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ELASTIC STABILITY OF CIRCULAR CYLINDRICAL SHELLS

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INTRODUCTION

Buckling of circular cylindrical shells has posed baffling problems to engineering for many years. In the elastic domain the problem may now be considered to be solved completely, thanks to the efforts of numerous authors including the writer of the present book Professor Yamaki who has contributed the most extensive and accurate theoretical and experimental data up to the present time. His work will be the standard reference for elastic stability, buckling and post-buckling behaviour of isotropic circular cylindrical shells for many years to come.

W.T. Koiter

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PREFACE

For the design of light-weight structures, it is of great technical importance to clarify the elastic stability of circular cylindrical shells under various loading conditions. Hence, numerous researches have been made on this subject since the beginning of this century along with the development of aircraft structures. In the early stage of the relevant researches, only approximate solutions were obtained under special loading and boundary conditions, owing to the inherent mathematical difficulty and physical complexity. Experimental studies had also been conducted with thin-walled metal test cylinders, but the results were not precise enough to examine and to improve the corresponding theoretical analyses, due to the deteriorating effect of both initial imperfections and plastic deformations.

With the advent of high-speed digital computers in the 1960s, it became possible to solve the buckling problem with sufficient accuracy and effects of boundary conditions and further those of prebuckling edge rotations have been pursued under various loading conditions. Experimental techniques have also made a great progress , and nearly perfect test cylinders as well as highly elastic cylinders sustainable fairly large deformations became available , leading to the verification of reasonable agreement between theory and experiment , not only for the buckling problem but also for the postbuckling behaviors.

This book presents a comprehensive treatise on the elastic stability of circular cylindrical shells, which represents the sum of the past 17 years of research conducted at the Institute of High Speed Mechanics, Tohoku University . Only the static conservative problems are treated concerning the unstiffened cylinders made of homogeneous, isotropic elastic material with constant thickness. Both theoretical and experimental studies were performed on the buckling, postbuckling and initial-postbuckling problems under typical single or combined loadings, paying due attention to the effect of boundary conditions. Emphases were placed on the accurate analyses, precise tests and extensive presentations of both theoretical and experimental results, to provide fundamental data for the basic problems on the elastic stability of cylindrical shells . No attempt is made to give a complete bibliography, but only the papers closely related to the specific problems studied in the book are cited at appropriate places.

In the first chapter, typical nonlinear theories of circular cylindrical shells are described which constitute the theoretical foundations of the ensuing analyses throughout the book. Chapter 2 deals with the buckling problem. First, the basic equations, the homogeneous linear equations for the eigenvalue problem, are derived on the basis of the relevant nonlinear theories, which are applied to the buckling of cylindrical shells subjected to one of the three fundamental loads, i.e., the torsional, pressure and axially compressive loads. Eight sets of boundary conditions are considered and the critical load and corresponding mode are clarified for a wide range of the shell geometry, taking the effect of prebuckling edge rotations into consideration. Most of the analyses are based on the Donnell equations , the validity of which is examined through application of the Flügge equations.

Chapter 3 is devoted to the postbuckling problems of completely clamped cylindrical shells subjected to one of the three fundamental loads. In each case, experimental results are first presented, carefully conducted by using six polyester test cylinders, and then the corresponding theoretical results are given , obtained by applying the Galerkin method to the Donnell nonlinear equations. Reasonable agreements between theory and experiment are revealed. Analyses for the initial postbuckling behaviors and imperfection sensitivities corresponding to the same cases as in the foregoing are presented in Chapter 4. Under each loading condition, the problem is first solved by applying the Galerkin procedure directly to the Donnell nonlinear equations and then asymptotic solutions are obtained through a perturbation procedure, thus clarifying the degrading effect of initial imperfections in the shape of the buckling mode as well as the range of applicability of the so-called initial postbuckling theory originated by Koiter and developed by Budiansky.

Buckling and postbuckling problems under combined loads are treated in Chapter 5, in which the combined actions of hydrostatic pressure together with the torsional, axial and transverse edge loads, respectively, are considered. Finally, effects of the contained liquid on the buckling and postbuckling of clamped cylindrical tanks under each of the three fundamental loads are examined in Chapter 6. In each case above stated, the buckling problem is theoretically analysed and experimental results are presented for typical postbuckling behaviors checking the accuracy of the critical load theoretically determined. Both theoretical and experimental results are given for the postbuckling problems under the first two loading conditions in Chapter 5, demonstrating fairly good agreement between theory and experiment.

Thin-walled circular cylindrical shells have been more and more extensively used in many different branches of engineering as most efficient structural members, and the author hopes this book to be beneficial to deepen the basic understanding of the complex stability characteristics of this structure and to assess the validity of other numerical procedures such as those utilizing the finite element method.

The author wishes to acknowledge his sincere gratitude to Professors Koiter and Budiansky , Editors of the North-Holland Series in Applied Mathematics and Mechanics, for their suggestion to write this volume and for their kind remarks on the manuscript. He is also thankful to Drs. Sevenster, Mathematical Editor at North-Holland, for his courteous and efficient collaboration.

The author is indebted to all of his associates, staffs and students for their contributions, cooperations and assistances during the past two decades. He appreciates the collaborations of Drs. J. Tani, S. Kodama and H. Doki, in writing the portions of the book related to sections 4.4 and 4.5 , 5.2 through 5.7 and 6.2 through 6.7 , respectively. He is especially thankful to Messrs. K. Otomo and T. Sato for preparing the drawings, to Mr. K. Asano for making the photographs, to Mrs. K. Tsuchiya and Miss H. Hoshi for typing the manuscripts and to Messrs. S. Kodama, K. Otomo and T. Sato for their help in editing the final manuscript.

Noboru YAMAKI

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CHAPTER 1

NONLINEAR THEORY OF CIRCULAR CYLINDRICAL SHELLS

1.1 INTRODUCTION

When an elastic body is subjected to a small deformation in which displacements as well as derivatives of displacements are small, that is, deformation with small rotations and small strains, we will have linear expressions for both displacement-strain relations and strain-stress relations and the equilibrium conditions can be derived at the original undeformed state neglecting the effect of displacements. Thus, the basic equations governing the deformation of the body become linear in terms of displacement, resulting in the classical linear theory of elasticity [1,2]. When the body is subjected to a large or finite deformation in which either the rotations or strains are not small enough in comparison with unity, the above assertions cease to hold in general and the linear theory becomes inadequate. In particular for deformation with small strains but large rotations, the linear stress-strain relations remain valid but the nonlinear effect of rotations should be considered in the displacement-strain relations. Further, the equilibrium conditions should be examined at the deformed state considering the effect of displacements. The resulting basic equations will be nonlinear in terms of displacement, leading to the nonlinear theory of elasticity [3,4]. In contrast to the cases under the linear theory of elasticity, uniqueness of solution as well as the stability of equilibrium state can not be generally assured on the basis of the nonlinear theory of elasticity. In other words, we may have several different equilibrium configurations under the same loading and boundary conditions, some of which are stable and the others unstable.

Of course only the stable equilibrium state can be realized in the physical world. There have been long debates on the classification, definition and criterion of the stability of elastic systems [5,6,7] and although the mathematical theory of elastic stability has been established by Liapounov for a discrete system, its extension and generalization to a continuous system, i.e., elastic bodies, does not seem to have been accomplished [8]. However, in case when an elastic body is subjected to a static conservative load, the so-called energy criterion is generally accepted for the verification of stability, which requires the total potential energy of the body to assume a relative minimum at the equilibrium position.

With the advent of aircraft in the beginning of this century, numerous researches have been conducted to develop most effective structures in weight and stiffness, leading to the present-day light-weight structures which are increasingly used in almost every field of industry. In general, the light-weight structures are composed of slender columns and thin-walled plates shells, which are stiff in axial or in-plane deformations but flexible in bending deformations. Since these structural members can be easily deformed into states with finite rotations within the range of small strains, they are susceptible to various instability phenomena. In fact, when they are subjected to axial or in-plane forces, they often lose stability at fairly low stress levels, resulting in large bending deformations. The loss of stability is usually associated with either an extremal of the equilibrium load or branching of a new equilibrium configuration, which are called limit point buckling and bifurcation buckling, respectively. Thus, the buckling problem to determine the critical load and to clarify the ensuing behavior after buckling has been one of the most important problems for the development of light-weight structures.

It is quite difficult to solve the foregoing buckling problem through a direct application of the general nonlinear theory of elasticity. On the other hand, the problem of practical interest is generally restricted to comparatively small finite deformation of elastic beams, plates and shells, and for

each of these structural members, linear bending theories have been established for approximate analyses within the small deformation range [9-15]. Hence, as the basic equations for the buckling problem, the corresponding nonlinear theories have been developed, taking the effect of the foregoing small finite deformation into consideration. Based on these, numerous buckling problems [10, 16-20] as well as postbuckling problems [21-24] have been formulated and solved under various loading and boundary conditions. Through these analyses, however, the general theory of the elastic stability had not been duly explored.
In 1945, Koiter [25] originated the so-called initial postbuckling theory concerning the bifurcation buckling of elastic bodies subjected to static conservative loads. In this theory, the stability at the bifurcation point is systematically clarified with the asymptotic analysis of the total potential energy of the system, through which the initial postbuckling behavior as well as the effect of small initial imperfections on the critical load are reasonably predicted. Later, the theory was further developed and refined in connection with elastic continuous system [26-28] as well as discrete system with generalized coordinates [29-31], which have been successfully applied to clarify the initial postbuckling behavior together with the imperfection sensitivity of a variety of elastic systems.

In addition to the afore-mentioned traditional buckling problems, we have the stability problems under non-conservative loads [32,33] as well as those under various dynamic loads [34,35]. Further, the problems associated with solid-fluid interaction have attracted increasing interests recently among structural researchers in various industrial fields [36,37]. In contrast to the problems under conservative static loading for which the static energy method is applicable, these problems should be solved by examining the dynamic response of the system after the application of perturbation, which makes the analysis much more complicated. Besides, it is more difficult to define the stability of motion properly. In spite of these difficulties, long-range intensive studies are expected to continue, because of the practical importance of these problems.

The purpose of this book is to clarify the whole aspect of the basic problems concerning the elastic stability of circular cylindrical shells under typical loading conditions. Numerous researches have been made on this subject since the thin-walled circular cylindrical shell constitutes a fundamental structural element most widely used in the light-weight structures. However, owing to its mathematical difficulty together with physical complexity, accurate results, both theoretical and experimental, have become available only recently with the advent of high speed computers and highly elastic test materials.

Because of space limitations, we shall deal with only the buckling problems under static conservative forces, that is, the buckling, postbuckling and initial postbuckling problems under one of the three fundamental loads as well as the buckling and postbuckling problems under the influence of either the combined loads or the contained liquid. The emphases are placed on the accurate analysis and comprehensive numerical results for the buckling problem, experimental verification of the theoretical analysis for the postbuckling problem and clarification of the range of applicability of the perturbation method for the initial postbuckling problem.

In this chapter, we shall briefly explain the typical nonlinear theories of circular cylindrical shells, that is, those developed by Donnell , Flügge and Sanders, which will provide the governing equations for the ensuing analyses throughout the book.

1.2 DONNELL THEORY

Donnell's nonlinear theory of circular cylindrical shells was established by Donnell in 1933, in connection with the analysis of torsional buckling of thin-walled tubes [38]. Owing to its relative simplicity and practical accuracy, this theory has been most widely used for analysing both buckling and postbuckling problems, despite criticisms concerning its applicability.

We shall consider moderately large deformation of a circular cylindrical shell with radius R, length L and thickness h, which is made of homogeneous, isotropic elastic material with Young's modules E and Poisson's ratio ν. Along the middle surface of the shell, the coordinate system is taken as shown in Fig. 1.1, and the displacement components will be denoted by U, V and W, respectively. The Donnell theory is based on the following assumptions:

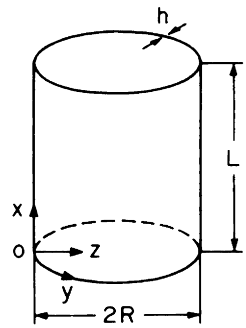

Fig. 1.1 Shell geometry and coordinate system.

(1) The shell is sufficiently thin, i.e., $h/R \ll 1$, $h/L \ll 1$.

(2) The strains ε are sufficiently small, $\varepsilon \ll 1$, and Hooke's law holds.

(3) Straight lines normal to the undeformed middle surface remain straight and normal to the deformed middle surface with their length unchanged.

(4) The normal stress acting in the direction normal to the middle surface may be neglected in comparison with the stresses acting in the direction parallel to the middle surface.

(5) Displacements U and V are infinitesimal, while W is of the same order as the shell thickness, that is, $|U| \ll h$, $|V| \ll h$, $|W| = 0(h)$.

(6) The derivatives of W are small, but their squares and productes are of the same order as the strain here considered. Hence,

$$\{|\frac{\partial W}{\partial x}|, |\frac{\partial W}{\partial y}|\} \ll 1, \quad \{(\frac{\partial W}{\partial x})^2, |\frac{\partial W}{\partial x}\cdot\frac{\partial W}{\partial y}|, (\frac{\partial W}{\partial y})^2\} = 0(\varepsilon).$$

(7) Curvature changes are small and the influences of U and V are negligible so that they can be represented by linear functions of W only.

The assumptions (3) and (4) constitute the so-called Kirchhoff -Love hypotheses while those from (5) to (7) correspond to the shallow shell approximations applicable for deformations dominated by the normal displacement W.

Based upon the foregoing assumptions, we have the strain-displacement relations in the shell as

$$\varepsilon_x = \varepsilon_{x0} + z\kappa_x\,, \quad \varepsilon_y = \varepsilon_{y0} + z\kappa_y\,, \quad \gamma_{xy} = \gamma_{xy0} + z\kappa_{xy}\,, \tag{1.2.1}$$

where

$$\left.\begin{aligned} \varepsilon_{x0} = U_{,x} + \frac{1}{2} W_{,x}^2, \qquad \varepsilon_{y0} = V_{,y} - R^{-1}W + \frac{1}{2} W_{,y}^2, \\ \gamma_{xy0} = U_{,y} + V_{,x} + W_{,x} W_{,y}\,, \end{aligned}\right\} \tag{1.2.2}$$

$$\kappa_x = -W_{,xx}, \quad \kappa_y = -W_{,yy}, \quad \kappa_{xy} = -2W_{,xy}\,. \tag{1.2.3}$$

In the foregoing, subscripts following a comma stand for partial differentiation. The stress-strain relations are given by

$$E\varepsilon_x = \sigma_x - \nu\sigma_y, \quad E\varepsilon_y = \sigma_y - \nu\sigma_x, \quad \frac{E}{2(1+\nu)}\gamma_{xy} = \tau_{xy},$$

from which the stresses in the shell become

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)\,, \quad \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)\,, \quad \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}\,. \tag{1.2.4}$$

Here we define the stress resultants and stress couples per unit length, acting along the x = const. and y = const. sections, as

$$\begin{aligned} (N_x, N_{xy}, Q_x) &= \int_{-h/2}^{h/2} (\sigma_x, \tau_{xy}, \tau_{xz})\,dz\,, \\ (N_{yx}, N_y, Q_y) &= \int_{-h/2}^{h/2} (\tau_{yx}, \sigma_y, \tau_{yz})\,dz\,, \\ (M_x, M_{xy}) &= \int_{-h/2}^{h/2} (\sigma_x, \tau_{xy})\,z\,dz\,, \\ (M_{yx}, M_y) &= \int_{-h/2}^{h/2} (\tau_{yx}, \sigma_y)\,z\,dz\,, \end{aligned} \tag{1.2.5}$$

which lead to

$$N_x = J(\varepsilon_{x0} + \nu\varepsilon_{y0})\,, \quad N_y = J(\varepsilon_{y0} + \nu\varepsilon_{x0})\,, \quad N_{xy} = N_{yx} = J\cdot\frac{1-\nu}{2}\gamma_{xy0}\,, \tag{1.2.6}$$

$$M_x = D(\kappa_x + \nu\kappa_y)\,, \quad M_y = D(\kappa_y + \nu\kappa_x)\,, \quad M_{xy} = M_{yx} = D\cdot\frac{1-\nu}{2}\kappa_{xy}\,. \tag{1.2.7}$$

In the foregoing, we have introduced the notations

$$J = \frac{Eh}{1-\nu^2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}, \tag{1.2.8}$$

which stand for the extensional and flexural rigidities of the shell, respectively.

Now we shall derive the basic equations through a variational principle. The elastic strain energy, $U_e(U, V, W)$, will be given by

$$\begin{aligned} U_e &= \frac{1}{2}\int_0^L\int_0^{2\pi R}\int_{-h/2}^{h/2}(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \tau_{xy}\gamma_{xy})\,dx\,dy\,dz \\ &= \frac{E}{2(1-\nu^2)}\int_0^L\int_0^{2\pi R}\int_{-h/2}^{h/2}(\varepsilon_x^2 + \varepsilon_y^2 + 2\nu\varepsilon_x\varepsilon_y + \frac{1-\nu}{2}\gamma_{xy}^2)\,dx\,dy\,dz, \end{aligned} \tag{1.2.9}$$

while, under the assumption of the conservative loading, the potential of external forces, $V_f(U, V, W)$, may be expressed as

$$\begin{aligned} V_f &= -\int_0^L\int_0^{2\pi R}(p_xU + p_yV + pW)\,dx\,dy \\ &\quad -\int_0^{2\pi R}[P_x^*U + P_y^*V + P_z^*W - M_x^*W_{,x}]_{x=0}^{x=L}\,dy, \end{aligned} \tag{1.2.10}$$

where p_x, p_y and p are the x, y and z components, respectively, of the distributed forces per unit area of the shell. Further, P_x^*, P_y^* and P_z^* are the components of the external loads, while M_x^* is the external bending moment, each per unit length, applied along the edges. The total potential energy $\Pi(U, V, W)$ is given by $\Pi = U_e + V_f$. When the shell is in equilibrium, the variation of the total potential energy assumes a stationary value in the virtual displacement consistent with the prescribed geometrical constraint along the boundaries. Thus, we have

$$\delta\Pi = \delta U_e + \delta V_f = 0, \tag{1.2.11}$$

where

$$\begin{aligned} \delta U_e &= \int_0^L\int_0^{2\pi R}\int_{-h/2}^{h/2}(\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy})\,dx\,dy\,dz \\ &= \int_0^L\int_0^{2\pi R}(N_x\delta\varepsilon_{x0} + N_y\delta\varepsilon_{y0} + N_{xy}\delta\gamma_{xy0} \\ &\qquad + M_x\delta\kappa_x + M_y\delta\kappa_y + M_{xy}\delta\kappa_{xy})\,dx\,dy, \end{aligned}$$

$$\delta V_f = -\int_0^L\int_0^{2\pi R}(p_x\delta U + p_y\delta V + p\delta W)\,dx\,dy$$
$$- \int_0^{2\pi R}[P_x^*\delta U + P_y^*\delta V + P_z^*\delta W - M_x^*\delta W_{,x}]_{x=0}^{x=L}\,dy\,.$$

With the help of a Gauss's theorem, the foregoing condition leads to an equation in the following form:

$$\int_0^L\int_0^{2\pi R}[L_1\delta U + L_2\delta V + L_3\delta W]\,dx\,dy$$
$$+ \int_0^{2\pi R}[B_1\delta U + B_2\delta V + B_3\delta W + B_4\delta W_{,x}]_{x=0}^{x=L}\,dy = 0. \qquad (1.2.12)$$

Hence, by setting $L_i = 0$ $(i = 1,2,3)$, we obtain the equilibrium equations as

$$\left.\begin{aligned} &N_{x,x} + N_{xy,y} + p_x = 0,\\ &N_{xy,x} + N_{y,y} + p_y = 0, \end{aligned}\right\} \qquad (1.2.13)$$

$$M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + R^{-1}N_y$$
$$+ (N_xW_{,x} + N_{xy}W_{,y})_{,x} + (N_{xy}W_{,x} + N_yW_{,y})_{,y} + p = 0,$$

where the last equation becomes

$$D\nabla^4W - R^{-1}N_y - N_xW_{,xx} - 2N_{xy}W_{,xy} - N_yW_{,yy} - p + p_xW_{,x} + p_yW_{,y} = 0, \qquad (1.2.14)$$

in which

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2. \qquad (1.2.15)$$

Further, the natural boundary conditions will be given by $B_i = 0$ $(i = 1\sim 4)$, along the boundary where the displacements and/or the rotation are not specified. Hence, appropriate boundary conditions along $x = 0$ and $x = L$ may be given by

$$\left.\begin{aligned} &N_x = P_x^* \quad \text{or} \quad U = U^*,\\ &N_{xy} = P_y^* \quad \text{or} \quad V = V^*,\\ &M_{x,x} + 2M_{xy,y} + N_xW_{,x} + N_{xy}W_{,y} = P_z^* \quad \text{or} \quad W = W^*,\\ &M_x = M_x^* \quad \text{or} \quad W_{,x} = W_x^*, \end{aligned}\right\} \qquad (1.2.16)$$

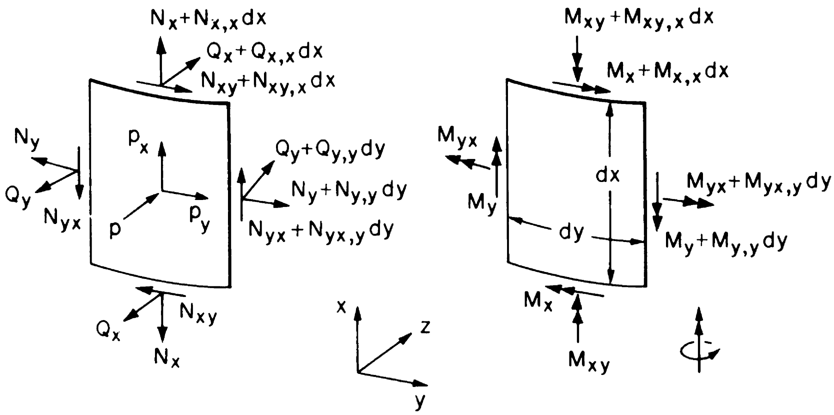

Fig. 1.2 Forces and moments acting on the shell element.

where U^*, V^*, W^* and W^*_x, respectively, are the prescribed values of the displacement components and the rotation along the boundary. Various boundary conditions can be constructed by selecting one condition from each pairs in equations (1.2.16). It is to be added that the foregoing equations (1.2.13) and (1.2.14) can also be derived by examining the equilibrium of an infinitesimal shell element after deformation. We notice that the intensity as well as the positive direction of the forces and moments acting on the shell element are as shown in Fig. 1.2. Then, the equilibrium conditions of the moments about the x and y axes yield the expressions

$$\left.\begin{aligned} Q_x &= M_{x,x} + M_{yx,y} = -D\frac{\partial}{\partial x}\nabla^2 W, \\ Q_y &= M_{xy,x} + M_{y,y} = -D\frac{\partial}{\partial y}\nabla^2 W, \end{aligned}\right\} \qquad (1.2.17)$$

while those of the forces in the x, y and z directions, together with these expressions, lead to the same results as before.

The aforestated equilibrium equations and the boundary conditions are the required Donnell basic equations for analysing moderately large deformations of cylindrical shells. The basic equations for analysing nonlinear free vibrations of the shell are given by replacing p_x, p_y and p with $-\rho h U_{,tt}$, $-\rho h V_{,tt}$ and $-\rho h W_{,tt}$, respectively, where ρ is the density of the shell and t is time. It is to be noted that the equilibrium equations represent a set of three nonlinear partial differential equations in terms of U, V and W.

In case when $p_x = p_y = 0$, both equilibrium equations (1.2.13) are identically satisfied with the use of the stress function F defined by

$$N_x = F_{,yy}, \qquad N_y = F_{,xx}, \qquad N_{xy} = -F_{,xy}, \tag{1.2.18}$$

while the following relations will be obtained from (1.2.6).

$$\left.\begin{aligned} &Eh[U_{,x} + (1/2)W_{,x}^2] = N_x - \nu N_y = F_{,yy} - \nu F_{,xx}, \\ &Eh[V_{,y} - R^{-1}W + (1/2)W_{,y}^2] = N_y - \nu N_x = F_{,xx} - \nu F_{,yy}, \\ &Eh(U_{,y} + V_{,x} + W_{,x}W_{,y}) = 2(1+\nu)N_{xy} = -2(1+\nu)F_{,xy}. \end{aligned}\right\} \tag{1.2.19}$$

Eliminating U and V from these, we obtain the compatibility condition as

$$\nabla^4 F + Eh(R^{-1}W_{,xx} - W_{,xy}^2 + W_{,xx}W_{,yy}) = 0, \tag{1.2.20}$$

while the remaining equilibrium equation is rewritten as

$$D\nabla^4 W - R^{-1}F_{,xx} - F_{,yy}W_{,xx} + 2F_{,xy}W_{,xy} - F_{,xx}W_{,yy} - p = 0. \tag{1.2.21}$$

Equations (1.2.20) and (1.2.21) constitute another set of the Donnell's basic equations with two unknown functions F and W, which seem to be more convenient in practical applications than the preceding ones. Finally, it is to be noted that in case when R becomes indefinitely large, these equations reduce to the well-known Kármán equations for large deflections of thin plates.

1.3 MODIFIED FLÜGGE THEORY

Donnell's theory has a deficiency inherent to the shallow shell approximation that is not applicable to the analysis of the deformations of a cylinder in which the magnitude of the in-plane displacement is of the same order as that of the deflection, for example, bending deformations of a long cylinder with the circumferential wave number N less than four. On the other hand, Flügge derived basic equations for the buckling of circular cylindrical shells under typical loading conditions [39], without resort to the shallow shell approximation. These equations are applicable to the problem with any buckling configuration, including the Euler buckling of long shells under axial compression. However, they are not sufficiently accurate in the sense that the prebuckling state is assumed to be a membrane stress state, neglecting the effect of bending deformation. With the main object of obtaining more accurate governing equations for the buckling problem, we shall derive the nonlinear basic equations of the cylindrical shell, on the basis of assumptions similar to those adopted in the fore-stated Flügge equations.

In addition to the assumptions (1) to (4) stated in the preceding section, i.e., those for thinness of the shell, small strains and Kirchhoff-Love hypotheses, we assume the following :

(1) In deriving expressions for stress resultants, we retain terms with orders up to $(h/R)^2$ from unity.

(2) The rotations are moderately small but the effect of their product and squares on the mid-surface strains will be considered.

(3) The curvature changes are small enough to allow linearized expressions for the bending moment.

The foregoing assumptions seem to be valid at least for finite deformations immediately after buckling.

We take the coordinate system of the cylinder as shown in Fig. 1.1 and denote the displacement components by U, V and W, as before. Letting $\tilde{U}$, $\tilde{V}$ and $\tilde{W}$ be the displacement components of the shell along the surface which is distant z from the middle surface, we have

$$\tilde{U} = U - zW_{,x}, \quad \tilde{V} = \frac{R-z}{R} V - zW_{,y}, \quad \tilde{W} = W. \tag{1.3.1}$$

For finite deformations, the corresponding strains may be generally expressed as

$$\varepsilon_x = \tilde{U}_{,x} + \frac{1}{2}(\tilde{U}^2_{,x} + \tilde{V}^2_{,x} + \tilde{W}^2_{,x}),$$

$$\varepsilon_y = \frac{R}{R-z}\tilde{V}_{,y} - \frac{1}{R-z}\tilde{W} + \frac{1}{2}\left(\frac{R}{R-z}\right)^2[\tilde{U}^2_{,y} + (\tilde{V}_{,y} - \frac{1}{R}\tilde{W})^2 + (\tilde{W}_{,y} + \frac{1}{R}\tilde{V})^2],$$

$$\gamma_{xy} = \tilde{V}_{,x} + \frac{R}{R-z}\tilde{U}_{,y} + \frac{R}{R-z}[\tilde{U}_{,x}\tilde{U}_{,y} + \tilde{V}_{,x}(\tilde{V}_{,y} - \frac{1}{R}\tilde{W}) + \tilde{W}_{,x}(\tilde{W}_{,y} + \frac{1}{R}\tilde{V})]. \tag{1.3.2}$$

Retaining the nonlinear terms for only the strain components along the middle surface of the shell, we have

$$\left.\begin{aligned} \varepsilon_x &= U_{,x} - zW_{,xx} + \varepsilon^{(2)}_{x0}, \\ \varepsilon_y &= V_{,y} - \frac{R}{R-z}zW_{,yy} - \frac{1}{R-z}W + \varepsilon^{(2)}_{y0}, \\ \gamma_{xy} &= \frac{R-z}{R}V_{,x} + \frac{R}{R-z}U_{,y} - (1 + \frac{R}{R-z})zW_{,xy} + \gamma^{(2)}_{xy0}, \end{aligned}\right\} \tag{1.3.3}$$

where

$$\left.\begin{aligned} \varepsilon^{(2)}_{x0} &= \frac{1}{2}(U^2_{,x} + V^2_{,x} + W^2_{,x}), \\ \varepsilon^{(2)}_{y0} &= \frac{1}{2}[U^2_{,y} + (V_{,y} - R^{-1}W)^2 + (W_{,y} + R^{-1}V)^2], \\ \gamma^{(2)}_{xy0} &= U_{,x}U_{,y} + V_{,x}(V_{,y} - R^{-1}W) + W_{,x}(W_{,y} + R^{-1}V). \end{aligned}\right\} \tag{1.3.4}$$

The corresponding stresses are

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y), \quad \sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x), \quad \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}, \tag{1.3.5}$$

while the stress resultants and stress couples are defined by

$$\left.\begin{aligned} (N_x, N_{xy}) &= \int_{-h/2}^{h/2}(\sigma_x, \tau_{xy})(1 - \frac{z}{R})\,dz, \\ (N_y, N_{yx}) &= \int_{-h/2}^{h/2}(\sigma_y, \tau_{yx})\,dz, \\ (M_x, M_{xy}) &= \int_{-h/2}^{h/2}(\sigma_x, \tau_{xy})(1 - \frac{z}{R})\,zdz, \\ (M_y, M_{yx}) &= \int_{-h/2}^{h/2}(\sigma_y, \tau_{yx})\,zdz. \end{aligned}\right\} \tag{1.3.6}$$

Performing integration, we finally obtain

$$
\begin{aligned}
N_x &= J[U_{,x} + \nu(V_{,y} - R^{-1}W) + \varepsilon_{x0}^{(2)} + \nu\varepsilon_{y0}^{(2)}] + R^{-1}DW_{,xx}, \\
N_y &= J[V_{,y} - R^{-1}W + \nu U_{,x} + \varepsilon_{y0}^{(2)} + \nu\varepsilon_{x0}^{(2)}] - R^{-1}D(W_{,yy} + R^{-2}W), \\
N_{xy} &= \frac{1-\nu}{2}[J(U_{,y} + V_{,x} + \gamma_{xy0}^{(2)}) + R^{-1}D(R^{-1}V_{,x} + W_{,xy})], \\
N_{yx} &= \frac{1-\nu}{2}[J(U_{,y} + V_{,x} + \gamma_{xy0}^{(2)}) + R^{-1}D(R^{-1}U_{,y} - W_{,xy})], \\
M_x &= -D[W_{,xx} + \nu W_{,yy} + R^{-1}(U_{,x} + \nu V_{,y})], \\
M_y &= -D(W_{,yy} + R^{-2}W + \nu W_{,xx}), \\
M_{xy} &= -(1-\nu)D(W_{,xy} + R^{-1}V_{,x}), \\
M_{yx} &= -(1-\nu)D[W_{,xy} + (1/2R)(V_{,x} - U_{,y})], \qquad (1.3.7)
\end{aligned}
$$

where J and D have been defined by (1.2.8).

The equilibrium equations and the appropriate boundary conditions will be obtained with the use of the stationary principle of the total potential energy as before. The variation of the elastic strain energy U_e is

$$
\delta U_e = \frac{1}{2}\int_0^L\int_0^{2\pi R}\int_{-h/2}^{h/2}(\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \tau_{xy}\delta\gamma_{xy})(1 - \frac{z}{R})\,dx\,dy\,dz\,, \qquad (1.3.8)
$$

while that of the potential of the external forces V_f may be expressed as

$$
\begin{aligned}
\delta V_f = &-\int_0^L\int_0^{2\pi R}\{p_x\delta U + p_y\delta V + p[-W_{,x}\delta U - (W_{,y} + R^{-1}V)\delta V \\
&+ (1 + U_{,x} + V_{,y} - R^{-1}W)\delta W]\}\,dx\,dy \\
&- \int_0^{2\pi R}[P_x^*\delta U + P_y^*\delta V + P_z^*\delta W - M_x^*\delta W_{,x}]_{x=0}^{x=L}\,dy\,, \qquad (1.3.9)
\end{aligned}
$$

where p_x and p_y, respectively, are the x and y components of the distributed force applied per unit area of the undeformed middle surface while p is the intensity of the lateral pressure acting normal to the deformed middle surface per unit deformed area. Further, P_x^*, P_y^*, P_z^* and M_x^*, respectively, are the components of the external load and bending moment applied per unit original length of the shell edges $x = 0$ and $x = L$. Then, from the variational principle

$$\delta U_e + \delta V_f = 0, \tag{1.3.10}$$

we finally obtain the equilibrium equations as follows:

$$[N_x(1+U_{,x})]_{,x} + [N_{yx}(1+U_{,x})]_{,y} + (N_y U_{,y})_{,y} + (N_{xy}U_{,y})_{,x}$$
$$+ p_x - pW_{,x} = 0, \tag{1.3.11a}$$

$$[N_{xy}(1+V_{,y}-R^{-1}W)]_{,x} + [N_y(1+V_{,y}-R^{-1}W)]_{,y}$$
$$- R^{-1}(M_{y,y}+M_{xy,x}) + (N_x V_{,x})_{,x} + (N_{yx}V_{,x})_{,y} - R^{-1}N_{yx}W_{,x}$$
$$+ p_y - (p+R^{-1}N_y)(W_{,y}+R^{-1}V) = 0, \tag{1.3.11b}$$

$$M_{x,xx} + (M_{xy}+M_{yx})_{,xy} + M_{y,yy} + R^{-1}N_y(1+V_{,y}-R^{-1}W)$$
$$+ [N_x W_{,x} + N_{xy}(W_{,y}+R^{-1}V)]_{,x} + [N_{yx}W_{,x} + N_y(W_{,y}+R^{-1}V)]_{,y}$$
$$+ R^{-1}N_{yx}V_{,x} + p(1+U_{,x}+V_{,y}-R^{-1}W) = 0. \tag{1.3.11c}$$

The appropriate boundary conditions at $x = 0$ and $x = L$ are also obtained as

$$N_x(1+U_{,x}) + N_{xy}U_{,y} = P_x^* \quad \text{or} \quad U = U^*,$$
$$N_{xy}(1+V_{,y}-R^{-1}W) + N_x V_{,x} - R^{-1}M_{xy} = P_y^* \quad \text{or} \quad V = V^*,$$
$$M_{x,x} + (M_{xy}+M_{yx})_{,y} + N_x W_{,x} + N_{xy}(W_{,y}+R^{-1}V) = P_z^* \quad \text{or} \quad W = W^*,$$
$$M_x = M_x^* \quad \text{or} \quad W_{,x} = W_x^*. \tag{1.3.12}$$

In the foregoing, U^*, V^*, W^* and W_x^*, respectively, are the prescribed values of the displacement components and the rotation along the boundary. Equations (1.3.11) together with (1.3.12) are the modified Flügge equations for the finite deformation of the cylindrical shell, which represent a set of three coupled nonlinear partial differential equations in U, V and W. The corresponding linear basic equations will be obtained by omitting the nonlinear terms in the expressions (1.3.3) as well as (1.3.7). In this case, the equilibrium equations become