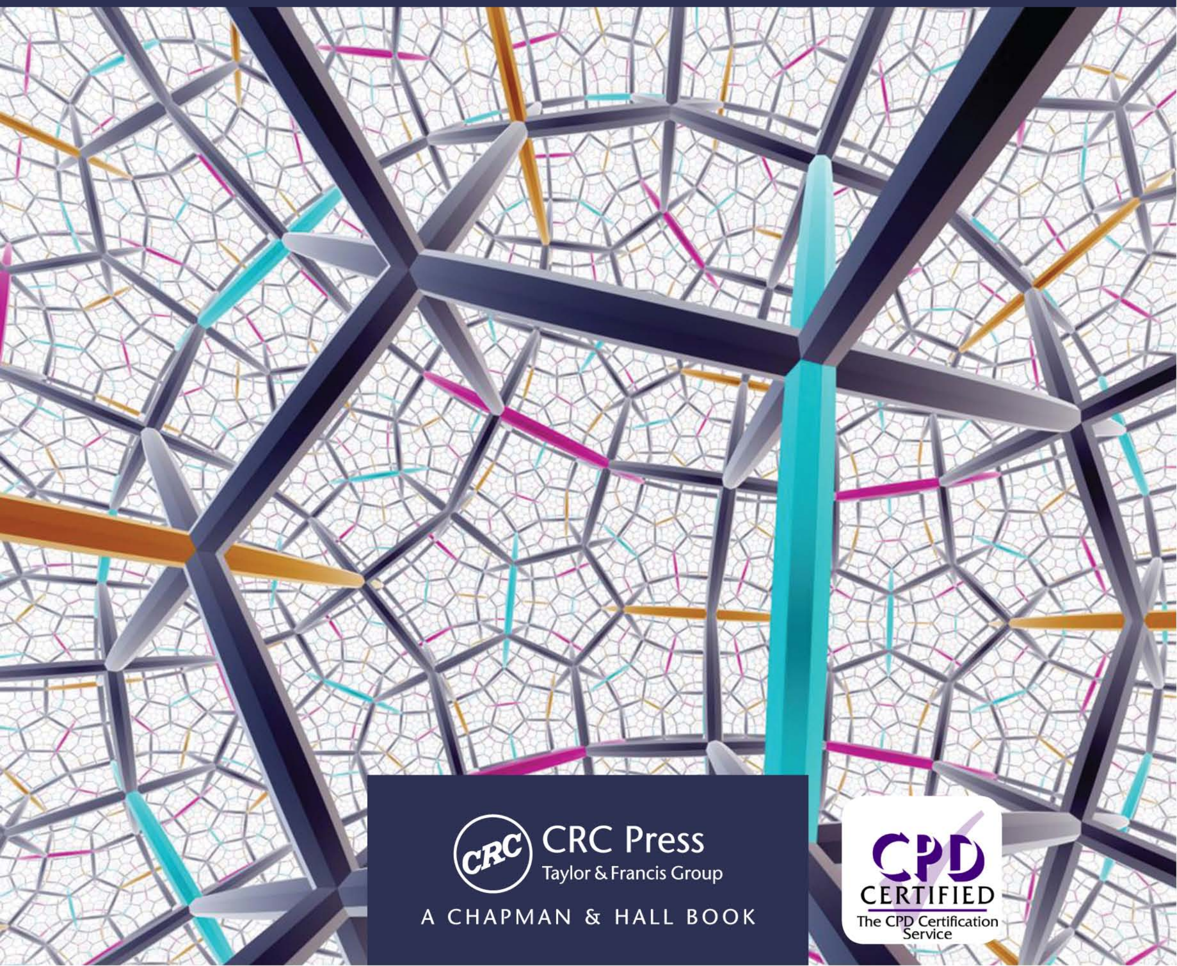
SECOND EDITION

YEARNING FOR THE IMPOSSIBLE

THE SURPRISING TRUTHS OF MATHEMATICS

JOHN STILLWELL



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Yearning for the Impossible

The Surprising Truths of Mathematics

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Preface to the Second Edition

For the past 12 years I have taught a course based on this book at the University of San Francisco. It is a first-year seminar, intended mainly for students in non-mathematical courses. In keeping with the theme of the book, the seminar is entitled "Mathematics and the Impossible," and it has been pitched as follows:

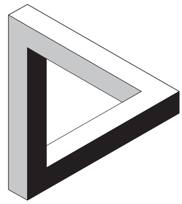
> This course is a novel introduction to mathematics and its history. It puts the difficulties of the subject upfront by enthusiastically tackling the most important ones: the seemingly impossible concepts of irrational and imaginary numbers, the fourth dimension, curved space, and infinity. Similar "impossibilities" arise in music, art, literature, philosophy, and physics—as we will see—but math has the precision to separate actual impossibilities from those that are merely apparent. In fact, "impossibility" has always been a spur to the creativity of mathematicians, and a major influence on the development of math. By focusing reason and imagination on several apparent impossibilities, the course aims to show interesting math to students whose major may be in another field, and to widen the horizons of math students whose other courses are necessarily rather narrowly focused.

Thus the aim of the seminar is to introduce students to the ideas of mathematics, rather than to drill them in mathematical techniques. But (as all mathematicians know) mathematics is not a spectator sport. So it has been my practice to give exercises alongside classroom discussion of the book, to ensure that students grapple with interesting ideas for themselves. Since other teachers may also decide to give courses based on the book, I have decided to produce this second edition, which includes the exercises I have used with my class and many more.

The exercises are distributed in small batches after most sections, and a batch should be attempted by students after the corresponding section has been discussed in class. Some of the exercises are quite routine—intended to reinforce the ideas just discussed—but other exercises serve to extend the ideas and develop them in interesting directions. There are also exercises that fill gaps by supplying proofs of results claimed without proof in the text, or by answering questions that are likely to arise. The more advanced exercises are accompanied by commentary that explains their context and background. In this way I have been able to cover several topics that will be of interest to more ambitious readers.

Of course, it is by no means necessary to use the book in the context of a course. I hope that it will continue to be used for recreational reading or self-study, and that even the casual reader will be tempted to try some of the exercises. However, for those teachers who have wished to give a course based on the book, I hope that teaching from it has now become much easier.

John Stillwell
San Francisco



Preface

The germ of this book was an article I called (somewhat tongue-in-cheek) “Mathematics Accepts the Impossible.” I wrote it for the Monash University magazine *Function* in 1984 and its main aim was to show that the “impossible” figure shown above (the Penrose tribar) is actually not impossible. The tribar exists in a perfectly reasonable space, different from the one we think we live in, but nevertheless meaningful and known to mathematicians. With this example, I hoped to show a general audience that mathematics is a discipline that demands imagination, perhaps even fantasy.

There are many instances of apparent impossibilities that are important to mathematics, and other mathematicians have been struck by this phenomenon. For example, Philip Davis wrote in *The Mathematics of Matrices* of 1965:

> It is paradoxical that while mathematics has the reputation of being the one subject that brooks no contradictions, in reality it has a long history of successfully living with contradictions. This is best seen in the extensions of the notion of number that have been made over a period of 2500 years … each extension, in its way, overcame a contradictory set of demands.

Mathematical language is littered with pejorative and mystical terms such as irrational, imaginary, surd, transcendental that were once used to ridicule supposedly impossible objects. And these are just terms applied to numbers. Geometry also has many concepts that seem impossible to most people, such as the fourth dimension, finite universes, and curved space—yet geometers (and physicists) cannot do without them. Thus there is no doubt that mathematics flirts with the impossible, and seems to make progress by doing so.

The question is: why?

I believe that the reason was best expressed by the Russian mathematician A. N. Kolmogorov in 1943 [31, p. 50]:

> At any given moment there is only a fine layer between the "trivial" and the impossible. Mathematical discoveries are made in this layer.

To put this another way: mathematics is a story of close encounters with the impossible because *all its great discoveries are close to the impossible*. The aim of this book is to tell the story, briefly and with few prerequisites, by presenting some representative encounters across the breadth of mathematics. With this approach I also hope to capture some of the feeling of *ideas in flux*, which is usually lost when discoveries are written up. Textbooks and research papers omit encounters with the impossible, and introduce new ideas without mentioning the confusion they were intended to clear up. This cuts long stories short, but we have to experience some of the confusion to see the need for new and strange ideas.

It helps to know why new ideas are needed, yet *there is still no royal road to mathematics*. Readers with a good mathematical background from high school should be able to appreciate all, and understand most, of the ideas in this book. But many of the ideas are hard and there is no way to soften them. You may have to read some passages several times, or reread earlier parts of the book. If you find the ideas attractive you can pursue them further by reading some of the suggested literature. (This applies to mathematicians too, some of whom may be reading this book to learn about fields outside their specialty.)

As a specific followup, I suggest my book *Mathematics and Its History*, which develops ideas of this book in more detail, and reinforces them with exercises. It also offers a pathway into the classics of mathematics, where you can experience "yearnings for the impossible" at first hand.

Several people have helped me write, and rewrite, this book. My wife, Elaine, as usual, was in the front line; reading several drafts and making the first round of corrections and criticisms. The book was also read carefully by Laurens Gunnarsen, David Ireland, James McCoy, and

Abe Shenitzer, who gave crucial suggestions that helped me clarify my general perspective.

Acknowledgments. I am grateful to the M. C. Escher Company-Baarn-Holland for permission to reproduce the Escher works *Waterfall*, shown in Figure 8.1, *Circle Limit IV*, shown in Figure 5.26, and the two transformations of *Circle Limit IV* shown in Figures 5.26 and 5.27.

I also thank the Artists Rights Society of New York for permission to reproduce the Magritte picture *La reproduction interdite* shown in Figure 8.9.

John Stillwell
South Melbourne, February 2005
San Francisco, December 2005

Contents

Chapter 1

The Irrational

Preview

What are numbers and what are they for? The simplest answer is that they are the *whole numbers* 1, 2, 3, 4, 5, ... (also called the *natural numbers*) and that they are used for counting. Whole numbers can also be used for *measuring* quantities such as length by choosing a unit of measurement (such as the inch or the millimeter) and counting how many units are in a given quantity.

Two lengths can be accurately compared if there is a unit that measures them both exactly—a *common measure*. Figure 1.1 shows an example, where a unit has been found so that one line is 5 units long and the other is 7 units long. We can then say that the lengths are in the *ratio* of 5:7.

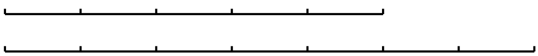


Figure 1.1: Finding the ratio of lengths.

If a common measure exists for any two lines, then any two lengths are in a natural number ratio. *Mathematicians once dreamed of such a world—in fact, a world so simple that natural numbers explain everything. However, this "rational" world is impossible.*

The ancient Greeks discovered that there is no common measure for the side and diagonal of a square. We know that when the side is 1 the diagonal is $\sqrt{2}$, hence $\sqrt{2}$ is *not* a ratio of natural numbers. For this reason, $\sqrt{2}$ is called *irrational*.

Thus $\sqrt{2}$ lies outside the rational world, but is it nevertheless possible to treat irrational quantities as numbers?

1.1 The Pythagorean Dream

> It is clear that two scientific methods will lay hold of and deal with the whole investigation of quantity; arithmetic, absolute quantity, and music, relative quantity.
>
> Nicomachus, *Arithmetic*, Chapter III

In ancient times, higher learning was divided into seven disciplines. The first three—grammar, logic, rhetoric—were considered easier and made up what was called the *trivium* (which is where our word "trivial" comes from). The remaining four—arithmetic, music, geometry, astronomy—made up the advanced portion, called the *quadrivium*. The four disciplines of the quadrivium are naturally grouped into two pairs: arithmetic and music, and geometry and astronomy. The connection between geometry and astronomy is clear enough, but how did arithmetic become linked with music?

According to legend, this began with Pythagoras and his immediate followers, the Pythagoreans. It comes down to us in the writings of later followers such as Nicomachus, whose *Arithmetic* quoted above was written around 100 CE.

Music was linked to arithmetic by the discovery that harmonies between the notes of plucked strings occur when the lengths of the strings are in small whole number ratios (given that the strings are of the same material and have the same tension). The most harmonious interval between notes, the *octave*, occurs when the ratio of the lengths is 2:1. The next most harmonious, the *fifth*, occurs when the ratio of the lengths is 3:2, and after that the *fourth*, when the ratio is 4:3. Thus musical intervals are "relative" quantities because they depend, not on actual lengths, but on the ratios between them. Seeing numbers in music was a revelation to the Pythagoreans. They thought it was a glimpse of something greater: the all-pervasiveness of number and harmony in the universe. In short, *all is number*.

We now know that there is a lot of truth in this Pythagorean dream, though the truth involves mathematical ideas far beyond the natural numbers. Still, it is interesting to pursue the story of natural numbers in music a little further, as later developments clarify and enhance their role.

The octave interval is so harmonious that we perceive the upper note in some way as the "same" as the lower. And we customarily divide the interval between the two into an eight-note scale (hence the terms "octave," "fifth," and "fourth")—do, re, mi, fa, so, la, ti, do—whose last note is named the same as the first so as to begin the scale for the next octave.

But why do notes an octave apart sound the "same"? An explanation comes from the relationship between the length of a stretched string and its *frequency of vibration*. Frequency is what we actually hear, because notes produced by (say) a flute and a guitar will have the same pitch provided only that they cause our eardrum to vibrate with the same frequency. Now if we halve the length of a string it vibrates twice as fast, and more generally if we divide the length of the string by n, its frequency is multiplied by n. This law was first formulated by the Dutch scientist Isaac Beeckman in 1615. When combined with knowledge of the way a string produces a tone (actually consisting of many notes, which come from the *modes of vibration* shown in Figure 1.2), it shows that each tone *contains* the tone an octave higher. Thus it is no wonder that the two sound very much the same.

A string has infinitely many simple modes of vibration: the fundamental mode in which only the endpoints remain fixed, and higher modes in which the string vibrates as if divided into $2, 3, 4, 5, \ldots$ equal parts. If the fundamental frequency is f, then the higher modes have frequencies $2f, 3f, 4f, 5f, \ldots$ by Beeckman's law.

When the string is plucked, it vibrates in all modes simultaneously, so in theory all these frequencies can be heard (though with decreasing volume as the frequency increases, and subject to the limitation that the human ear cannot detect frequencies above about 20,000 vibrations per second). A string with half the length has fundamental frequency $2f$—an octave higher—and higher modes with frequencies $4f, 6f, 8f, 10f, \ldots$. Thus all the frequencies of the half-length string are among those of the full-length string.

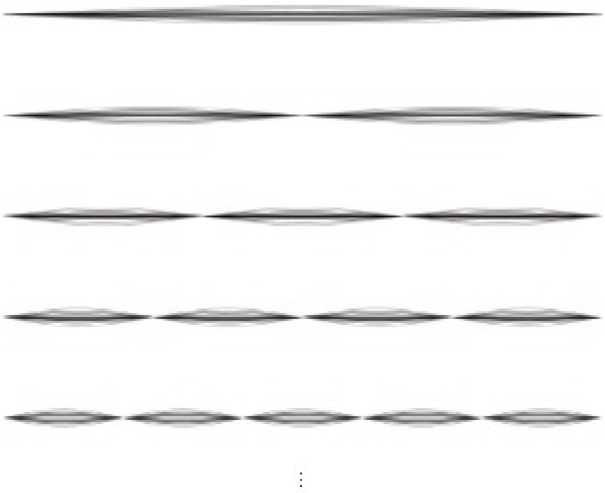


Figure 1.2: Modes of vibration.

Since frequency doubling produces a tone that is "the same only higher," repeated doubling produces tones that are perceived to *increase in equal steps.* This was the first observation of another remarkable phenomenon: *multiplication perceived as addition.* This property of perception is known in psychology as the *Weber-Fechner law.* It also applies, approximately, to the perception of volume of sound and intensity of light. But for pitch the perception is peculiarly exact and it has the octave as a natural unit of length.

The Pythagoreans knew that addition of pitch corresponds to multiplication of ratios (from their viewpoint, ratios of lengths). For example, they knew that a fifth (multiplication of frequency by $3/2$) "plus" a fourth (multiplication of frequency by $4/3$) equals an octave because

$$\frac{3}{2} \times \frac{4}{3} = 2.$$

Thus the fifth and fourth are natural steps, smaller than the octave. Where do the other steps of the eight-note scale come from? By adding more fifths, the Pythagoreans thought, but in doing so they also found some limitations in the world of natural number ratios.

If we add two fifths, we multiply the frequency twice by 3/2. Since

$$\frac{3}{2} \times \frac{3}{2} = \frac{9}{4},$$

the frequency is multiplied by 9/4, which is a little greater than 2. Thus the pitch is raised by a little over an octave. To find the size of the step over the octave we divide by 2, obtaining 9/8. The interval in pitch corresponding to multiplication of frequency by 9/8 corresponds to the second note of the scale, so it is called a *second.* The other notes are found similarly: we "add fifths" by multiplying factors of 3/2 together, and "subtract octaves" by dividing by 2 until the "difference" is an interval less than an octave (that is, a frequency ratio between 1 and 2).

After 12 fifths have been added, the result is very close to 7 octaves, and one also has enough intervals to form an eight-note scale, so it would be nice to stop. The trouble is, 12 fifths are not *exactly* the same as 7 octaves. The interval between them corresponds to the frequency ratio

$$\left(\frac{3}{2}\right)^{12} \div 2^7 = \frac{3^{12}}{2^{19}} = \frac{531441}{524288} = 1.0136\ldots.$$

This is a very small interval, called the *Pythagorean comma.* It is about 1/4 of the smallest step in the scale, so one fears that the scale is not exactly right. Moreover, the problem cannot be fixed by adding a larger number of fifths. A sum of fifths is *never* exactly equal to a sum of octaves. Can you see why? The explanation is given in the last section of this chapter.

This situation seems to threaten the dream of a "rational" world, a world governed by ratios of natural numbers. However, we do not know whether the Pythagoreans noticed this threat in the heart of their favorite creation, the arithmetical theory of music. What we do know is that the threat became clear to them when they looked at the world of geometry.

Exercises

The interval from the first note of the scale to the second is also called a *tone*, and the seven intervals in the usual scale C, D, E, F, G, A, B, C (from C to D, from D to E, . . . , from B to C) are tone, tone, semitone, tone, tone, tone, semitone

respectively. Thus the interval from C to the next C is six tones, which should correspond to the frequency ratio of 2.

1.1.1 If a tone corresponds to a frequency ratio of 9/8, as the Pythagoreans thought, explain why an interval of six Pythagorean tones corresponds to a frequency ratio of $9^6/8^6$.

1.1.2 Show that $9^6/8^6$ is *not* equal to 2.

1.1.3 Show that, in fact, $9^6/8^6$ divided by 2 is $\frac{531441}{524288}$ (a "Pythagorean comma").

In music today the interval from C to the C one octave higher is divided into 12 equal semitones with the help of extra notes called $C^\#$ (lying between C and D, and pronounced "C sharp"), $D^\#$, $F^\#$, $G^\#$, and $A^\#$. These are the black keys on the piano.

1.1.4 Which note divides the octave from C to C into two equal intervals?

1.1.5 Find notes which divide the octave from C to C into

a. Three equal intervals.

b. Four equal intervals.

c. Six equal intervals.

1.2 The Pythagorean Theorem

The role of natural numbers in music may be the exclusive discovery of the Pythagoreans, but the equally remarkable role of natural numbers in geometry was discovered in many other places—Babylonia, Egypt, China, India—in some cases before the Pythagoreans noticed it. As everyone knows, the *Pythagorean theorem* about right-angled triangles states that the square on the hypotenuse c equals the sum of the squares on the other two sides a and b (Figure 1.3).

The word "square" denotes the *area* of the square of the side in question. If the side of the square has length l units, then its area is

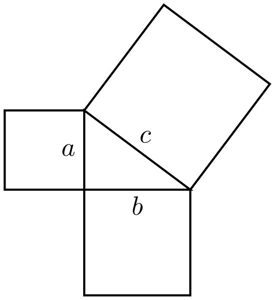


Figure 1.3: The Pythagorean theorem.

naturally divided into $l \times l = l^2$ unit squares, which is why l^2 is called "l squared." Figure 1.4 shows this for a side of length 3 units, where the area is clearly $3 \times 3 = 9$ square units.

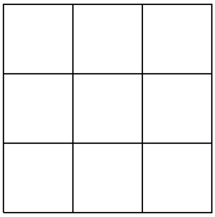


Figure 1.4: Area of a square.

Thus if a and b are the perpendicular sides of the triangle, and c is the third side, the Pythagorean theorem can be written as the equation

$$a^2 + b^2 = c^2.$$

Conversely, any triple (a, b, c) of positive numbers satisfying this equation is the triple of sides of a right-angled triangle. The story of natural numbers in geometry begins with the discovery that the equation has many solutions with natural number values of a, b, and c, and hence there are many right-angled triangles with natural number sides. The simplest has $a = 3$, $b = 4$, $c = 5$, which corresponds to the equation

$$3^2 + 4^2 = 9 + 16 = 25 = 5^2.$$

The next simplest solutions for (a,b,c) are (5,12,13), (8,15,17), and (7,24,25), among infinitely many others, called *Pythagorean triples*. As long ago as 1800 BCE, the Babylonians discovered Pythagorean triples with values of a and b in the thousands.

The Babylonian triples appear on a famous clay tablet known as Plimpton 322 (from its museum catalog number). Actually only the b and c values appear, but the a values can be inferred from the fact that in each case $c^2 - b^2$ is the square of a natural number—something that could hardly be an accident! Also, the pairs (b,c) are listed in an order corresponding to the values of b/a, which steadily decrease, as you can see from Figure 1.5.

a	b	c	b/a
120	119	169	0.9917
3456	3367	4825	0.9742
4800	4601	6649	0.9585
13500	12709	18541	0.9414
72	65	97	0.9028
360	319	481	0.8861
2700	2291	3541	0.8485
960	799	1249	0.8323
600	481	769	0.8017
6480	4961	8161	0.7656
60	45	75	0.7500
2400	1679	2929	0.6996
240	161	289	0.6708
2700	1771	3229	0.6559
90	56	106	0.6222

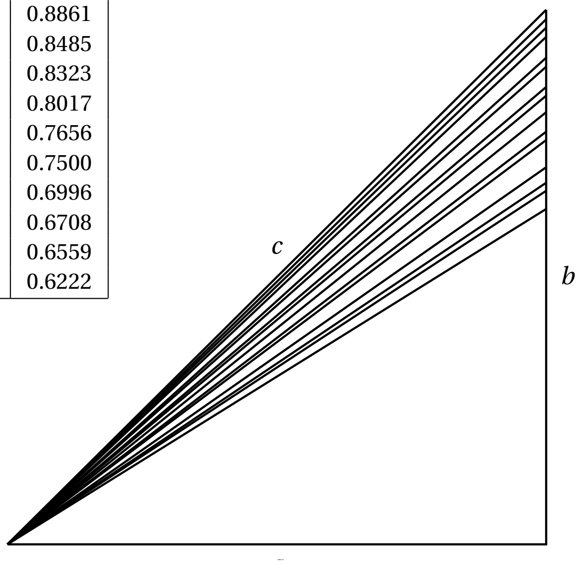


Figure 1.5: Triangles derived from Plimpton 322.

As you can also see, the slopes form a rough "scale," rather densely filling a range of angles between 30° and 45°. It looks as though the

Babylonians believed in a world of natural number ratios, rather like the Pythagoreans, and this could be an exercise like subdividing the octave by natural number ratios. But if so, there is a glaring hole in the rational geometric world: right at the top of the scale there is no triangle with $a = b$.

It is to the credit of the Pythagoreans that they alone—of all the discoverers of the Pythagorean theorem—were bothered by this hole in the rational world. They were sufficiently bothered that they tried to understand it, and in doing so discovered an *irrational* world.

Exercises

Each Pythagorean triple (a, b, c) in Plimpton 322 can be "explained" in terms of a simpler number x, given in the following table. (The numbers x are *not* in Plimpton 322 but, as we explain below, they provide a very plausible explanation of it.)

a	b	c	x
120	119	169	12/5
3456	3367	4825	64/27
4800	4601	6649	75/32
13500	12709	18541	125/54
72	65	97	9/4
360	319	481	20/9
2700	2291	3541	54/25
960	799	1249	32/15
600	481	769	25/12
6480	4961	8161	81/40
60	45	75	2
2400	1679	2929	48/25
240	161	289	15/8
2700	1771	3229	50/27
90	56	106	9/5

For each line in the table,

$$\frac{b}{a} = \frac{1}{2}\left(x - \frac{1}{x}\right).$$

1.2.1 Check that $\frac{1}{2}\left(x-\frac{1}{x}\right)=\frac{119}{120}$ when $x=12/5$.

1.2.2 Also check that
$$\frac{b}{a}=\frac{1}{2}\left(x-\frac{1}{x}\right).$$
for three other lines in the table.

The numbers x are not only "shorter" than the numbers b/a, they are "simple" in the sense that they are built from the numbers 2, 3, and 5. For example

$$\frac{12}{5}=\frac{2^2\times 3}{5} \quad \text{and} \quad \frac{125}{54}=\frac{5^3}{2\times 3^3}\ .$$

Numbers divisible by 2, 3, or 5 were "round" numbers in the view of the Babylonians, whose number system was based on the number 60. Remnants of this number system are still in use today; for example, we divide the circle into 360 degrees, the degree into 60 minutes, and the minute into 60 seconds.

1.2.3 Check that every other fraction x in the table can be written with both numerator and denominator as a product of powers of 2, 3, or 5.

The formula $\frac{1}{2}\left(x-\frac{1}{x}\right)=\frac{b}{a}$ gives us whole numbers a and b from a rational number x. But why should there be a whole number c such that $a^2+b^2=c^2$? Let us see:

1.2.4 Verify by algebra that
$$\left[\frac{1}{2}\left(x-\frac{1}{x}\right)\right]^2+1=\left[\frac{1}{2}\left(x+\frac{1}{x}\right)\right]^2.$$

1.2.5 Deduce from 1.2.4 that $a^2+b^2=c^2$, where
$$\frac{1}{2}\left(x+\frac{1}{x}\right)=\frac{c}{a}\ .$$

1.2.6 Check that the formula in 1.2.5 gives $c=169$ when $x=12/5$ (the first line of the table), and also check three other lines in the table.

1.3 Irrational Triangles

Surely the simplest triangle in the world is the one that is half a square, that is, the triangle with two perpendicular sides of equal length (Figure 1.6). If we take the perpendicular sides to be of length 1, then the hypotenuse c satisfies $c^2 = 1^2 + 1^2 = 2$, by the Pythagorean theorem. Hence c is what we call $\sqrt{2}$, the *square root* of 2.

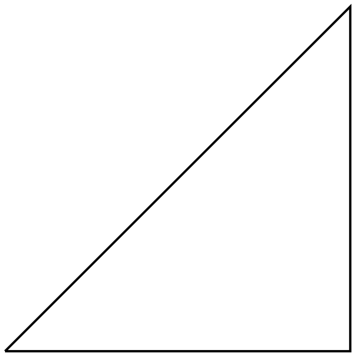


Figure 1.6: The simplest triangle.

Is $\sqrt{2}$ a ratio of natural numbers? No one has ever found such a ratio, but perhaps this is simply because we have not looked far enough. The Pythagoreans found that *no such ratio exists*, probably using some simple properties of even and odd numbers. They knew that the square of an odd number is odd, for example, and hence that an even square is necessarily the square of an even number. However, this is the easy part. The hard part is just to imagine proving that $\sqrt{2}$ is not among the ratios of natural numbers, when such ratios are the only numbers we know.

This calls for a daring method of proof known as *proof by contradiction* or *reductio ad absurdam* ("reduction to an absurdity"). To show that $\sqrt{2}$ is not a ratio of natural numbers we *suppose it is* (for the sake of argument), and deduce a contradiction. The assumption is therefore false, as we wanted to show.

In this case we begin by supposing that

$$\sqrt{2} = m/n \quad \text{for some natural numbers } m \text{ and } n.$$

We also suppose that any common factors have been cancelled from m and n. In particular, m and n are not both even, otherwise we could cancel a common factor of 2 from them both. It follows, by squaring both sides, that

	$2 = m^2/n^2$	
hence	$2n^2 = m^2$,	multiplying both sides by n^2,
hence	m^2 is even,	being a multiple of 2,
hence	m is even,	since its square is even,
hence	$m = 2l$	for some natural number l,
hence	$m^2 = 4l^2 = 2n^2$	because $m^2 = 2n^2$,
hence	$n^2 = 2l^2$,	dividing both sides by 2,
hence	n^2 is even,	
hence	n is even.	

But this contradicts our assumption that m and n are not both even, so $\sqrt{2}$ is *not a ratio* m/n of natural numbers. For this reason, we call $\sqrt{2}$ *irrational.*

The "Irrational" and the "Absurd"

In ordinary speech "irrational" means illogical or unreasonable—rather a prejudicial term to apply to numbers, one would think, so how can mathematicians do it without qualms? The way this came about is an interesting story, which shows in passing how accidental the evolution of mathematical terminology can be.

In ancient Greece the word *logos* covered a cluster of concepts involving speech: language, reason, explanation, and number. It is the root of our word *logic* and all the words ending in *-ology.* As we know, the Pythagoreans regarded number as the ultimate medium for explanation, so *logos* also meant ratio or calculation. Conversely, the opposite word *alogos* meant the opposite of rational, both in the general sense and in geometry, where Euclid used it to denote quantities not expressible as ratios of natural numbers.

Logos and *alogos* were translated into Latin as *rationalis* and *irrationalis,* and first used in mathematics by Cassiodorus, secretary of the

Ostrogoth king Theodoric, around 500 CE. The English words *rational* and *irrational* came from the Latin, with both the mathematical and general meaning intact.

Meanwhile, *logos* and *alogos*, because they can mean "expressible" and "inexpressible," were translated into Arabic with the slightly altered meanings "audible" and "inaudible" in the writings of the mathematician al-Khwārizmī around 800 CE. Later Arabic translators bent "inaudible" further to "dumb," from which it re-entered Latin as *surdus*, meaning "silent." Finally, *surdus* became the English word *surd* in Robert Recorde's *The Pathwaie to Knowledge* of 1551. The derived word *absurd* comes from the Latin *absurdus* meaning unmelodious or discordant, so the word has actually not strayed far from its Pythagorean origins.

However, we have come a long way from Pythagorean philosophy. It is no longer "irrational" to look for explanations outside the world of natural numbers, so there is now a conflict between the everyday use of the word "irrational" and its use in mathematics. We surely cannot stop calling unreasonable actions "irrational," so it would be better to stop calling numbers "irrational." Unfortunately, this seems to be a lost cause.

As long ago as 1585, the Dutch mathematician Simon Stevin railed against using the words "irrational" and "absurd" for numbers, but his advice has not been followed. In this broadside, Stevin avoids using absolute terms for numbers, like "irrational," by using the relative term *incommensurable* ("no common measure") for any pair of numbers not in natural number ratio. He also calls rational numbers "arithmetical." Here is a paraphrase by D. J. Struik of Stevin's words from *l'Arithmetique*, in [49, vol. IIB, p. 533].

> *That there are no absurd, irrational, irregular, inexplicable or surd numbers*
>
> It is true that $\sqrt{8}$ is incommensurable with an arithmetical number, but this does not mean it is absurd etc. … if $\sqrt{8}$ and an arithmetical number are incommensurable, then it is as much (or as little) the fault of $\sqrt{8}$ as of the arithmetical number.

Exercises

There is one step in the irrationality proof above that perhaps needs further justification: if m^2 is even then m is even.

1.3.1 Suppose on the contrary that m is *odd*; that is, $m = 2p + 1$ for some whole number p. If so, show that

$$m^2 = 2(2p^2 + 2p) + 1,$$

which contradicts the fact that m^2 is even.

It is useful to have this nitpicking explanation why m is even when m^2 is even, because we can use the same idea to deal with the similar problem that comes up in proving $\sqrt{3}$ irrational: proving that m is a multiple of 3 when m^2 is a multiple of 3.

The *multiples of 3* are numbers $3n$, where n is a whole number. Numbers that are *not* multiples of 3 are numbers of the form $3n + 1$ and $3n + 2$ (which leave remainder 1 and 2, respectively, when divided by 3).

1.3.2 Show that

$$(3n + 1)^2 = 3(3n^2 + 2n) + 1,$$

and explain why this implies that *the square of a number that leaves remainder 1 (when divided by 3) also leaves remainder 1.*

1.3.3 It is *not* true that the square leaves remainder 2 when the number itself leaves remainder 2 (when divided by 3). Give an example.

1.3.4 Using algebra similar to that in 1.3.2, show that *the square of a number that leaves remainder 2 (when divided by 3) leaves remainder 1.*

1.3.5 Deduce from 1.3.2 and 1.3.4 that if m^2 is a multiple of 3 then so is m.

1.3.6 Use 1.3.5 to give a proof that $\sqrt{3}$ is irrational.

1.4 The Pythagorean Nightmare

The discovery of irrationality in geometry was a terrible blow to the dream of a world governed by natural numbers. The diagonal of the unit square is surely real—as real as the square itself—yet its length in units is not a ratio of natural numbers, so from the Pythagorean viewpoint it cannot be expressed by number at all. Later Greek mathematicians coped with this nightmare by developing geometry as a non-numerical subject: the study of quantities called *magnitudes*.

Magnitudes include quantities such as length, area, and volume. They also include numbers, but, in the ancient Greek view, length does not enjoy all the properties of numbers. For example, the product of two numbers is itself a number, but the product of two lengths is *not* a length—it is a rectangle.

It is true that a rectangle with sides 2 and 3 consists of $2 \times 3 = 6$ unit squares, reflecting the fact that the product of the *numbers* 2 and 3 is the number 6. Today we exploit this parallel between area and multiplication by calling the rectangle a “2×3 rectangle.” But the rectangle with sides $\sqrt{2}$ and $\sqrt{3}$ does not consist of any “number” of unit squares in the Pythagorean sense of number (Figure 1.7).

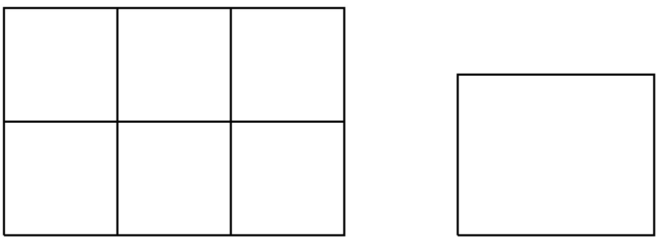


Figure 1.7: The 2×3 rectangle versus the $\sqrt{2} \times \sqrt{3}$ rectangle.

This blocks the general idea of *algebra*, where addition and multiplication are unrestricted, and it also causes mischief with the normally straightforward concept of *equality*. Two figures are shown to have equal area by cutting one figure into pieces and reassembling them to form the other. It turns out that any polygon can be “measured” in this way by a unique square, so, with some difficulty, the Greek theory of area gives the same results as ours. In fact, “equality by cutting and pasting" even gives neat proofs of a few algebraic identities. Figure 1.8 shows why $a^2 - b^2 = (a-b)(a+b)$.

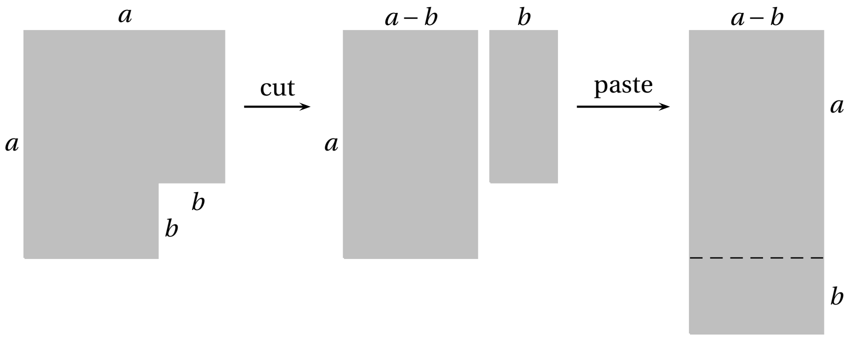


Figure 1.8: Difference of two squares: why $a^2 - b^2 = (a-b)(a+b)$.

But there is worse trouble with the concept of volume. The Greeks viewed the product of three lengths a, b, and c as a *box* with perpendicular sides a, b, and c, and they measured volumes by boxes. There are at least two problems with this train of thought.

- Volume cannot be determined by a finite number of cuts. The appropriate measure of volume is the cube, but not every polyhedron can be cut into a finite number of pieces that reassemble to form a cube. In fact, to measure the volume of a tetrahedron, it is necessary to cut it into infinitely many pieces. (See Section 4.3 for a way to do this.) The Greeks did not know it, but irrationality is the problem here too. In 1900, the German mathematician Max Dehn showed that the reason for the difficulty with the tetrahedron is that the angle between its faces is not a rational multiple of a right angle.

- We are at a loss to decide what the product of four lengths means, because we cannot visualize space with more than three dimensions.

This led to a split between geometry and number theory in Greek mathematics, ultimately to the detriment of geometry. The split is clear in Euclid's *Elements*, the most influential mathematics book of all time.