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an introduction to thermomechanics

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AN INTRODUCTION TO THERMOMECHANICS

Hans ZIEGLER

Swiss Federal Institute of Technology, Zürich
and
University of Colorado, Boulder

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PREFACE

Continuum mechanics deals with deformable bodies. In its early stages it was confined to a few special materials and to particular situations, namely to ideal liquids or to elastic solids under isothermal or adiabatic conditions. In these special cases it is possible to solve the basic problem, i.e., to determine the flow and pressure distributions or the deformation and stress fields in purely mechanical terms. This is due to the fact that the solution can be developed from a set of differential equations which does not contain the energy balance.

From the viewpoint of general continuum mechanics, however, problems of this type are singular. Anyone working in this field knows that sooner or later he gets involved in thermodynamics. The reason for this is that in general a complete set of differential equations contains the energy balance. Since part of the energy exchange takes place as heat flow, the appropriate form of the energy balance is the first fundamental law of thermodynamics, and it becomes clear therefore that it is generally impossible to separate the mechanical aspect of a problem from the thermodynamic processes accompanying the motion. To obtain a solution, the fundamental laws of both mechanics and thermodynamics must be applied. In gas dynamics and in thermoelasticity this has long been recognized.

This situation has its counterpart in thermodynamics. Until recently the interest in this field was almost exclusively focused on particularly simple bodies, mainly on inviscid gases, characterized by certain state variables as, e.g., volume, pressure and temperature. In other bodies, however, or if viscosity is to be taken into account, one is compelled to use concepts from continuum mechanics, replacing the volume by the strain tensor and the pressure by the stress tensor. It may even be necessary to have recourse to the momentum theorems, and to account for the kinetic energy in formulating the first fundamental law. In short, thermodynamics cannot be separated from continuum mechanics.

In view of these statements it becomes clear that continuum mechanics and thermodynamics are inseparable: a general theory of continuum mechanics always includes thermodynamics and vice versa. The entire field is truly interdisciplinary and requires a unified treatment, which may properly be denoted as *thermomechanics*. Such a unified treatment is the topic of this book.

In order to amalgamate two branches of science, one needs a common language. Continuum mechanics has always been a field theory, even in its rudimentary forms like hydraulics or strength of materials. To treat even such a simple problem as bending of a beam, one must recognize that the states of strain and stress depend on position and possibly on time. The object of thermodynamics, on the other hand, has always been a finite volume, e.g., a mole, and the state within the body has been tacitly assumed to be the same throughout the entire volume. It is surprising that this philosophy has been maintained even at the age of statistical and quantum mechanics, although it is clearly inconsistent with the first fundamental law in its common form: At least part of the heat supply appearing in this law is due to heat flow through the surface of the body. As long as this process goes on, the temperature of the elements near the surface differs from the one of the elements further inside the body; the state of the body is therefore not homogeneous.

There are two ways out of this dilemma.

The historical way, still dominating vast areas of teaching in thermodynamics, consists in the restriction to infinitely slow processes. In place of actual processes one considers sequences of (homogeneous) equilibrium states. Except for a few special cases, such idealized processes are practically reversible, and this explains why in classical thermodynamics (or rather thermostatics) the limiting case of reversibility plays such a dominant role. However, the engineer engaged in the construction of thermomechanical machinery cannot limit himself to infinitely slow processes and hence has never taken this restriction seriously. The situation strongly resembles the one in pre-Newtonian mechanics with its attempts to develop dynamics from purely static concepts.

The modern way out of the dilemma is different but surprisingly simple: instead of infinitely slow processes one considers infinitesimal elements of the body in which a process takes place, admitting that the state variables differ from element to element. In other words: one conceives thermodynamics as a field theory in much the same way as continuum mechanics has been treated for more than 200 years. In such a field theory, reasonably fast processes can be treated with the same ease as slow ones, and restriction to reversible processes becomes unnecessary. Finally, this field theory is the proper form in which thermodynamics and continuum mechanics are easily amalgamated.

The strong interdependence of continuum mechanics and thermodynamics was generally recognized about three decades ago. Various schools have since contributed to thermomechanics, each from its point of view and in its own language or formalism. It is not the aim of this book to report on the various approaches nor to compare them. The book is intended as an *introduction* to this fascinating field, based on the simplest possible approach.

Except for an introduction to the theory of cartesian tensors the first three chapters are concerned with the mechanical laws governing the motion of a continuum. They are based on considerations of mass geometry, on the principle of virtual power and on a general form of the reaction principle. It is well known that the most general approach to continuum mechanics makes use of the displacement field and of material, and hence curvilinear, coordinates. For a beginner, however, this approach presents considerable mathematical difficulties that are apt to obscure the physical contents. Since physics deserves priority in an introduction of this type, a treatment based on the velocity field has many advantages and has therefore been preferred. This kind of approach has been presented in a masterly fashion by Prager in his "Introduction to Mechanics of Continua", and since there is not much point in making changes just for the sake of originality, the first three chapters and certain portions of the subsequent applications are similar to the corresponding parts of Prager's book.

Chapter 4 deals with thermodynamics. It starts from the classical representation, familiar from textbooks in this field, introduces and discusses the concept of (independent and dependent) state variables, and shows how the fundamental laws can be formulated in terms of a field theory. A characteristic point of the present treatment is the fact that the stress appears as the sum of a quasiconservative and a dissipative stress. The first is a state function, dependent on the free energy, the second is connected with the dissipation function. In view of later developments (Chapter 14) the role of the two functions is emphasized. The deformation history is represented in the simplest possible manner, namely by internal parameters.

Chapter 5 deals with the characteristic properties of various materials. A rough classification of bodies is presented, and the constitutive equations of some continua are discussed. The general theorems established in the preceding chapters, supplemented by the proper constitutive relations, determine the thermomechanical behavior of a given body. This is illustrated in Chapters 6 through 11, which deal with the application of the theory to various types of continua.

Chapters 12 and 13 contain a short outline of general tensors and their application in the study of large displacements. The representation follows the lines of Green and Zerna in their excellent book on "Theoretical Elasticity". The inclusion of this material makes it possible, in particular, to point out (a) the importance of a proper choice of the strain measure and of the corresponding stress, and (b) the difference between covariant and contravariant components of a tensor, essential for the proof of the orthogonality condition in Chapter 14.

Up to and including Chapter 13 the subject matter, in spite of a personal tinge in the presentation, remains within confines that appear to be generally accepted by now. The remainder of the book transgresses these traditional limits. It may be considered, together with Chapter 4, as a synopsis of the author's contributions to thermomechanics, published from 1957 onwards, occasionally with the assistance of Dr. Jürg Nänni and Professor Christoph Wehrli. It is clear that in a synopsis of this type many points which once seemed essential but have lost their importance can be dropped, and it is equally obvious that many thoughts which once appeared vague have since assumed a more concise form. Incidentally, in a field which is still in a state of development a certain amount of controversy cannot be avoided; in this respect I assume full responsibility for the final chapters.

Chapter 14 returns to the basis of thermodynamics. The classical theory, restricted to reversible processes, tacitly excludes gyroscopic forces. With exactly the same right they may be excluded in the irreversible case. The obvious way of doing this is to assume that the dissipative stresses are determined by the dissipation function alone much in the same way as the quasiconservative forces depend on the free energy. For certain systems, to be called elementary, the connection between dissipative stresses and dissipation function then turns out to have the form of an *orthogonality condition,* and it follows that two scalar functions, the free energy and the dissipation function (or the rate of entropy production) completely govern any kind of process.

Chapter 15 shows that the orthogonality condition is equivalent to a number of extremum principles, among them a principle of maximal rate of entropy production. This last principle suggests a generalization of the orthogonality condition for systems of the so-called complex type. This generalization will be referred to as the *orthogonality principle*, and it is easy to see that it reduces to Onsager's symmetry relations in the linear case. Finally, Chapters 16 through 18 are concerned with applications of the orthogonality condition and the orthogonality principle to various types of continua.

As already mentioned, I have tried to keep the mathematical formalism as simple as possible. I assume, however, that the reader is familiar with vector algebra and analysis, with the basic laws of mechanics and thermodynamics, with the elements of geometry in n-dimensional space and of the theory of functions, and with the notion of convexity. To provide the reader with a means of testing his grasp of the matter, problems have been added at the end of each section wherever this was possible.

In the second edition of this book the thermodynamic aspect of continuum mechanics has been stressed wherever this seemed desirable; besides, some weak points have been strengthened. In Chapter 1 a section dealing mainly with invariants has been added, and in this context the basic invariants of second-order tensors have been redefined. Chapters 11 and 18, dealing with viscoelasticity, have been extended to include thermal effects. The first one appears supplemented by a section, the second one has been completely rewritten. Section 14.4 appears in a new form, as do Chapter 16, on non-Newtonian liquids, and Chapter 17, on plasticity. In Chapter 15 a section dealing with the derivation of the second fundamental law from the orthogonality condition has been added. On the whole, the terminology has been simplified, particularly in connection with the classification of materials (fluids, solids and viscoelastic bodies). Many minor changes have been made, and misprints of the first edition have been eliminated. Most of the problems have been reformulated in such a way that they now show the main results.

I am greatly indebted to Professors William Prager and Warner T. Koiter, who have both critically read the manuscript of the first edition and provided numerous suggestions for improvement. I am also grateful to Professors Ralph C. Koeller and William L. Wainwright for pointing out that some of the applications in Section 15.3 and Chapter 16 lacked

generality. A special word of thanks is due to my son, Professor Hansheinrich Ziegler, for his valuable linguistic assistance in the preparation of the text. I finally express my gratitude to Dr. Carlo Spinedi for his help, particularly in proofreading, and to the Daniel Jenny Foundation for support in the preparation of the drawings.

Zürich, July 1982 | Hans Ziegler

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CHAPTER 1

MATHEMATICAL PRELIMINARIES

In order to describe the *configuration* of an arbitrary body, we need a *reference system,* e.g., a rigid body or frame serving as a basis for the observer. Any quantitative treatment requires a *coordinate system* fixed to this reference frame. Our first task is to develop the mathematical tools needed for the description of the motion or, more generally, of any process in which the body in consideration takes part. The mathematical framework must be consistent with the fact that the choice of the coordinate system is arbitrary. In consequence, our starting point must be the study of coordinate transformations. Restricting ourselves in this chapter to cartesian coordinate systems, we will develop the concept of the cartesian tensor.

1.1. Cartesian tensors

Let us refer (Fig. 1.1) the three-dimensional physical space to a given

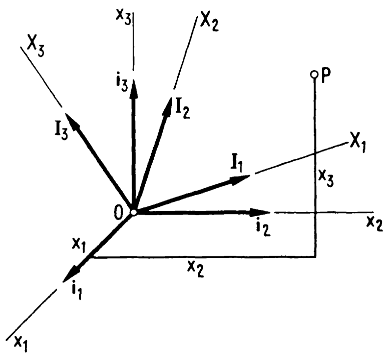

Fig. 1.1. Cartesian coordinate systems.

reference frame and here to a *cartesian,* i.e., rectangular and rectilinear, *coordinate system* x_1, x_2, x_3 with unit vectors $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ along the coordinate axes. The axes X_p ($p = 1, 2, 3$) with unit vectors $\boldsymbol{I}_p$ define another cartesian coordinate system with the same origin O. Denoting the cosines between the axes X_p and x_i by c_{pi}, we have, for arbitrary indices p and i between 1 and 3,

$$c_{pi} = \cos(X_p, x_i) = \boldsymbol{I}_p \cdot \boldsymbol{i}_i. \tag{1.1}$$

Let P be a point with coordinates x_i in the first system. Its coordinates in the second system are the projections of the radius vector (or, equivalently, of the sequence of straight segments representing the x_i) onto the axes X_p. Making use of (1.1), we obtain

$$\begin{aligned} X_1 &= c_{11}x_1 + c_{12}x_2 + c_{13}x_3, \\ X_2 &= c_{21}x_1 + c_{22}x_2 + c_{23}x_3, \\ X_3 &= c_{31}x_1 + c_{32}x_2 + c_{33}x_3 \end{aligned} \tag{1.2}$$

as coordinate transformations between the two coordinate systems. It is easy to see that the inversions are

$$\begin{aligned} x_1 &= c_{11}X_1 + c_{21}X_2 + c_{31}X_3, \\ x_2 &= c_{12}X_1 + c_{22}X_2 + c_{32}X_3, \\ x_3 &= c_{13}X_1 + c_{23}X_2 + c_{33}X_3. \end{aligned} \tag{1.3}$$

A more compact way to write (1.2) and (1.3) is

$$X_p = \sum_{i=1}^{3} c_{pi}x_i, \qquad x_i = \sum_{p=1}^{3} c_{pi}X_p, \tag{1.4}$$

where p is free in the first equation, and i in the second one. We may even dispense of the summation symbol by adopting, once and for all, the so-called *summation convention* stipulating that whenever a letter index appears twice in a product the sum is to be taken over this index. We thus write, in place of (1.4),

$$X_p = c_{pi}x_i, \qquad x_i = c_{pi}X_p. \tag{1.5}$$

It is clear that an index appearing once in a term of an equation like (1.5) must appear in every single term. On the other hand, the summation index is sometimes called a *dummy index* since it may be replaced by any other letter. Such a replacement may become necessary to avoid indices appearing more than twice. To insert $(1.5)_2$ into $(1.5)_1$, e.g., it is necessary to write $(1.5)_2$ in the form

$$x_i = c_{qi} X_q. \tag{1.6}$$

Thus,

$$X_p = c_{pi} c_{qi} X_q \quad \text{and similarly} \quad x_i = c_{pi} c_{pj} x_j, \tag{1.7}$$

where the right-hand sides are double sums.

It is obvious that the coefficient of X_q in $(1.7)_1$ must be 1 for $q=p$ and 0 for $q \neq p$. A similar statement holds for $(1.7)_2$. Introducing the so-called *Kronecker symbol*

$$\delta_{pq} = \begin{cases} 1 & \text{for } p=q, \\ 0 & \text{for } p \neq q, \end{cases} \tag{1.8}$$

we thus have

$$c_{pi} c_{qi} = \delta_{pq}, \qquad c_{pi} c_{pj} = \delta_{ij}. \tag{1.9}$$

These equations might be interpreted as orthonormality conditions; they are valid only in orthogonal coordinate systems.

The c_{pi} may be written as a matrix,

$$c_{pi} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}. \tag{1.10}$$

Here the first index indicates the line, the second the column in which a given element is situated. For any fixed value of p the c_{pi}, appearing in the p-th line of the matrix (1.10), are, according to (1.1), the components of the unit vector $\boldsymbol{I}_p$ in the coordinate system x_i. Thus, the determinant of the matrix is the triple product

$$\det c_{pi} = \boldsymbol{I}_1 \cdot (\boldsymbol{I}_2 \times \boldsymbol{I}_3). \tag{1.11}$$

It follows that

$$\det c_{pi} = \pm 1, \tag{1.12}$$

where the positive sign corresponds to the case where both coordinate systems are right- or left-handed, the negative sign to the case where one of them is right-handed and the other one left-handed. In the first case the second coordinate system is obtained from the first one by a rotation about

O, in the second case a reflection on a plane passing through O must be added.

Making once more use of (1.1), we obtain

$$\boldsymbol{I}_p = (\boldsymbol{I}_p \cdot \boldsymbol{i}_i)\boldsymbol{i}_i = c_{pi}\boldsymbol{i}_i, \qquad \boldsymbol{i}_i = (\boldsymbol{i}_i \cdot \boldsymbol{I}_p)\boldsymbol{I}_p = c_{pi}\boldsymbol{I}_p. \tag{1.13}$$

Comparing this to (1.5), we note that the base vectors of the two cartesian coordinate systems transform as the coordinates of a point (or, equivalently, as the components of its radius vector). In non-cartesian coordinate systems, this would not be true.

Our present interpretation of (1.2) is this: P is a point fixed in space, i.e., in our reference frame, and (1.2) connects its coordinates in different cartesian systems. Another interpretation, to be used later, considers (1.2) as representing a displacement with respect to the reference frame: the coordinate system is fixed and the X_p are the instantaneous positions of the points with original positions x_i. The displacement is obviously a rotation about O, possibly combined with a reflection on a plane passing through O.

A *scalar* λ is a quantity which is independent of the coordinate system. Denoting the corresponding quantity in the system X_p by Λ, we thus have

$$\Lambda = \lambda. \tag{1.14}$$

A *vector* $\boldsymbol{v}$ has a direction and hence three components v_i. The vector itself is independent of the coordinate system; its components transform as the coordinates of a point (the end point of $\boldsymbol{v}$ when the coordinate origin is chosen as the starting point), i.e., according to (1.5),

$$V_p = c_{pi} v_i, \qquad v_i = c_{pi} V_p. \tag{1.15}$$

Thus, a vector might be defined as a triplet of components transforming according to (1.15), and this definition might be used to obtain some of the rules of vector algebra, supplying, e.g., the product $\lambda\boldsymbol{v}$ of a scalar and a vector or the scalar product $\boldsymbol{u}\cdot\boldsymbol{v}$ of two vectors.

Generalizing (1.15), let us define a *cartesian tensor* of order n as a set of 3^n components $t_{ij\ldots l}$ transforming according to

$$T_{pq\ldots s} = c_{pi}c_{qj}\ldots c_{sl}t_{ij\ldots l}, \qquad t_{ij\ldots l} = c_{pi}c_{qj}\ldots c_{sl}T_{pq\ldots s}. \tag{1.16}$$

Note that the order of the tensor is given by the number of its indices. In accordance with this definition, a scalar λ may be considered as a tensor of order zero. A vector is a tensor of order one, *symbolically* denoted by $\boldsymbol{v}$.