FRONTIERS IN PHYSICS

SECOND EDITION

LIE ALGEBRAS IN PARTICLE PHYSICS

From Isospin to Unified Theories

Howard Georgi

CRC Press
Taylor & Francis Group

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Lie Algebras in Particle Physics

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Second Edition

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business

SCOAP3
Sponsoring Consortium for
Open Access Publishing in Particle Physics

Open Access funded by SCOAP3

First published 1999 by Westview Press

Published 2019 by CRC Press
Taylor & Francis Group
6000 Broken Sound Parkway NW, Suite 300
Boca Raton, FL 33487-2742

First issued in hardback 2019

CRC Press is an imprint of the Taylor & Francis Group, an informa business

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Library of Congress Catalog Card Number: 99-64878

ISBN 13: 978-0-367-09172-9 (hbk)
ISBN 13: 978-0-7382-0233-4 (pbk)

DOI: 10.1201/9780429499210

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Preface to the Revised Edition

Lie Algebras in Particle Physics has been a very successful book. I have long resisted the temptation to produce a revised edition. I do so finally, because I find that there is so much new material that should be included, and so many things that I would like to say slightly differently. On the other hand, one of the good things about the first edition was that it did not do too much. The material could be dealt with in a one semester course by students with good preparation in quantum mechanics. In an attempt to preserve this advantage while including new material, I have flagged some sections that can be left out in a first reading. The titles of these sections begin with an asterisk, as do the problems that refer to them.

I may be prejudiced, but I think that this material is wonderful fun to teach, and to learn. I use this as a text for what is formally a graduate class, but it is taken successfully by many advanced undergrads at Harvard. The important prerequisite is a good background in quantum mechanics and linear algebra.

It has been over five years since I first began to revise this material and typeset it in LaTeX. Between then and now, many many students have used the evolving manuscript as a text. I am grateful to many of them for suggestions of many kinds, from typos to grammar to pedagogy.

As always, I am enormously grateful to my family for putting up with me for all this time. I am also grateful for their help with my inspirational epilogue.

Howard Georgi
Cambridge, MA
May, 1999

Contents

Why Group Theory?

Group theory is the study of symmetry. It is an incredible labor saving device. It allows us to say interesting, sometimes very detailed things about physical systems even when we don't understand exactly what the systems are! When I was a teenager, I read an essay by Sir Arthur Stanley Eddington on the Theory of Groups and a quotation from it has stuck with me for over 30 years:[1]

> We need a super-mathematics in which the operations are as unknown as the quantities they operate on, and a super-mathematician who does not know what he is doing when he performs these operations. Such a super-mathematics is the Theory of Groups.

In this book, I will try to convince you that Eddington had things a little bit wrong, as least as far as physics is concerned. A lot of what physicists use to extract information from symmetry is not the groups themselves, but group representations. You will see exactly what this means in more detail as you read on. What I hope you will take away from this book is enough about the theory of groups and Lie algebras and their representations to use group representations as labor-saving tools, particularly in the study of quantum mechanics.

The basic approach will be to alternate between mathematics and physics, and to approach each problem from several different angles. I hope that you will learn that by using several techniques at once, you can work problems more efficiently, and also understand each of the techniques more deeply.

[1] in **The World of Mathematics**, Ed. by James R. Newman, Simon & Schuster, New York, 1956.

DOI: 10.1201/9780429499210-1

Chapter 1

Finite Groups

We will begin with an introduction to finite group theory. This is not intended to be a self-contained treatment of this enormous and beautiful subject. We will concentrate on a few simple facts that are useful in understanding the compact Lie algebras. We will introduce a lot of definitions, sometimes proving things, but often relying on the reader to prove them.

1.1 Groups and representations

A Group, G, is a set with a rule for assigning to every (ordered) pair of elements, a third element, satisfying:

(1.A.1) If $f, g \in G$ then $h = fg \in G$.

(1.A.2) For $f, g, h \in G$, $f(gh) = (fg)h$.

(1.A.3) There is an identity element, e, such that for all $f \in G$, $ef = fe = f$.

(1.A.4) Every element $f \in G$ has an inverse, f^{-1}, such that $ff^{-1} = f^{-1}f = e$.

Thus a group is a multiplication table specifying $g_1 g_2 \; \forall g_1, g_2 \in G$. If the group elements are discrete, we can write the multiplication table in the form

$$\begin{array}{c||c|c|c|c} \backslash & e & g_1 & g_2 & \cdots \\ \hline\hline e & e & g_1 & g_2 & \cdots \\ \hline g_1 & g_1 & g_1 g_1 & g_1 g_2 & \cdots \\ \hline g_2 & g_2 & g_2 g_1 & g_2 g_2 & \cdots \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots \end{array} \tag{1.1}$$

DOI: 10.1201/9780429499210-2

A Representation of G is a mapping, D of the elements of G onto a set of linear operators with the following properties:

1.B.1 $D(e) = 1$, where 1 is the identity operator in the space on which the linear operators act.

1.B.2 $D(g_1)D(g_2) = D(g_1 g_2)$, in other words the group multiplication law is mapped onto the natural multiplication in the linear space on which the linear operators act.

1.2 Example - Z_3

A group is **finite** if it has a finite number of elements. Otherwise it is **infinite**. The number of elements in a finite group G is called the **order** of G. Here is a finite group of order 3.

\	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

(1.2)

This is Z_3, the cyclic group of order 3. Notice that every row and column of the multiplication table contains each element of the group exactly once. This must be the case because the inverse exists.

An Abelian group in one in which the multiplication law is commutative

$$g_1 g_2 = g_2 g_1 \,. \tag{1.3}$$

Evidently, Z_3 is Abelian.

The following is a representation of Z_3

$$D(e) = 1 \,, \quad D(a) = e^{2\pi i/3} \,, \quad D(b) = e^{4\pi i/3} \tag{1.4}$$

The **dimension** of a representation is the dimension of the space on which it acts — the representation (1.4) is 1 dimensional.

1.3 The regular representation

Here's another representation of Z_3

$$D(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad D(a) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$
$$D(b) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{1.5}$$

This representation was constructed directly from the multiplication table by the following trick. Take the group elements themselves to form an orthonormal basis for a vector space, $|e\rangle$, $|a\rangle$, and $|b\rangle$. Now define

$$D(g_1)|g_2\rangle = |g_1 g_2\rangle \tag{1.6}$$

The reader should show that this is a representation. It is called the **regular** representation. Evidently, the dimension of the regular representation is the order of the group. The matrices of (1.5) are then constructed as follows.

$$|e_1\rangle \equiv |e\rangle\,, \quad |e_2\rangle \equiv |a\rangle\,, \quad |e_3\rangle \equiv |b\rangle \tag{1.7}$$

$$[D(g)]_{ij} = \langle e_i|D(g)|e_j\rangle \tag{1.8}$$

The matrices are the **matrix elements** of the linear operators. (1.8) is a simple, but very general and very important way of going back and forth from operators to matrices. This works for any representation, not just the regular representation. We will use it constantly. The basic idea here is just the insertion of a complete set of intermediate states. The matrix corresponding to a product of operators is the matrix product of the matrices corresponding to the operators —

$$\begin{aligned} [D(g_1 g_2)]_{ij} &= [D(g_1)D(g_2)]_{ij} \\ &= \langle e_i|D(g_1)D(g_2)|e_j\rangle \\ &= \sum_k \langle e_i|D(g_1)|e_k\rangle\,\langle e_k|D(g_2)|e_j\rangle \\ &= \sum_k [D(g_1)]_{ik}[D(g_2)]_{kj} \end{aligned} \tag{1.9}$$

Note that the construction of the regular representation is completely general for any finite group. For any finite group, we can define a vector space in which the basis vectors are labeled by the group elements. Then (1.6) defines the regular representation. We will see the regular representation of various groups in this chapter.

1.4 Irreducible representations

What makes the idea of group representations so powerful is the fact that they live in linear spaces. And the wonderful thing about linear spaces is we are free to choose to represent the states in a more convenient way by making a linear transformation. As long as the transformation is invertible, the new states are just as good as the old. Such a transformation on the states produces a **similarity transformation** on the linear operators, so that we can always make a new representation of the form

$$D(g) \to D'(g) = S^{-1} D(g) S \tag{1.10}$$

Because of the form of the similarity transformation, the new set of operators has the same multiplication rules as the old one, so D' is a representation if D is. D' and D are said to be **equivalent** representations because they differ just by a trivial choice of basis.

Unitary operators (O such that $O^\dagger = O^{-1}$) are particularly important. A representation is unitary if all the $D(g)$s are unitary. Both the representations we have discussed so far are unitary. It will turn out that all representations of finite groups are equivalent to unitary representations (we'll prove this later - it is easy and neat).

A representation is **reducible** if it has an **invariant subspace**, which means that the action of any $D(g)$ on any vector in the subspace is still in the subspace. In terms of a projection operator P onto the subspace this condition can be written as

$$PD(g)P = D(g)P \;\forall g \in G \tag{1.11}$$

For example, the regular representation of Z_3 (1.5) has an invariant subspace projected on by

$$P = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{1.12}$$

because $D(g)P = P\;\forall g$. The restriction of the representation to the invariant subspace is itself a representation. In this case, it is the **trivial representation for which** $D(g) = 1$ (the trivial representation, $D(g) = 1$, is always a representation — every group has one).

A representation is **irreducible** if it is not reducible.

A representation is **completely reducible** if it is equivalent to a representation whose matrix elements have the following form:

$$\begin{pmatrix} D_1(g) & 0 & \cdots \\ 0 & D_2(g) & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{1.13}$$

where $D_j(g)$ is irreducible $\forall j$. This is called **block diagonal form.**

A representation in block diagonal form is said to be the **direct sum** of the subrepresentations, $D_j(g)$,

$$D_1 \oplus D_2 \oplus \cdots \tag{1.14}$$

In transforming a representation to block diagonal form, we are decomposing the original representation into a direct sum of its irreducible components. Thus another way of defining complete reducibility is to say that **a completely reducible representation can be decomposed into a direct sum of irreducible representations.** This is an important idea. We will use it often.

We will show later that any representation of a finite group is completely reducible. For example, for (1.5), take

$$S = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \tag{1.15}$$

where

$$\omega = e^{2\pi i/3} \tag{1.16}$$

then

$$D'(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad D'(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$
$$D'(b) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \tag{1.17}$$

1.5 Transformation groups

There is a natural multiplication law for transformations of a physical system. If g_1 and g_2 are two transformations, $g_1 g_2$ means first do g_2 and then do g_1.

Note that it is purely convention whether we define our composition law to be right to left, as we have done, or left to right. Either gives a perfectly consistent definition of a transformation group.

If this transformation is a symmetry of a quantum mechanical system, then the transformation takes the Hilbert space into an equivalent one. Then for each group element g, there is a unitary operator $D(g)$ that maps the Hilbert space into an equivalent one. These unitary operators form a representation of the transformation group because the transformed quantum states represent the transformed physical system. Thus for any set of symmetries, there is a representation of the symmetry group on the Hilbert space — we say that the Hilbert space transforms according to some representation of the group. Furthermore, because the transformed states have the same energy as the originals, $D(g)$ commutes with the Hamiltonian, $[D(g), H] = 0$. As we will see in more detail later, this means that we can always choose the energy eigenstates to transform like irreducible representations of the group. It is useful to think about this in a simple example.

1.6 Application: parity in quantum mechanics

Parity is the operation of reflection in a mirror. Reflecting twice gets you back to where you started. If p is a group element representing the parity reflection, this means that $p^2 = e$. Thus this is a transformation that together with the identity transformation (that is, doing nothing) forms a very simple group, with the following multiplication law:

\	e	p
e	e	p
p	p	e

(1.18)

This group is called Z_2. For this group there are only two irreducible representations, the trivial one in which $D(p) = 1$ and one in which $D(e) = 1$ and $D(p) = -1$. Any representation is completely reducible. In particular, that means that the Hilbert space of any parity invariant system can be decomposed into states that behave like irreducible representations, that is on which $D(p)$ is either 1 or -1. Furthermore, because $D(p)$ commutes with the Hamiltonian, $D(p)$ and H can be simultaneously diagonalized. That is we can assign each energy eigenstate a definite value of $D(p)$. The energy eigenstates on which $D(p) = 1$ are said to transform according to the trivial representation. Those on which $D(p) = -1$ transform according to the other representation. This should be familiar from nonrelativistic quantum mechanics in one dimension. There you know that a particle in a potential that is symmetric about $x = 0$ has energy eigenfunctions that are either symmetric under $x \to -x$ (corresponding to the trivial representation), or antisymmetric (the representation with $D(p) = -1$).

1.7 Example: S_3

The **permutation group** (or **symmetric group**) on 3 objects, called S_3 where

$$\begin{gathered} a_1 = (1,2,3) \qquad a_2 = (3,2,1) \\ a_3 = (1,2) \qquad a_4 = (2,3) \qquad a_5 = (3,1) \end{gathered} \tag{1.19}$$

The notation means that a_1 is a cyclic permutation of the things in positions 1, 2 and 3; a_2 is the inverse, anticyclic permutation; a_3 interchanges the objects in positions 1 and 2; and so on. The multiplication law is then determined by the transformation rule that $g_1 g_2$ means first do g_2 and then do g_1. It is

\	e	a_1	a_2	a_3	a_4	a_5
e	e	a_1	a_2	a_3	a_4	a_5
a_1	a_1	a_2	e	a_5	a_3	a_4
a_2	a_2	e	a_1	a_4	a_5	a_3
a_3	a_3	a_4	a_5	e	a_1	a_2
a_4	a_4	a_5	a_3	a_2	e	a_1
a_5	a_5	a_3	a_4	a_1	a_2	e

(1.20)

We could equally well define it to mean first do g_1 and then do g_2. These two rules define different multiplication tables, but they are related to one another by simple relabeling of the elements, so they give the same group. There is another possibility of confusion here between whether we are permuting the objects in positions 1, 2 and 3, or simply treating 1, 2 and 3 as names for the three objects. Again these two give different multiplication tables, but only up to trivial renamings. The first is a little more physical, so we will use that. The permutation group is an another example of a **transformation group** on a physical system.

S_3 is **non-Abelian** because the group multiplication law is not commutative. We will see that it is the lack of commutativity that makes group theory so interesting.

Here is a unitary irreducible representation of S_3

$$
\begin{aligned}
D(e) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D(a_1) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \\
D(a_2) &= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad D(a_3) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\
D(a_4) &= \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad D(a_5) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}
\end{aligned} \tag{1.21}
$$

The interesting thing is that the irreducible unitary representation is more than 1 dimensional. It is necessary that at least some of the representations of a non-Abelian group must be matrices rather than numbers. Only matrices can reproduce the non-Abelian multiplication law. Not all the operators in the representation can be diagonalized simultaneously. It is this that is responsible for a lot of the power of the theory of group representations.

1.8 Example: addition of integers

The integers form an infinite group under addition.

$$xy = x + y \tag{1.22}$$

This is rather unimaginatively called the additive group of the integers. Since this group is infinite, we can't write down the multiplication table, but the rule above specifies it completely.

Here is a representation:

$$D(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tag{1.23}$$

This representation is reducible, but you can show that it is not completely reducible and it is not equivalent to a unitary representation. It is reducible because

$$D(x)P = P \tag{1.24}$$

where

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1.25}$$

However,

$$D(x)(I - P) \neq (I - P) \tag{1.26}$$

so it is not completely reducible.

The additive group of the integers is infinite, because, obviously, there are an infinite number of integers. For a finite group, all reducible representations are completely reducible, because all representations are equivalent to unitary representations.

1.9 Useful theorems

Theorem 1.1 *Every representation of a finite group is equivalent to a unitary representation.*

Proof: Suppose $D(g)$ is a representation of a finite group G. Construct the operator

$$S = \sum_{g\in G} D(g)^\dagger D(g) \tag{1.27}$$

S is hermitian and positive semidefinite. Thus it can be diagonalized and its eigenvalues are non-negative:

$$S = U^{-1} d U \tag{1.28}$$

where d is diagonal

$$d = \begin{pmatrix} d_1 & 0 & \cdots \\ 0 & d_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} \tag{1.29}$$

where $d_j \geq 0 \; \forall j$. Because of the group property, all of the d_js are actually positive. Proof — suppose one of the d_js is zero. Then there is a vector λ such that $S\lambda = 0$. But then

$$\lambda^\dagger S \lambda = 0 = \sum_{g\in G} ||D(g)\lambda||^2 . \tag{1.30}$$

Thus $D(g)\lambda$ must vanish for all g, which is impossible, since $D(e) = 1$. Therefore, we can construct a square-root of S that is hermitian and invertible

$$X = S^{1/2} \equiv U^{-1} \begin{pmatrix} \sqrt{d_1} & 0 & \cdots \\ 0 & \sqrt{d_2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} U \tag{1.31}$$

X is invertible, because none of the d_js are zero. We can now define

$$D'(g) = X\, D(g)\, X^{-1} \tag{1.32}$$

Now, somewhat amazingly, this representation is unitary!

$$D'(g)^\dagger D'(g) = X^{-1} D(g)^\dagger S D(g) X^{-1} \tag{1.33}$$

but

$$\begin{aligned} D(g)^\dagger S D(g) &= D(g)^\dagger \left(\sum_{h\in G} D(h)^\dagger D(h) \right) D(g) \\ &= \sum_{h\in G} D(hg)^\dagger D(hg) \\ &= \sum_{h\in G} D(h)^\dagger D(h) = S = X^2 \end{aligned} \tag{1.34}$$

where the last line follows because hg runs over all elements of G when h does. QED.

We saw in the representation (1.23) of the additive group of the integers an example of a reducible but not completely reducible representation. The way it works is that there is a P that projects onto an invariant subspace, but $(1 - P)$ does not. This is impossible for a unitary representation, and thus representations of finite groups are always completely reducible. Let's prove it.

Theorem 1.2 *Every representation of a finite group is completely reducible.*

Proof: By the previous theorem, it is sufficient to consider unitary representations. If the representation is irreducible, we are finished because it is already in block diagonal form. If it is reducible, then $\exists$ a projector P such that $PD(g)P = D(g)P \; \forall g \in G$. This is the condition that P be an invariant subspace. Taking the adjoint gives $PD(g)^\dagger P = PD(g)^\dagger \; \forall g \in G$. But because $D(g)$ is unitary, $D(g)^\dagger = D(g)^{-1} = D(g^{-1})$ and thus since g^{-1} runs over all G when g does, $PD(g)P = PD(g) \; \forall g \in G$. But this implies that $(1 - P)D(g)(1 - P) = D(g)(1 - P) \; \forall g \in G$ and thus $1 - P$ projects onto an invariant subspace. Thus we can keep going by induction and eventually completely reduce the representation.

1.10 Subgroups

A group H whose elements are all elements of a group G is called a **subgroup** of G. The identity, and the group G are **trivial subgroups** of G. But many groups have **nontrivial subgroups** (which just means some subgroup other than G or e) as well. For example, the permutation group, S_3, has a Z_3 subgroup formed by the elements $\{e, a_1, a_2\}$.

We can use a subgroup to divide up the elements of the group into subsets called cosets. A **right-coset of the subgroup H in the group G** is a set of elements formed by the action of the elements of H on the left on a given element of G, that is all elements of the form Hg for some fixed g. You can define left-cosets as well.

For example, $\{a_3, a_4, a_5\}$ is a coset of Z_3 in S_3 in (1.20) above. The number of elements in each coset is the order of H. Every element of G must belong to one and only one coset. Thus for finite groups, the order of a subgroup H must be a factor of order of G. It is also sometimes useful to think about the **coset-space,** G/H defined by regarding each coset as a single element of the space.

A subgroup H of G is called an **invariant** or **normal subgroup** if for every $g \in G$

$$gH = Hg \tag{1.35}$$

which is (we hope) an obvious short-hand for the following: for every $g \in G$ and $h_1 \in H$ there exists an $h_2 \in H$ such that $h_1 g = g h_2$, or $g h_2 g^{-1} = h_1$. The trivial subgroups e and G are invariant for any group. It is less obvious but also true of the subgroup Z_3 of S_3 in (1.20) (you can see this by direct computation or notice that the elements of Z_3 are those permutations that involve an even number of interchanges). However, the set $\{e, a_4\}$ is a subgroup of G which is not invariant. $a_5\{e, a_4\} = \{a_5, a_1\}$ while $\{e, a_4\}a_5 = \{a_5, a_2\}$.

If H is invariant, then we can regard the coset space as a group. The multiplication law in G gives the natural multiplication law on the cosets, Hg:

$$(Hg_1)(Hg_2) = (Hg_1 H g_1^{-1})(g_1 g_2) \tag{1.36}$$

But if H is invariant $Hg_1 H g_1^{-1} = H$, so the product of elements in two cosets is in the coset represented by the product of the elements. In this case, the coset space, G/H, is called the **factor group of G by H.**

What is the factor group S_3/Z_3? The answer is Z_2.

The **center** of a group G is the set of all elements of G that commute with all elements of G. The center is always an Abelian, invariant subgroup of G. However, it may be trivial, consisting only of the identity, or of the whole group.

There is one other concept, related to the idea of an invariant subgroup, that will be useful. Notice that the condition for a subgroup to be invariant can be rewritten as

$$gHg^{-1} = H \; \forall g \in G \tag{1.37}$$

This suggests that we consider sets rather than subgroups satisfying same condition.

$$g^{-1}S\,g = S\;\forall g \in G \tag{1.38}$$

Such sets are called conjugacy classes. We will see later that there is a one-to-one correspondence between them and irreducible representations. A subgroup that is a union of conjugacy classes is invariant.

Example —

The conjugacy classes of S_3 are $\{e\}$, $\{a_1, a_2\}$ and $\{a_3, a_4, a_5\}$.

The mapping

$$G \to gGg^{-1} \tag{1.39}$$

for a fixed g is also interesting. It is called an **inner automorphism**. An **isomorphism** is a one-to-one mapping of one group onto another that preserves the multiplication law. An **automorphism** is a one-to-one mapping of a group onto itself that preserves the multiplication law. It is easy to see that (1.39) is an automorphism. Because $g^{-1}g_1g\,g^{-1}g_2g = g^{-1}g_1g_2g$, it preserves the multiplication law. Since $g^{-1}g_1g = g^{-1}g_2g \Rightarrow g_1 = g_2$, it is one to one. An automorphism of the form (1.39) where g is a group element is called an **inner automorphism**). An **outer automorphism** is one that cannot be written as $g^{-1}Gg$ for any group element g.

1.11 Schur's lemma

Theorem 1.3 *If $D_1(g)A = AD_2(g)\;\forall g \in G$ where D_1 and D_2 are inequivalent, irreducible representations, then $A = 0$.*

Proof: This is part of **Schur's lemma**. First suppose that there is a vector $|\mu\rangle$ such that $A|\mu\rangle = 0$. Then there is a non-zero projector, P, onto the subspace that annihilates A on the right. But this subspace is invariant with respect to the representation D_2, because

$$AD_2(g)P = D_1(g)AP = 0\;\forall g \in G \tag{1.40}$$

But because D_2 is irreducible, P must project onto the whole space, and A must vanish. If A annihilates one state, it must annihilate them all. A similar argument shows that A vanishes if there is a $\langle\nu|$ which annihilates A. If no vector annihilates A on either side, then it must be an invertible square matrix. It must be square, because, for example, if the number of rows were larger than the number of columns, then the rows could not be a complete set of states, and there would be a vector that annihilates A on the right. A square matrix is invertible unless its determinant vanishes. But if the determinant vanishes, then the set of homogeneous linear equations

$$A|\mu\rangle = 0 \tag{1.41}$$

has a nontrivial solution, which again means that there is a vector that annihilates A. But if A is square and invertible, then

$$A^{-1}D_1(g)A = D_2(g) \ \forall g \in G \tag{1.42}$$

so D_1 and D_2 are equivalent, contrary to assumption. QED.

The more important half of Schur's lemma applies to the situation where D_1 and D_2 above are equivalent representations. In this case, we might as well take $D_1 = D_2 = D$, because we can do so by a simple change of basis. The other half of Schur's lemma is the following.

Theorem 1.4 *If $D(g)A = AD(g)$ $\forall g \in G$ where D is a finite dimensional irreducible representation, then $A \propto I$.*

In words, if a matrix commutes with all the elements of a finite dimensional irreducible representation, it is proportional to the identity.

Proof: Note that here the restriction to a finite dimensional representation is important. We use the fact that any finite dimensional matrix has at least one eigenvalue, because the characteristic equation $\det(A - \lambda I) = 0$ has at least one root, and then we can solve the homogeneous linear equations for the components of the eigenvector $|\mu\rangle$. But then $D(g)(A - \lambda I) = (A - \lambda I)D(g)$ $\forall g \in G$ and $(A - \lambda I)|\mu\rangle = 0$. Thus the same argument we used in the proof of the previous theorem implies $(A - \lambda I) = 0$. QED.

A consequence of Schur's lemma is that the form of the basis states of an irreducible representation are essentially unique. We can rewrite theorem 1.4 as the statement

$$A^{-1}D(g)A = D(g) \ \forall g \in G \Rightarrow A \propto I \tag{1.43}$$

for any irreducible representation D. This means once the form of D is fixed, there is no further freedom to make nontrivial similarity transformations on the states. The only unitary transformation you can make is to multiply all the states by the same phase factor.

In quantum mechanics, Schur's lemma has very strong consequences for the matrix elements of any operator, O, corresponding to an observable that is invariant under the symmetry transformations. This is because the matrix elements $\langle a, j, x|O|b, k, y\rangle$ behave like the A operator in (1.40). To see this, let's consider the complete reduction of the Hilbert space in more detail. The symmetry group gets mapped into a unitary representation

$$g \to D(g) \; \forall g \in G \tag{1.44}$$

where D is the (in general very reducible) unitary representation of G that acts on the entire Hilbert space of the quantum mechanical system. But if the representation is completely reducible, we know that we can choose a basis in which D has block diagonal form with each block corresponding to some unitary irreducible representation of G. We can write the orthonormal basis states as

$$|a, j, x\rangle \tag{1.45}$$

satisfying

$$\langle a, j, x \mid b, k, y\rangle = \delta_{ab}\, \delta_{jk}\, \delta_{xy} \tag{1.46}$$

where a labels the irreducible representation, $j = 1$ to n_a labels the state within the representation, and x represents whatever other physical parameters there are.

Implicit in this treatment is an important assumption that we will almost always make without talking about it. We assume that have chosen a basis in which all occurences of each irreducible representation a, is described by the same set of unitary representation matrices, $D_a(g)$. In other words, for each irreducible representation, we choose a canonical form, and use it exclusively

In this special basis, the matrix elements of $D(g)$ are

$$\langle a, j, x|\, D(g)\, |b, k, y\rangle = \delta_{ab}\, \delta_{xy}\, [D_a(g)]_{jk} \tag{1.47}$$

This is just a rewriting of (1.13) with explicit indices rather than as a matrix. We can now check that our treatment makes sense by writing the representation D in this basis by inserting a complete set of intermediate states on both sides:

$$I = \sum_{a,j,x} |a, j, x\rangle \langle a, j, x| \tag{1.48}$$

Then we can write

$$\begin{aligned} D(g) &= \sum_{a,j,x} |a, j, x\rangle \langle a, j, x|\, D(g) \sum_{b,k,y} |b, k, y\rangle \langle b, k, y| \\ &= \sum_{\substack{a,j,x \\ b,k,y}} |a, j, x\rangle\, \delta_{ab}\, \delta_{xy}\, [D_a(g)]_{jk}\, \langle b, k, y| \\ &= \sum_{a,j,k,x} |a, j, x\rangle\, [D_a(g)]_{jk}\, \langle a, k, x| \end{aligned} \tag{1.49}$$