+\varepsilon-1}$ (having for $\varepsilon = 1$ a negation of the condition (i), and for $\varepsilon = 0$ a negation of the condition (ii)) etc. As a result we obtain an $\omega \cdot 2$-sequence $R_0 \subseteq R_1 \subseteq \cdots \subseteq R_\omega \subseteq R_{\omega+1} \subseteq \cdots$ such that $A = \bigcup_{i=0}^{\infty}(R_{2i+1} - R_{2i}) \cup \bigcup_{i=0}^{\infty}(R_{\omega+2i+1} - R_{\omega+2i})$, which means that $A \in \Sigma^{-1}_{\omega \cdot 2}$. ⊣

Corollary 5. *Let $\omega \leq \alpha \leq \omega^\omega$. Then for any set A,*

if $A \leq_{gtt(\alpha)} K \,\&\, \bar{A} \leq_{gtt(\alpha)} K$ then $A \in \Delta^{-1}_\alpha$, and

if $A \in \Delta^{-1}_\alpha$, then $A \leq_{gtt^\star(\alpha)} K \,\&\, \bar{A} \leq_{gtt^\star(\alpha)} K$.

The following theorem shows that the Turing reducibility is not exhausted by any collection of $gtt(\alpha)$- and $gtt^\star(\alpha)$-reducibilities.

Theorem 13. *There is a set $A \leq_T \emptyset''$ such that for all α $A \not\leq_{gtt(\alpha)} \emptyset''$.*

Proof. The proof is based on the following lemma.

Lemma 2. *If $B \leqslant_{gtt(\alpha)} C$ for some α, then there exists a $\emptyset'$-computable function $\Phi^{\emptyset'}_e$ such that*

$$(\forall x)\left(x \in B \leftrightarrow C \models \sigma^{\omega}_{\Phi^{\emptyset'}_e(x)}\right).$$

PROOF. (of lemma). Let $B \leqslant_{gtt(\alpha)} C$ by a computable function f, i.e. for any x,

$$x \in B \leftrightarrow C \models \sigma^{\alpha}_{f(x)}.$$

From α and an index for f we can effectively find the following presentation

$$\sigma^{\alpha}_{f(x)} = \Big(\bigvee_{i \leqslant m} \sigma^{\omega}_{g(i,x)} \,\&\, \sigma^{\beta}_{\psi(i,x)}\Big) \vee \sigma^{\gamma}_{k(x)}$$

If $\beta > \omega$ then we find such a presentation also for $\sigma^{\beta}_{h(i,x)}$ via new β' and γ' and similarly for $\sigma^{\gamma}_{k(x)}$ etc. Finally, we obtain an extended presentation

$$\sigma^{\alpha}_{f(x)} = \bigvee_{i \leqslant m} \Big(\Big(\bigwedge_{j_i \leqslant n_i} \sigma^{\omega}_{p_{i,j_i}(x)}\Big) \,\&\, \sigma^{\omega}_{q_i(x)}\Big),$$

where all $p_{i,j_i}(x), 0 \le i \le m, 0 \le j_i \le n_i$, are defined, but some $q_i(x), 0 \le i \le m$, for some i can be undefined. Now let for $0 \le i \le m$,

$$\tau_s(x) = \begin{cases} \Big(\bigwedge_{j_i \leqslant n_i} \sigma^{\omega}_{p_{i,j_i}(x)}\Big) \,\&\, \sigma^{\omega}_{q_i(x)}, & \text{if } q_i(x)\downarrow; \\ \sigma^{\omega}_{false}, & \text{if } q_i(x)\uparrow. \end{cases}$$

We have $\sigma^{\alpha}_{f(x)} = \tau_0(x) \vee \tau_1(x) \cdots \vee \tau_k(x)$. This is obviously a tt-condition whose index in the enumeration of all tt-conditions can be computed using an oracle for $\emptyset'$. $\dashv$

Now let $B = \{x | (\exists y)[\varphi^{\emptyset'}_x(x) = y \,\&\, \emptyset'' \models \sigma^{\omega}_y]\}$ and let $A = \omega - B$.

The reducibility $A \le_T \emptyset''$ is obvious. If $A \leqslant_{gtt(\alpha)} \emptyset''$ for some α, then there exists $\Phi^{\emptyset'}_e(x)$ from the lemma. Then

$$e \in A \leftrightarrow \emptyset'' \models \sigma^{\omega}_{\Phi^{\emptyset'}_e(x)} \leftrightarrow e \in B \leftrightarrow e \notin A \qquad \dashv$$

At last, the following theorem shows that the weak truth-table reducibility is a special case of the $gtt(\omega^2)$-reducibility.

DEFINITION 9. $A \le_{wtt} B$, if $A = \Phi^B_e$ for some e and for all x $\varphi^B_e(x) \le f(x)$ for some computable function f.

THEOREM 14. *If $A \le_{wtt} B$, then $A \le_{gtt(\omega^2)} B$.*

PROOF. Let $A = \Phi^B$ and let g be a computable function such that $\varphi^B(x) < g(x)$ for all x. There are $2^{g(x)}$ subsets $X_i \subseteq \{0, 1, \dots, g(x) - 1\}$. For each of them we compose a tt-formula $\sigma^{\omega}_{p(i)}$, $i \le 2^{g(x)}$, as follows:

$$X \models \sigma^{\omega}_{p(i)} \leftrightarrow X\lceil g(x) = X_i.$$

Now consider the formula

$$\sigma^{\omega^2}_{h(x)} \leqslant \sigma^{\omega}_{p(1)} \,\&\, \sigma^{\omega}_{q(1)} \vee \cdots \vee \sigma^{\omega}_{p(2^{g(x)})} \,\&\, \sigma^{\omega}_{q(2^{g(x)})},$$

where $\sigma^{\omega}_{p(i)}$ from above and $q(x)$ is defined as follows:

$$q(x) = \begin{cases} \text{true}, & \text{if } \Phi^{X_i}(x)\downarrow = 1; \\ \text{false}, & \text{if } \Phi^{X_i}(x)\downarrow = 0; \\ \uparrow . & \text{if } \Phi^{X_i}(x)\uparrow . \end{cases}$$

For any given x we can effectively compute an index $f(x)$ of the gtt-formula $\sigma^{\omega^2}_{h(x)}$. Now $\Phi^B(x) = 1 \leftrightarrow B \models \sigma^{\omega^2}_{f(x)}$, i.e. $A \leqslant_{gtt(\omega^2)} B$ by function $f(x)$. $\dashv$

The converse of this theorem is not true. Indeed, let $A \in \Delta^{-1}_{\omega^2} - \Delta^{-1}_{\omega}$. Then $A \leq_{gtt(\omega^2)} K$ but $A \not\leq_{wtt} K$, since $A \leq_{wtt} K$ if and only if $A \leq_{tt} K$ (see, for example, Rogers [18, exercise 9.45, page 159]) if and only if $A \in \Delta^{-1}_{\omega}$.

REFERENCES

[1] M. M. Arslanov, *Some generalizations of a fixed-point theorem*, ***Izvestiya Vysshikh Uchebnykh Zavedeniĭ. Matematika***, (1981), no. 5, pp. 9–16.

[2] ———, *Lattice properties of the degrees below* $\mathbf{0}'$, ***Doklady Akademii Nauk SSSR***, vol. 283 (1985), no. 2, pp. 270–273.

[3] ———, *The lattice of the degrees below* $\mathbf{0}'$, ***Izvestiya Vysshikh Uchebnykh Zavedeniĭ. Matematika***, (1988), no. 7, pp. 27–33.

[4] ———, *Generalized tabular reducibilities in infinite levels of the ershov difference hierarchy*, ***Logical Approaches of Computational Barriers*** (A. Beckmann, U. Berger, B. Lowe, and J. Tucker, editors), Swansea, UK, 2006, pp. 15–23.

[5] M. M. Arslanov, I. Sh. Kalimullin, and L. Lempp, *On Downey's conjecture*, to appear.

[6] M. M. Arslanov, Geoffrey L. LaForte, and Theodore A. Slaman, *Relative enumerability in the difference hierarchy*, ***The Journal of Symbolic Logic***, vol. 63 (1998), no. 2, pp. 411–420.

[7] M. M. Arslanov, S. Lempp, and R. A. Shore, *On isolating r.e. and isolated d-r.e. degrees*, ***Computability, Enumerability, Unsolvability***, London Mathematical Society Lecture Note Series, vol. 224, Cambridge University Press, Cambridge, 1996, pp. 61–80.

[8] H. G. Carstens, Δ^0_2-*Mengen*, ***Archiv für Mathematische Logik und Grundlagenforschung***, vol. 18 (1976/77), no. 1–2, pp. 55–65.

[9] S. B. Cooper, *A splitting theorem for the n-r.e. degrees*, ***Proceedings of the American Mathematical Society***, vol. 115 (1992), no. 2, pp. 461–471.

[10] S. B. Cooper, L. Harrington, A. H. Lachlan, S. Lempp, and R. I. Soare, *The d.r.e. degrees are not dense*, ***Annals of Pure and Applied Logic***, vol. 55 (1991), no. 2, pp. 125–151.

[11] S. B. Cooper and A. Li, *Turing definability in the Ershov hierarchy*, ***Journal of the London Mathematical Society. Second Series***, vol. 66 (2002), no. 3, pp. 513–528.

[12] S. B. Cooper and X. Yi, *Isolated d-r.e. degrees*, Preprint, 1993.

[13] R. Downey, *D.r.e. degrees and the Nondiamond theorem*, ***The Bulletin of the London Mathematical Society***, vol. 21 (1989), no. 1, pp. 43–50.

[14] Y. L. Ershov, *On a hierarchy of sets I*, ***Algebra i Logika***, vol. 7 (1968), no. 1, pp. 47–73.

[15] ———, *On a hierarchy of sets II*, ***Algebra i Logika***, vol. 7 (1968), no. 3, pp. 15–47.

[16] M. Jamaleev, *Splitting and cupping in the d-c. e. degrees*, submitted to ***Izvestiya Vysshikh Uchebnykh Zavedeniĭ. Matematika***.

[17] C. G. Jockusch, Jr., M. Lerman, R. I. Soare, and R. M. Solovay, *Recursively enumerable sets modulo iterated jumps and extensions of Arslanov's completeness criterion*, ***The Journal of Symbolic Logic***, vol. 54 (1989), no. 4, pp. 1288–1323.

[18] H. Rogers, Jr., ***Theory of Recursive Functions and Effective Computability***, McGraw-Hill, New York, 1967.

[19] V. L. Selivanov, *The Ershov hierarchy*, ***Akademiya Nauk SSSR. Sibirskoe Otdelenie. Sibirskiĭ Matematicheskiĭ Zhurnal***, vol. 26 (1985), no. 1, pp. 134–149.

[20] R. I. Soare, ***Recursively Enumerable Sets and Degrees***, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1987.

[21] Y. Yang and L. Yu, *On differences among elementary theories of finite levels of Ershov hierarchies*, ***Theory and Applications of Models of Computation***, Lecture Notes in Computer Science, vol. 3959, Springer, Berlin, 2006, pp. 765–771.

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A UNIFIED APPROACH TO ALGEBRAIC SET THEORY

BENNO VAN DEN BERG AND IEKE MOERDIJK

§1. **Introduction.** This short paper provides a summary of the tutorial on categorical logic given by the second named author at the Logic Colloquium in Nijmegen. Before we go into the subject matter, we would like to express our thanks to the organisers for an excellent conference, and for offering us the opportunity to present this material.

Categorical logic studies the relation between category theory and logical languages, and provides a very efficient framework in which to treat the syntax and the model theory on an equal footing. For a given theory T formulated in a suitable language, both the theory itself and its models can be viewed as categories with structure, and the fact that the models are models of the theory corresponds to the existence of canonical functors between these categories. This applies to ordinary models of first order theories, but also to more complicated topological models, forcing models, realisability and dialectica interpretations of intuitionistic arithmetic, domain-theoretic models of the λ-calculus, and so on. One of the best worked out examples is that where T extends the theory HHA of higher order Heyting arithmetic [24], which is closely related to the Lawvere-Tierney theory of elementary toposes. Indeed, every elementary topos (always taken with a natural numbers object here) provides a categorical model for HHA, and the theory HHA itself also corresponds to a particular topos, the "free" one, in which the true sentences are the provable ones.

The logic of many particular toposes shares features of independence results in set theory. For example, there are very natural constructions of toposes which model HHA plus classical logic in which the axiom of choice fails, or in which the continuum hypothesis is refuted. In addition, one easily finds topological sheaf toposes which model famous consistency results of intuitionistic logic, such as the consistency of HHA plus the continuity of all real-valued functions on the unit interval, and realisability toposes validating HHA plus "Church's thesis" (all functions from the natural numbers to itself are recursive). It took some effort (by Freyd, Fourman, McCarthy, Blass and Scedrov [13, 12, 6, 7] and many others), however, to modify the constructions

Logic Colloquium '06
Edited by Barry Cooper, Herman Geuvers, Anand Pillay, and Jouko Väänänen
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so as to provide models proving the consistency of such statements with HHA replaced by an appropriate set theory such as ZF or its intuitionistic counterpart IZF. This modification heavily depended on the fact that the toposes in question, namely various so-called Grothendieck toposes and Hyland's effective topos [18], were in some sense defined in terms of sets.

The original purpose of "algebraic set theory" [22] was to identify a categorical structure independently of sets, which would allow one to construct models of set theories like (I)ZF. These categorical structures were pairs $(\mathcal{E}, \mathcal{S})$ where $\mathcal{E}$ is a category much like a topos, and $\mathcal{S}$ is a class of arrows in $\mathcal{E}$ satisfying suitable axioms, and referred to as the class of "small maps". It was shown in loc. cit. that any such structure gave rise to a model of (I)ZF. An important feature of the axiomisation in terms of such pairs $(\mathcal{E}, \mathcal{S})$ is that it is preserved under the construction of categories of sheaves and of realisability categories, so that the model constructions referred to above become special cases of a general and "elementary" preservation result.

In recent years, there has been a lot of activity in the field of algebraic set theory, which is well documented on the web site www.phil.cmu.edu/projects/ast. Several variations and extensions of the the original Joyal-Moerdijk axiomatisation have been developed. In particular, Alex Simpson [30] developed an axiomatisation in which $\mathcal{E}$ is far from a topos (in his set-up, $\mathcal{E}$ is not exact, and is only assumed to be a regular category). This allowed him to include the example of classes in IZF, and to prove completeness for IZF of models constructed from his categorical pairs $(\mathcal{E}, \mathcal{S})$. This approach has been further developed by Awodey, Butz, Simpson and Streicher in their paper [3], in which they prove a categorical completeness theorem characterising the category of small objects in such a pair $(\mathcal{E}, \mathcal{S})$ (cf. Theorem 3.9 below), and identify a weak "basic" intuitionistic set theory BIST corresponding to the core of the categorical axioms in their setting.

In other papers, a variant has been developed which is adequate for constructing models of *predicative* set theories like Aczel's theory CZF [1, 2]. The most important feature of this variant is that in the structure $(\mathcal{E}, \mathcal{S})$, the existence of suitable power objects is replaced by that of inductive W-types. These W-types enabled Moerdijk and Palmgren in [28] to prove the existence of a model V for CZF out of such a structure $(\mathcal{E}, \mathcal{S})$ on the basis of some exactness assumptions on $\mathcal{E}$, and to derive the preservation of (a slight extension of) the axioms under the construction of sheaf categories. This result was later improved by Van den Berg [35]. It is precisely at this point, however, that we believe our current set-up to be superior to the ones in [28] and [35], and we will come back to this in some detail in Section 6 below. We should mention here that sheaf models for CZF have also been considered by Gambino [15] and to some extent go back to Grayson [17]. Categorical pairs $(\mathcal{E}, \mathcal{S})$ for weak predicative set theories have also been considered by Awodey-Warren [5] and

Simpson [31]. (Note, however, that these authors do not consider W-types and only deal with set theories weaker than Aczel's CZF.)

The purpose of this paper is to outline an axiomatisation of algebraic set theory which combines the good features of all the approaches mentioned above. More precisely, we will present axioms for pairs $(\mathcal{E}, \mathcal{S})$ which

- imply the existence in $\mathcal{E}$ of a universe V, which models a suitable set theory (such as IZF) (cf. Theorem 4.1 below);
- allow one to prove completeness theorems of the kind in [30] and [3] (cf. Theorem 3.7 and Theorem 3.9 below);
- work equally well in the predicative context (to construct models of CZF);
- are preserved under the construction of sheaf categories, so that the usual topological techniques automatically yield consistency results for IZF, CZF and similar theories;
- hold for realisability categories (cf. Examples 5.3 and 5.4 and Theorem Theorem 7.1).

Before we do so, however, we will recall the axioms of the systems IZF and CZF of set theory. In the next Section, we will then present our axioms for small maps, and compare them (in Subsection 3.4) to those in the literature. One of the main features of our axiomatisation is that we do not require the category $\mathcal{E}$ to be exact, but only to possess quotients of "small" equivalence relations. This restricted exactness axiom is consistent with the fact that every object is separated (in the sense of having a small diagonal), and is much easier to deal with in many contexts, in particular those of sheaves. Moreover, together with the Collection axiom this weakened form of exactness suffices for many crucial constructions, such as that of the model V of set theory from the universal small map $E \to U$, or of the associated sheaf of a given presheaf. In Section 4 we will describe the models of set theory obtained from pairs $(\mathcal{E}, \mathcal{S})$ satisfying our axioms, while Section 5 discusses some examples. Finally in Sections 6 and 7, we will discuss in some detail the preservation of the axioms under the construction of sheaf and realisability categories.

Like the tutorial given at the conference, this exposition is necessarily concise, and most of the proofs have been omitted. With the exception perhaps of Sections 6 and 7, these proofs are often suitable adaptations of existing proofs in the literature, notably [22, 30, 28, 3, 33]. A complete exposition with full proofs will appear as [37, 38, 39].

We would like to thank Thomas Streicher, Jaap van Oosten and the anonymous referees for their comments on an earlier draft of this paper, and Thomas Streicher in particular for suggesting the notion of a display map defined in Section 7.

§2. Constructive set theories. In this Section we recall the axioms for the two most prominent constructive variants of Zermelo-Fraenkel set theory, IZF and CZF. Like ordinary ZF, these two theories are formulated in first-order logic with one non-logical symbol ε. But unlike ordinary set theory, these theories are constructive, in that their underlying logic is intuitionistic.

In the formulation of the axioms, we use the following standard abbreviations: $\exists x \,\varepsilon\, a\, (\dots)$ for $\exists x\, (x \,\varepsilon\, a \wedge \dots)$, and $\forall x \,\varepsilon\, a\, (\dots)$ for $\forall x\, (x \,\varepsilon\, a \to \dots)$. Recall also that a formula is called *bounded*, when all the quantifiers it contains are of one of these two forms. Finally, a formula of the form $\forall x \,\varepsilon\, a\, \exists y \,\varepsilon\, b\, \phi \wedge \forall y \,\varepsilon\, b\, \exists x \,\varepsilon\, a\, \phi$ will be abbreviated as:

$$\mathrm{B}(x \,\varepsilon\, a, y \,\varepsilon\, b)\, \phi.$$

The axioms which both theories have in common are (the universal closures of):

Extensionality: $\forall x\, (x \,\varepsilon\, a \leftrightarrow x \,\varepsilon\, b) \to a = b$.
Empty set: $\exists x\, \forall y\, \neg y \,\varepsilon\, x$.
Pairing: $\exists x\, \forall y\, (y \,\varepsilon\, x \leftrightarrow y = a \vee y = b)$.
Union: $\exists x\, \forall y\, (y \,\varepsilon\, x \leftrightarrow \exists z \,\varepsilon\, a\; y \,\varepsilon\, z)$.
ε-induction: $\forall x\, (\forall y \,\varepsilon\, x\, \phi(y) \to \phi(x)) \to \forall x\, \phi(x)$
Bounded separation: $\exists x\, \forall y\, (y \,\varepsilon\, x \leftrightarrow y \,\varepsilon\, a \wedge \phi(y))$, for any bounded formula ϕ in which a does not occur.
Strong collection: $\forall x \,\varepsilon\, a\, \exists y\, \phi(x, y) \to \exists b\, \mathrm{B}(x \,\varepsilon\, a, y \,\varepsilon\, b)\, \phi$.
Infinity: $\exists a\, (\exists x\; x \,\varepsilon\, a) \wedge (\forall x \,\varepsilon\, a\, \exists y \,\varepsilon\, a\; x \,\varepsilon\, y)$.

One can obtain an axiomatisation for the constructive set theory IZF by adding to the axioms above the following two statements:

Full separation: $\exists x\, \forall y\, (y \,\varepsilon\, x \leftrightarrow y \,\varepsilon\, a \wedge \phi(y))$, for any formula ϕ in which a does not occur.
Power set axiom: $\exists x\, \forall y\, (y \,\varepsilon\, x \leftrightarrow y \subseteq a)$.

To obtain the predicative constructive set theory CZF, one should add instead the following axiom (which is a weakening of the Power Set Axiom):

Subset collection: $\exists c \forall z (\forall x \,\varepsilon\, a \exists y \,\varepsilon\, b \phi(x, y, z) \to \exists d \,\varepsilon\, c \mathrm{B}(x \,\varepsilon\, a, y \,\varepsilon\, d) \phi(x, y, z))$.

The Subset Collection Axiom has a more palatable formulation (equivalent to it relative to the other axioms), called Fullness (see [2]). Write $\boldsymbol{mv}(a, b)$ for the class of all multi-valued functions from a set a to a set b, i.e. relations R such that $\forall x \,\varepsilon\, a\, \exists y \,\varepsilon\, b\, (x, y) \,\varepsilon\, R$.

Fullness: $\exists u\, (u \subseteq \boldsymbol{mv}(a, b) \wedge \forall v \,\varepsilon\, \boldsymbol{mv}(a, b)\, \exists w \,\varepsilon\, u\, (w \subseteq v))$.

Using this formulation, it is also easier to see that Subset Collection implies Exponentiation, the statement that the functions from a set a to a set b form a set.

§3. Categories with small maps. Here we introduce the categorical structure which is necessary to model set theory. The structure is that of a category $\mathcal{E}$ equipped with a class of morphisms $\mathcal{S}$, satisfying certain axioms and being referred to as the *class of small maps*. The canonical example is the one where $\mathcal{E}$ is the category of classes in a model of some weak set theory, and morphisms between classes are small in case all the *fibres* are sets. More examples will follow in Section 5. In Section 4, we will show that these axioms actually provide us with the means of constructing models of set theory.

3.1. Axioms. In our work, the underlying category $\mathcal{E}$ is a Heyting category with sums. More precisely, $\mathcal{E}$ satisfies the following axioms (for an excellent account of the notions involved, see [20, Part A1]):

- $\mathcal{E}$ is cartesian, i.e. it has finite limits.
- $\mathcal{E}$ is regular, i.e. every morphism factors as a cover followed by a mono and covers are stable under pullback.
- $\mathcal{E}$ has finite disjoint and stable coproducts.
- $\mathcal{E}$ is Heyting, i.e. for any morphism $f : X \longrightarrow Y$ the functor $f^* : \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(X)$ has a right adjoint $\forall_f$.

This expresses precisely that $\mathcal{E}$ is a categorical structure suitable for modelling a typed version of first-order intuitionistic logic with finite product and sum types.

We now list the axioms that we require to hold for a class of small maps, extending the axioms for a class of open maps (see [22]). We will comment on the relation between our axiomatisation and existing alternatives in Subsection 3.4 below.

The axioms for a class of open maps $\mathcal{S}$ are:

(A1) (Pullback stability) In any pullback square

$$\begin{tikzcd} D \arrow[r] \arrow[d, "g"'] & B \arrow[d, "f"] \\ C \arrow[r, "p"'] & A, \end{tikzcd}$$

where $f \in \mathcal{S}$, also $g \in \mathcal{S}$.

(A2) (Descent) Whenever in a pullback square as above, $g \in \mathcal{S}$ and p is a cover, $f \in \mathcal{S}$.

(A3) (Sums) Whenever $X \longrightarrow Y$ and $X' \longrightarrow Y'$ belong to $\mathcal{S}$, so does $X + X' \longrightarrow Y + Y'$.

(A4) (Finiteness) The maps $0 \longrightarrow 1$, $1 \longrightarrow 1$ and $2 = 1 + 1 \longrightarrow 1$ belong to $\mathcal{S}$.

(A5) (Composition) $\mathcal{S}$ is closed under composition.

(A6) (Quotients) In any commutative triangle

$$\begin{tikzcd} Z \arrow[rr, two heads, "p"] \arrow[dr, "g"'] & & Y \arrow[dl, "f"] \\ & X, & \end{tikzcd}$$

where p is a cover and g belongs to $\mathcal{S}$, so does f.

These axioms are of two kinds: the axioms (A1-3) express that the property we are interested in is one of the *fibres* of maps in $\mathcal{S}$. The others are more set-theoretic: (A4) says that the collections containing 0, 1 or 2 elements are sets. (A5) is a union axiom: the union of a small disjoint family of sets is again a set. Finally, (A6) is a form of replacement: the image of a set is again a set.

We will always assume that a class of small maps $\mathcal{S}$ satisfies the following two additional axioms, familiar from [22]:

(C) (Collection) Any two arrows $p : Y \longrightarrow X$ and $f : X \longrightarrow A$ where p is a cover and f belongs to $\mathcal{S}$ fits into a quasi-pullback diagram[1] of the form

$$\begin{tikzcd} Z \arrow[r] \arrow[d, "g"'] & Y \arrow[r, two heads, "p"] & X \arrow[d, "f"] \\ B \arrow[rr, two heads, "h"'] & & A, \end{tikzcd}$$

where h is a cover and g belongs to $\mathcal{S}$.

(R) (Representability, see Remark 3.4) There exists a small map $\pi : E \longrightarrow U$ (a "universal small map") such that for every small map $f : X \longrightarrow Y$ there is a diagram of the shape

$$\begin{tikzcd} X \arrow[d, "f"'] & A \arrow[l, two heads] \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ Y & B \arrow[l, two heads, "p"] \arrow[r] & U, \end{tikzcd}$$

where the left square is a quasi-pullback, the right square is a pullback and p is a cover.

The collection principle (C) expresses that in the internal logic it holds that for any cover $p : Y \longrightarrow X$ with small codomain there is a cover $Z \longrightarrow X$ with small domain that factors through p, while (R) says that there is a (necessarily class-sized) family of sets $(E_u)_{u \in U}$ such that any set is covered by one in this family.

[1] Recall that a commutative square in a regular category is called a quasi-pullback if the unique arrow from the initial vertex of the square to the inscribed pullback is a cover.

The next requirement is also part of the axioms in [22]. For a morphism $f : X \longrightarrow Y$, the pullback functor $f^* : \mathcal{E}/Y \longrightarrow \mathcal{E}/X$ always has a left adjoint Σ_f given by composition[2]. It has a right adjoint Π_f only when f is exponentiable.

(ΠE) (Existence of Π) The right adjoint Π_f exists, whenever f belongs to $\mathcal{S}$.

This intuitively means that for any set A and class X there is a class of functions from A to X.

When f is exponentiable, one can define an endofunctor P_f (the polynomial functor associated with f) as the composition:

$$P_f = \Sigma_Y \Pi_f X^*.$$

Its initial algebra (whenever it exists) is called the W-type associated to f. For extensive discussion and examples of these W-types we refer the reader to [27, 33, 16]. We impose the axiom (familiar from [27, 14]):

(WE) (Existence of W) The W-type associated to any map $f : X \longrightarrow Y$ in $\mathcal{S}$ exists.

In non-categorical terms this means that for a signature consisting of a (possibly class-sized) number of term constructors each of which has an arity forming a set, the free term algebra exists (but maybe not as a set).

The following two axioms are necessary to have bounded separation as an internally valid principle (see Remark 3.3). For this purpose we need a piece of terminology: call a subobject

$$m : A \rightarrowtail X$$

$\mathcal{S}$-bounded, whenever m belongs to $\mathcal{S}$; note that the $\mathcal{S}$-bounded subobjects form a submeetsemilattice of $\mathrm{Sub}(X)$. We impose the following axiom:

(HB) (Heyting axiom for bounded subobjects) For any small map $f : Y \longrightarrow X$ the functor $\forall_f : \mathrm{Sub}(Y) \longrightarrow \mathrm{Sub}(X)$ maps $\mathcal{S}$-bounded subobjects to $\mathcal{S}$-bounded subobjects.

In addition, we require that all equalities are bounded. Call an object X separated, when the diagonal $\Delta : X \longrightarrow X \times X$ is small. We furthermore impose (see [3]):

(US) (Universal separation) All objects are separated.

We finally demand a limited form of *exactness*, by requiring the existence of quotients for a restricted class of equivalence relations. To formulate this categorically, we recall the following definitions. Two parallel arrows

$$R \overset{r_0}{\underset{r_1}{\rightrightarrows}} X$$

[2]We will write X^* and Σ_X for f^* and Σ_f, where f is the unique map $X \longrightarrow 1$.

in category $\mathcal{E}$ form an *equivalence relation* when for any object A in $\mathcal{E}$ the induced function

$$\mathrm{Hom}(A, R) \longrightarrow \mathrm{Hom}(A, X) \times \mathrm{Hom}(A, X)$$

is an injection defining an equivalence relation on the set $\mathrm{Hom}(A, X)$. We call an equivalence relation bounded, when R is a bounded subobject of $X \times X$. A morphism $q : X \longrightarrow Q$ is called the *quotient* of the equivalence relation, if the diagram

$$\begin{tikzcd} R \arrow[r, shift left, "r_0"] \arrow[r, shift right, "r_1"'] & X \arrow[r, "q"] & Q \end{tikzcd}$$

is both a pullback and a coequaliser. In this case, the diagram is called *exact*. The diagram is called *stably exact*, when for any $p : P \longrightarrow Q$ the diagram

$$\begin{tikzcd} p^*R \arrow[r, shift left, "p^*r_0"] \arrow[r, shift right, "p^*r_1"'] & p^*X \arrow[r, "p^*q"] & p^*Q \end{tikzcd}$$

is also exact. If the quotient completes the equivalence relation to a stably exact diagram, we call the quotient stable.

In the presence of (US), any equivalence relation that has a (stable) quotient, must be bounded. So our last axiom imposes the maximum amount of exactness that can be demanded:

(BE) (Bounded exactness) All $\mathcal{S}$-bounded equivalence relations have stable quotients.

This completes our definition of a class of small maps. A pair $(\mathcal{E}, \mathcal{S})$ satisfying the above axioms now will be called *a category with small maps*.

When a class of small maps $\mathcal{S}$ has been fixed, we call a map f small if it belongs to $\mathcal{S}$, an object A small if $A \longrightarrow 1$ is small, a subobject $m : A \longrightarrow X$ small if A is small, and a relation $R \subseteq C \times D$ small if the composite

$$R \subseteq C \times D \longrightarrow D$$

is small.

We conclude this Subsection with some remarks on a form of exact completion relative to a class of small maps. As a motivation, notice that axiom (BE) is not satisfied in our canonical example, where $\mathcal{E}$ is the category of classes in a model of some weak set theory. To circumvent this problem, we will prove the following theorem in our companion paper [37]:

THEOREM 3.1. *The axiom* (BE) *is conservative over the other axioms, in the following precise sense. Any category $\mathcal{E}$ equipped with a class of maps $\mathcal{S}$ satisfying all axioms for a class of small maps except* (BE) *can be embedded in a category $\overline{\mathcal{E}}$ equipped with a class of small maps $\overline{\mathcal{S}}$ satisfying all the axioms, including* (BE). *Moreover, the embedding $\mathbf{y} : \mathcal{E} \longrightarrow \overline{\mathcal{E}}$ is fully faithful, bijective on subobjects and preserves the structure of a Heyting category with sums, hence* *preserves and reflects validity of statements in the internal logic. Finally, it also preserves and reflects smallness, in the sense that* $\mathbf{y}f$ *belongs to* $\overline{\mathcal{S}}$ *iff* f *belongs to* $\mathcal{S}$.

The category $\overline{\mathcal{E}}$ is obtained by formally adjoining quotients for bounded equivalence relations, as in [11, 9]. Furthermore, a map $g : B \longrightarrow A$ in $\overline{\mathcal{E}}$ belongs to $\overline{\mathcal{S}}$ iff it fits into a quasi-pullback square

$$\begin{tikzcd} \mathbf{y}D \arrow[r, two heads] \arrow[d, "\mathbf{y}f"'] & B \arrow[d, "g"] \\ \mathbf{y}C \arrow[r, two heads] & A, \end{tikzcd}$$

with f belonging to $\mathcal{S}$ in $\mathcal{E}$.

3.2. Consequences. Among the consequences of these axioms we list the following.

Remark 3.2. For any object X in $\mathcal{E}$, the slice category $\mathcal{E}/X$ is equipped with a class of small maps $\mathcal{S}/X$, by declaring that an arrow $p \in \mathcal{E}/X$ belongs to $\mathcal{S}/X$ whenever $\Sigma_X f$ belongs to $\mathcal{S}$. Any further requirement for a class of small maps should be stable under slicing in this sense, if it is to be a sensible addition. We will not explicitly check this every time we introduce a new axiom, and leave this to the reader.

Remark 3.3. In a category $\mathcal{E}$ with small maps the following internal form of "bounded separation" holds. If $\phi(x)$ is a formula in the internal logic of $\mathcal{E}$ with free variable $x \in X$, all whose basic predicates are bounded, and contains existential and universal quantifications $\exists_f$ and $\forall_f$ only along small maps f, then

$$A = \{x \in X \mid \phi(x)\} \subseteq X$$

defines a bounded subobject of X. In particular, smallness of X implies smallness of A.

Remark 3.4. It follows from the axioms that any class of small maps $\mathcal{S}$ is also representable in the stronger sense that there is a universal small map $\pi : E \longrightarrow U$ such that for every small map $f : X \longrightarrow Y$ there is a diagram of the shape

$$\begin{tikzcd} X \arrow[d, "f"'] & A \arrow[l, two heads] \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ Y & B \arrow[l, two heads, "p"] \arrow[r] & U, \end{tikzcd}$$

where the left square is a pullback, the right square is a pullback and p is a cover. Actually, this is how representability was stated in [22]. We have chosen the weaker formulation (R), because it is easier to check in some examples.

REMARK 3.5. Using the axioms (ΠE), (R), (HB) and (BE), it can be shown along the lines of Theorem 3.1 in [22] that for any class of small maps the following axiom holds:

(PE) (Existence of power class functor) For any object C in $\mathcal{E}$ there exists a power object $\mathcal{P}_s C$ and a small relation $\in_C \subseteq C \times \mathcal{P}_s C$ such that, for any D and any small relation $R \subseteq C \times D$, there exists a unique map $\rho : D \longrightarrow \mathcal{P}_s C$ such that the square:

$$\begin{tikzcd} R \arrow[r] \arrow[d, tail] & \in_C \arrow[d, tail] \\ C \times D \arrow[r, "1\times\rho"'] & C \times \mathcal{P}_s C \end{tikzcd}$$

is a pullback.

In addition, one can show that the object $\mathcal{P}_s C$ is unique (up to isomorphism) with this property, and that the assignment $C \mapsto \mathcal{P}_s C$ is functorial.

A special role is played by $\Omega_b = \mathcal{P}_s 1$, what one might call the object of bounded truth-values, or the bounded subobject classifier. There are a couple of observations one can make: bounded truth-values are closed under small infima and suprema, implication, and truth and falsity are bounded truth-values. A subobject $m : A \rightarrowtail X$ is bounded, when the assertion "$x \in A$" has a bounded truth-value for any $x \in X$, as such bounded subobjects are classified by maps $X \longrightarrow \Omega_b$.

REMARK 3.6 (See [5]). When $\mathcal{E}$ is a category with a class of small maps $\mathcal{S}$, and we fix an object $X \in \mathcal{E}$, we can define a full subcategory $\mathcal{S}_X$ of $\mathcal{E}/X$, whose objects are small maps into X. The category $\mathcal{S}_X$ is a Heyting pretopos, and the inclusion into $\mathcal{E}/X$ preserves this structure; this was proved in [5]. This result can be regarded as a kind of categorical "soundness" theorem, in view of the following corresponding "completeness" theorem, which is analogous to Grothendieck's result that every pretopos arises as the coherent objects in a coherent topos (see [21, Section D.3.3]).

THEOREM 3.7. *Any Heyting pretopos $\mathcal{H}$ arises as the category of small objects $\mathcal{S}_1$ in a category $\mathcal{E}$ with a class of small maps $\mathcal{S}$.*

This theorem was proved in [5], where, following [3], the objects in $\mathcal{E}$ were called the *ideals* over $\mathcal{H}$.

3.3. Strengthenings. For the purpose of constructing models of important (constructive) set theories, we will consider the following additional properties which a class of small maps may enjoy.

(NE) (Existence of nno) The category $\mathcal{E}$ possesses a natural numbers object.

(NS) (Smallness of nno) In addition, it is small.