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Economic Dynamics: Methods and Models

G. Gandolfo

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PREFACE TO THE SECOND EDITION

This is a new edition of the author's previous book, *Mathematical Methods and Models in Economic Dynamics*, which has been thoroughly revised on the basis of the suggestions communicated to him by colleagues and students or contained in reviews of the book, and on the basis of the author's own teaching and research experience. The new Part III incorporates the material previously contained in the Appendices. The revisions – besides the routine ones, such as the clarification of several points, the updating of the references, and the correction of misprints – comprise additions, too numerous to be listed here, to the chapters and sections (including exercises) on both mathematical methods and economic applications. Suffice it to say that these additions concern useful methods which are new (in the sense that they were not in use when the first edition was written, either because they did not then exist or because economists had not yet discovered their existence or usefulness), economic models which serve to illustrate both old and new methods, and new exercises in the form of more substantive and challenging problems (as Part III is aimed at more advanced students, no exercises were included there: in the author's experience, for these readers the best exercise is to work through the original models listed in the references). These revisions, it is hoped, will ensure that the book will better achieve its purpose.

Finally, non-incriminating thanks are due to Dr. Pietro Carlo Padoan, who helped to check the revisions, and to Miss Anna Maria Olivari, who so competently carried out the secretarial work.

University of Rome, June 1979

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PREFACE TO THE FIRST EDITION

This book has evolved from undergraduate and graduate courses given by the author at the Universities of Rome and of Siena in the last five years. Criticism and comments by the students on a provisional Italian edition helped in the preparation of the English edition.

The book aims at giving a simple but comprehensive treatment of some mathematical methods commonly used in economic dynamics and at showing how they are utilised to build and to analyse dynamic models. Accordingly, the book focuses on methods, and every new mathematical technique introduced is followed by its application to selected models. The unifying principle in the exposition of the different economic models is then seen to be the common mathematical technique. The process should ultimately enable the student to build and to analyse his own models.

The material is arranged in two Parts and four Appendixes. The latter contain relatively more advanced material (from the mathematical point of view) and also the treatment is relatively less simple. The two Parts, as far as the mathematics is concerned, follow the same scheme. Although a unified treatment of both difference and differential equations (linear and with constant coefficients) would have been more elegant, the author has preferred to keep them apart, at the cost of some repetition, in order to avoid confusion to the beginner and to make it possible to teach and to study them separately. In the appropriate places the formal similarities (and dissimilarities) between the two kinds of equations are pointed out. The Appendixes are also independent of one another (though each requires the knowledge of some of the material contained in the text) so that the teacher (and the student) has freedom of choice.

The various economic models can usually be read independently; where necessary or useful, the connections with other models (whether or not included in the book) are indicated. The models included in the book were selected to serve the purpose stated at the beginning of this preface; other models might often have served equally well. The author thinks, however, that the selection – which includes both old and new contributions – offers a general idea of the scope of modern economic dynamics.

The exercises are problems involving the solution of economic models with numerically given values of the parameters. Some of them are fully worked out in order to serve both as numerical examples of what has been explained and as a guide for the solution of the proposed exercises.

The reader of this book is assumed to have an elementary knowledge of the basic principles of economic theory (such as that provided by any good *general* introductory textbook). As far as the mathematics is concerned, no previous familiarity with the topics treated is assumed, so that everything is worked out in great detail and no essential steps in the argument are omitted. The required background for the text consists of elementary algebra (including the notion of complex numbers) and (for Part II) of the rudiments of calculus. Knowledge of some advanced matrix algebra is needed to understand a few proofs in Part I, ch. 8, and Part II, chs. 8 and 9; such proofs, however, are given in footnotes (and can be omitted without loss to the main argument, which is developed in non-matricial terms). Some more mathematical background is needed for the Appendixes (e.g., the implicit function theorem and the first- and second-order conditions for a free or constrained extremum in *n* variables are used in Appendix I) where the treatment may also, in some places, be a bit harder than in the text.

University of Siena, 1970

CONTENTS

INTRODUCTION

"A system is dynamical if its behavior over time is determined by functional equations in which variables at different points of time are involved in an essential way" (Frisch's and Samuelson's definition). ★

Before commenting on this definition, it may be recalled that, according to another definition, economic dynamics is identified with those parts of economic theory where every quantity must be dated, whereas in economic statics we need not trouble about dating. But in this way we would include in dynamics many non-dynamic phenomena. As an example, think of a case where all quantities have the same date. This may mean that a certain phenomenon has taken place at a certain point of time (and this may be important, but it is not dynamics), or that a variable at time t depends on another variable at the same time t (and this too may be important – e.g., consumption is assumed to depend on current income and not on lagged income – but, again, it is not dynamics). The definition based on 'dating', then, is too vague and cannot be accepted.

Let us now turn back to the initial definition and explain what a *functional equation* is. The general theory of functional equations is outside the scope of this book, and we shall give only some basic notions, which are sufficient for our purposes.

The basic concept is the following: *a functional equation is an equation where the unknown is a function*. Everybody knows that to solve an equation means to find that value (or those values) of the unknown which satisfy the equation. Now, to solve a functional equation means to find an unknown

★ This definition is based on the *formal* characteristics common to all problems studied by economic dynamics. Other definitions, based on the economic substance of those problems, are possible (e.g., economic dynamics is concerned with growth, or stability, etc.: see the interesting survey by Machlup). But these definitions are inevitably partial (a complete definition of this type would reduce to a cumbersome list of problems, with the danger of omitting some of them). The formal definition, on the contrary, is precise and general.

function ★ which satisfies the functional equation *identically*. It is important to understand that '*to satisfy identically*' means that the function we have to find must satisfy the functional equation for *any* admissible value of the independent variable appearing in the function. The following simple example may clarify this point.

Let us consider the functional equation $y'(x) - y(x) = 0$. We must find a specific function (in one independent variable) which satisfies identically the stated equation, i.e. a function such that, for any value of its argument, the value of the function and the value of its first derivative are equal. It is easy to check that this function is $y(x) = Ae^x$, since, from elementary calculus, $y'(x) = Ae^x = y(x)$ for any x. Now consider the function $y = ax + b$, which gives $y' = a$; if we put $x = (a - b)/a$, we have also $y = a$, i.e. $y' = y$. However, for any other value of x the value of the function will be different from a; therefoıe, the function $y = ax + b$ does *not* satisfy identically our functional equation. As a matter of terminology, from now on we shall usually omit the adjective 'identically', it being understood that 'to satisfy' a functional equation means to satisfy it identically.

Now, if we suppose that the symbol x stands for time★★, we are ready to understand the second part of the definition of economic dynamics. In fact, $y'(x) = y(x)$ can be considered as a relation which involves the value of y at any point of time and the value it has at an arbitrarily close point, determined by y'. The 'different points of time' clause is necessary to exclude the case, already mentioned above, of quantities dated at the same point of time. Time must enter in an 'essential' way: for example, if it enters only as a unit of measurement (i.e. because we are dealing with quantities which are flows per unit of time) the system is not dynamic.

The types of functional equations most widely used in economic dynamics are linear, constant-coefficient difference and differential equations (the meaning of these words will be clarified in the following treatment), and to

★ It must be stressed that by 'function' we mean the *form* of the function, apart from arbitrary constants (e.g., $y = Ae^x$, where A is an arbitrary constant). As we shall see when expounding the various functional equations appearing in this book, the solution of a functional equation determines the form of the unknown function, and the determination of the arbitrary constant(s) requires additional conditions.

★★ Of course, the symbol x can stand for any variable. This obvious remark is useful to avoid the mistake of believing that in economics functional equations are used only in dynamic problems (as an example of a case outside economic dynamics, the classic problem of obtaining a utility function knowing the marginal rate(s) of substitution may be recalled). Since this is a book on economic dynamics only, from now on we shall use t instead of x, as this convention is commonly adopted.

them Parts I and II are devoted; chapters 3 and 4 of Part III are aimed at those wanting to know some more types.

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PART I

DIFFERENCE EQUATIONS (LINEAR AND WITH CONSTANT COEFFICIENTS)

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1

General Principles

Given a function $y = f(t)$, its first difference is defined as the difference between the value of the function when the argument assumes the value $t + h$ and the value of the function corresponding to the value t of the argument. In symbols, $\Delta y = f(t+h) - f(t)$. Without loss of generality we can assume unit increments of the independent variable, i.e. $\Delta y = f(t+1) - f(t)$.

If we consider successive equally-spaced values of the independent variable ($t+1, t+2, t+3$, etc.)*, we can obtain successive first differences:

$$\Delta y_t \quad = f(t+1) - f(t) = y_{t+1} - y_t ,$$

$$\Delta y_{t+1} = f(t+2) - f(t+1) = y_{t+2} - y_{t+1} ,$$

$$\Delta y_{t+2} = f(t+3) - f(t+2) = y_{t+3} - y_{t+2} ,$$

and so on. We can now compute the *second differences*, i.e. the sequence of differences between two successive first differences:

* It makes no difference whether the values run forwards or backwards ($t - 1, t - 2, t - 3$, etc.).

$$
\begin{aligned}
\Delta^2 y_t &= \Delta y_{t+1} - \Delta y_t = (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t) \\
&= y_{t+2} - 2y_{t+1} + y_t\,, \\
\Delta^2 y_{t+1} &= \Delta y_{t+2} - \Delta y_{t+1} = y_{t+3} - 2y_{t+2} + y_{t+1}\,, \\
\Delta^2 y_{t+2} &= \Delta y_{t+3} - \Delta y_{t+2} = y_{t+4} - 2y_{t+3} + y_{t+2}\,,
\end{aligned}
$$

and so on. Note that the superscript 2 means that the operation of computing the difference has been repeated twice, i.e. that the operator Δ has been applied twice.

Proceeding similarly, we can compute the differences between two successive second differences and obtain the *third differences* of the function:

$$
\begin{aligned}
\Delta^3 y_t &= \Delta^2 y_{t+1} - \Delta^2 y_t = (\Delta y_{t+2} - \Delta y_{t+1}) - (\Delta y_{t+1} - \Delta y_t) \\
&= \Delta y_{t+2} - 2\Delta y_{t+1} + \Delta y_t \\
&= (y_{t+3} - y_{t+2}) - 2(y_{t+2} - y_{t+1}) + (y_{t+1} - y_t) \\
&= y_{t+3} - 3y_{t+2} + 3y_{t+1} - y_t\,, \\
\Delta^3 y_{t+1} &= \Delta^2 y_{t+2} - \Delta^2 y_{t+1} = y_{t+4} - 3y_{t+3} + 3y_{t+2} - y_{t+1}\,,
\end{aligned}
$$

and so on. Higher-order differences can be computed by the reader as an exercise.

We can now define an *ordinary difference equation* as a functional equation involving one or more of the differences Δy, $\Delta^2 y$, etc., of an unknown function of time. Since the argument t varies in a discontinuous way, taking on equally spaced values, it follows that our unknown function will be defined only corresponding to these values of t (i.e. the graph of the function will be a succession of separated points, as we shall see in detail in ch. 2).

We have called this equation *ordinary* because the unknown function is a function of only one argument. When the partial differences of a function having more than one argument are involved, the equation becomes a partial difference equation, a type of functional equation that will not be treated in this book.

The *order* of a difference equation is that of the highest difference appearing in the equation. If, for example, the highest difference contained is the third difference, the equation is of the third order; note that the equation is of the third order independently of the fact that lower-order differences are or are not contained in the equation.

Since the differences of any order can be expressed, as we have seen above, in terms of various values of the function, a difference equation may also be defined as a functional equation involving two or more of the values y_t, y_{t+1}, etc., of an unknown function of time. As an example, the difference equation $a\Delta y_t + by_t = 0$ transforms, if we substitute $\Delta y_t = y_{t+1} - y_t$, into $ay_{t+1} + (b-a)y_t = 0$. In this form, the order of the equation is given by the highest difference between time subscripts: if the equation, for example, contains y_{t+3}, y_{t+1} and y_t, it is of the third order. We shall consider difference equations expressed in this second form, as it is the form they commonly take in economic models.

Let us note again that it makes no difference whether the equally spaced values of t are computed forwards or backwards, so long as the structure of time lags remains unaltered. The equation $ay_{t+1} + (b-a)y_t = 0$, for example, is identical with the equation $ay_t + (b-a)y_{t-1} = 0$. The reason is that to solve a difference equation means, as we know from the Introduction, to find a function (or functions) which satisfies (satisfy) the equation for any admissible value of t. This allows us to shift all the time subscripts as we like, provided that they are all shifted by the same amount (neglecting this proviso would alter the structure of the equation).

Consider now the equation $\Delta y_t = a$, i.e. $y_{t+1} - y_t = a$. In words, the problem is: find a function such that its first difference equals the given constant a for any value of t. It can be checked that the linear function $y = at + b$ satisfies the equation, since

$$y_{t+1} - y_t = [a(t+1) + b] - (at + b) = a\ . \star$$

Note that in the solution function an arbitrary constant (b) appears. This is not surprising, since the constancy of first differences is not affected by a parallel shift of the straight line. More generally, in the operation of differencing, the presence of an arbitrary constant, that is eliminated in the course of the operation, does not alter the result. Therefore, an arbitrary constant always appears in the solution of a first-order difference equation, and no more than one can appear.

★ Actually, this function is also the only one that satisfies the equation. This is shown by the 'existence and uniqueness' theorem, which we shall not treat. All types of equations considered in this book are 'well-behaved', i.e. their solution exists and is unique.

Proceeding further, consider the equation $\Delta^2 y_t = 0$ (find a function such that its second difference equals zero for any value of t). The solution is always the linear function $y = at + b$, but now both a and b are arbitrary constants; in fact, *any* straight line has a zero second difference. In general, the computation of second differences eliminates in succession *two* (and only two) arbitrary constants.

We shall see in the following chapters how the arbitrary constant(s) can be determined through additional conditions; what interests us here is to note that we can induce, from the reasoning given above, the important principle that *the general solution of a difference equation of order n is a function of t involving exactly n arbitrary constants.*

We can now summarize precisely the scope of our treatment. In Part I we shall be concerned with linear, constant-coefficient difference equations. The general n-th order form of such equations is

$$c_n y_{t+n} + c_{n-1} y_{t+n-1} + \ldots + c_1 y_{t+1} + c_0 y_t = g(t) \ , \tag{1.1}$$

where the c's are given constants and $g(t)$ is a known function. Some c's may be zero, but of course *both* c_n and c_0 must be different from zero if the equation is of order n.

In order to avoid cumbersome sentences, from now on we shall use the expression 'difference equations' (or even, when there is no danger of misunderstanding, simply 'equations') in the sense of 'ordinary difference equations, linear and with constant coefficients'.

We must now distinguish between homogeneous and non-homogeneous equations. Eq. (1.1) is non-homogeneous; the n-th order homogeneous equation is

$$c_n y_{t+n} + c_{n-1} y_{t+n-1} + \ldots + c_1 y_{t+1} + c_0 y_t = 0 \ . \tag{1.2}$$

The following theorems are fundamental in the theory of difference equations:

(1) *If $y_1(t)$ is a solution of the homogeneous equation, then $Ay_1(t)$, where A is an arbitrary constant, is also a solution.*

The proof is simple. Assume that $y_1(t)$ satisfies eq. (1.2). Substitute $Ay_1(t)$ in the same equation, obtaining

$$c_n A y_1(t+n) + c_{n-1} A y_1(t+n-1) + \ldots + c_1 A y_1(t+1) + c_0 A y_1(t) = 0 \ ;$$

therefore

$$A[c_n y_1(t+n) + c_{n-1}y_1(t+n-1) + \ldots + c_1 y_1(t+1) + c_0 y_1(t)] = 0\,.$$

If $Ay_1(t)$ has to be a solution, the last relationship must be satisfied. Since $y_1(t)$ is a solution of eq. (1.2), the expression in square brackets vanishes, and so the relationship

$$A[c_n y_1(t+n) + c_{n-1}y_1(t+n-1) + \ldots + c_1 y_1(t+1) + c_0 y_1(t)] = 0$$

is satisfied. This proves the theorem.

(2) *If $y_1(t), y_2(t)$ are two distinct*★ *solutions of the homogeneous equation $(n > 1)$, then $A_1 y_1(t) + A_2 y_2(t)$ is also a solution for any two constants A_1, A_2.*

The proof is similar to that of theorem (1) and is left as an exercise. Theorem (2) – called the 'superposition theorem' – can easily be extended to any number $k \leqslant n$ of distinct solutions of eq. (1.2).★★

To obtain the general solution of eq. (1.2), find n distinct solutions $y_1(t)$, $y_2(t), \ldots, y_n(t)$ and combine them (theorem (2)) into the function

$$f(t; A_1, A_2, \ldots, A_n) = A_1 y_1(t) + A_2 y_2(t) + \ldots + A_n y_n(t)\,, \qquad (1.3)$$

where $A_1, A_2, \ldots, A_n$ are arbitrary constants. Since this function contains exactly n arbitrary constants, we can conclude – from the general principle expounded before – that it is the general solution of eq. (1.2). The practical problem of how to find the n functions $y_1(t), y_2(t), \ldots, y_n(t)$ will be tackled in the following chapters.

(3) *If $\bar{y}(t)$ is any particular solution of the non-homogeneous equation, the general solution of this same equation is obtained adding $\bar{y}(t)$ to the general*

★ By 'distinct solutions' we mean *linearly independent* solutions. Let us recall that m functions $y_1(t), y_2(t), \ldots, y_m(t)$ are *linearly dependent* if m constants $A_1, A_2, \ldots, A_m$ exist, which do not all vanish, and such that the equation $A_1 y_1(t) + A_2 y_2(t) + \ldots + A_m y_m(t) = 0$ is identically satisfied for all admissible values of t. Otherwise the functions are *linearly independent*.

★★ Given a homogeneous equation of order n, a set of n linearly independent solutions is called a *fundamental set*.

*solution** *of the corresponding homogeneous equation, i.e.*

$$\bar{y}(t) + f(t; A_1, A_2, ..., A_n) \tag{1.4}$$

is the general solution of the non-homogeneous equation.

The proof of this theorem can be given substituting (1.4) into (1.1) and checking that the latter is satisfied. Since the function (1.4) contains exactly n arbitrary constants, it is the general solution of eq. (1.1).

Theorem (3) contains the method to solve the non-homogeneous equation:

(a) find a particular solution $\bar{y}(t)$ of the non-homogeneous equation;

(b) put $g(t) \equiv 0$ and solve the resulting homogeneous equation (often called the 'reduced' equation);

(c) add the two results.

Steps (a) and (b) can be taken in any order; step (c) gives the general solution of the non-homogeneous equation.

The particular solution of the non-homogeneous equation will depend, *ceteris paribus*, on the form of the known function $g(t)$. This suggests the following general approach: *to find a particular solution of the non-homogeneous equation, try a function having the same form of $g(t)$ but with undetermined coefficient(s) (e.g., if $g(t)$ is a constant, try an undetermined constant; if it is an exponential function, try the same exponential function with an undetermined multiplicative constant, and so on). Substitute this function in the non-homogeneous equation and determine the coefficient(s) so that the equation is satisfied.* This method – called *method of undetermined coefficients* – will be expounded in more detail in the following chapter.

We now have enough general principles to pass on to a detailed treatment of the difference equations of the various orders.

* The general solution of the homogeneous equation is then only a part of the general solution of the non-homogeneous equation, and so it is not 'general' with respect to the latter. This means that the expression 'general solution' must always be qualified. As a matter of terminology, note the following: (1) some authors use the word 'integral' (particular or general) instead of 'solution' but with the same meaning; (2) the expression 'particular solution' is also used (a) in the sense of a solution obtained from the general solution by giving specific values to the arbitrary constants, and (b) in the sense of any single non-general solution of the homogeneous equation (i.e., to indicate any one of $y_1(t)$, $y_2(t)$, etc.); (3) the expression 'complementary function' is used to indicate the general solution of the homogeneous equation when considered as a part of the general solution of the non-homogeneous equation, and the expression 'reduced equation' is used to indicate the homogeneous part of a non-homogeneous equation, i.e. the corresponding homogeneous equation obtained putting $g(t) \equiv 0$ in the course of the procedure to solve a non-homogeneous equation. To avoid confusion, we shall not adopt these uses.

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2

First-order Equations

The general form of these equations is

$$c_1 y_t + c_0 y_{t-1} = g(t)\,, \tag{2.1}$$

where c_0, c_1 are given constants and $g(t)$ is a known function. The constants c_0, c_1 must be both different from zero, since if even only one of them is zero the equation is no longer a difference equation.

Let us begin with the study of the homogeneous equation, whose form is

$$c_1 y_t + c_0 y_{t-1} = 0\,, \tag{2.2}$$

or

$$y_t + b y_{t-1} = 0\,, \tag{2.3}$$

where $b \equiv c_0/c_1$. Suppose that in the initial period (i.e. for $t = 0$) the function y takes on an arbitrary value A; from eq. (2.3) we can then compute the following sequence:

$$y_1 = -by_0 = -bA\,,$$

$$y_2 = -by_1 = -b(-bA) = b^2A \ ,$$

$$y_3 = -by_2 = -b(b^2A) = -b^3A \ ,$$

$$y_4 = -by_3 = -b(-b^3A) = b^4A \ ,$$

$$\cdots\cdots\cdots\cdots$$

and so the solution appears to be

$$y_t = A(-b)^t \ . \tag{2.4}$$

As a check, substitute this function in eq. (2.3):

$$A(-b)^t + bA(-b)^{t-1} = 0 \ . \tag{2.5}$$

If our function is a solution, eq. (2.5) must hold identically. Now, since

$$bA(-b)^{t-1} = -(-b)A(-b)^{t-1} = -A(-b)^t \ ,$$

eq. (2.5) can be written as

$$A(-b)^t - A(-b)^t = 0 \ , \tag{2.6}$$

and is indeed satisfied for any value of t.

Since the function we have found satisfies the difference equation and contains one arbitrary constant, we may conclude from general principles that it is the general solution.

The problem remains of how to determine the arbitrary constant. To do this we need an additional condition. This need derives from the fact that relation (2.4) gives only the *form* of the function y_t but not its position in the Cartesian plane (t, y_t). As soon as the function is constrained to pass through a given point, say (t^*, y^*), its position, which depends on one arbitrary constant only, is determined and the arbitrariness of the constant disappears. More formally, the additional condition says that $y_t = y^*$ for $t = t^*$, where t^* and y^* are known values. Substituting these values in (2.4) we get $y^* = A(-b)^{t^*}$ and so

$$A = y^*/(-b)^{t^*} \ . \tag{2.7}$$

In economic problems the value of y in the initial period is usually assumed as known, at least in principle, i.e. $y_t = y_0$ for $t = 0$, which gives $A = y_0$. In this case, we speak of the *initial condition*.

The behaviour over time of the function $y_t = A(-b)^t$ depends on the sign and on the absolute value of the parameter b.

As for the sign, if b is negative then $-b$ is positive and the movement is monotonic. On the other hand, if b is positive then $-b$ is negative and the values of the function will alternate in sign, since the power of a negative number is positive (negative) if the exponent is even (odd). This case is usually described as an 'oscillatory' movement. However, to distinguish terminologically this kind of movement from the trigonometric (sine or cosine) oscillations (which, as we shall see, can arise only in second- or higher-order equations), we suggest the expression 'improper oscillations' or 'alternations'. 'Proper oscillations' or simply 'oscillations' would then specifically indicate trigonometric oscillations.

As for the absolute value, if b is in absolute value less (greater) than unity, the movement will be convergent (divergent). This conclusion is a consequence of the properties of powers: the absolute value of a power, as the exponent increases, tends to zero (to infinity) if the absolute value of the base is less (greater) than one. In the particular case of $|b| = 1$, the function shows improper oscillations of constant amplitude (when $b = 1$) or takes on the constant value A (when $b = -1$).

In fig. 2.1 all kinds of movements are shown (A is assumed to be positive; if it were negative, the qualitative behaviour of the solution would not change). Note that the diagrams show only a succession of points. This is because, as we know, t varies over a set of equally spaced values (0, 1, 2, 3, etc.), and so the solution function is defined only corresponding to equally spaced values of t. The graphical counterpart of this is a succession of points.

Of course, in reality time is a continuous variable. When we formalize an economic problem in difference equations terms, we (implicitly or explicitly) assume that, to all relevant purposes, only what happens at the end of each time interval does matter, so that the variables we are analysing may be thought of as varying by discrete 'jumps'. What happens during the period is not considered, in the sense that all relevant economic activity of each period is assumed to be concentrated in a single point of time (the end of the period, which is the same as the beginning of the following period). These assumptions may or may not be justified according to the nature of the problem we are examining; for some further comments on this point, as well as on the related point of the use of discrete or of continuous time tools in economics, see Part II, ch. 1 (at the end), Part II, ch. 3, §2 (at the end) and Part III, ch. 4, §3 (at the beginning).

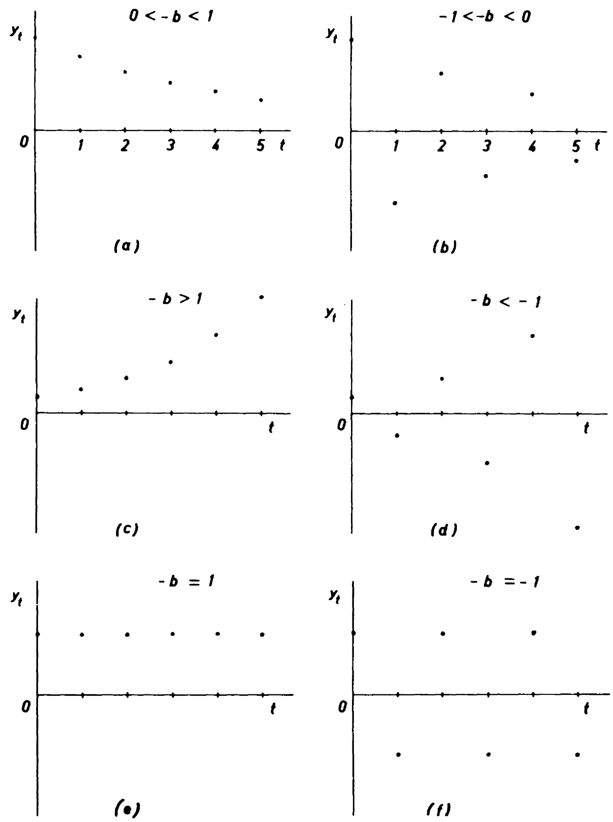

Fig. 2.1. (a) Monotonic and convergent; (b) Oscillatory and convergent; (c) Monotonic and divergent; (d) Oscillatory and divergent; (e) Constant; (f) Oscillatory with constant amplitude.

Going back to the diagrams, the points are usually joined with segments. Fig. 2.2 shows two alternative ways of doing this (diagram (b) of fig. 2.1 is exemplified). It must be emphasized that *the joining of the successive points is performed only to help the eye to follow the movement of the solution over time*. It would be a gross mistake to interpret the segments as describing the movement of y_t in <u>each</u> instant of the <u>period</u>: it is *not* possible to say, for example, that for $t = \overline{0B}$ the value of y is $\overline{0C}$. Such an inference would be wrong, since y_t is defined only for $t = 0, 1, 2, 3, \ldots$, as represented in fig. 2.1. If that is understood, graphical representations of the kind depicted in fig. 2.2 may be adopted safely as a visual aid.