

# THE THEORY OF QUANTUM LIQUIDS



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DAVID PINES  
PHILIPPE NOZIÈRES

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# THE THEORY OF QUANTUM LIQUIDS



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# THE THEORY OF QUANTUM LIQUIDS

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# Editor's Foreword

Perseus Books's *Advanced Book Classics* series has been designed to make available, at modest cost and in an attractive format, graduate-level lecture notes, texts and monographs that are classics, in that the depth and insight they have provided in the past may be expected to continue to play a significant role in the education of the present and future generations of graduate students, post-doctoral research associates, and their more senior scientific colleagues. It is hoped that although books in the series may, in some cases, have been written twenty-five or more years ago, the unique perspective and pedagogical clarity provided by the authors will make them as attractive and useful to today's reader as they were to the generations of readers who received them enthusiastically at the time of their original publication.

*The Theory of Quantum Liquids* brings together, for the first time, the first parts of what the authors had intended to be a two volume series dealing with three basic quantum liquids: the normal Fermi liquids found in conventional metals and liquid  $^3\text{He}$ , which are described by Landau's elegant theory; superfluid Bose liquids, such as liquid  $^4\text{He}$ ; and the conventional superconductors, so completely described by the microscopic theory of s-wave pairing developed by Bardeen, Cooper, and Schrieffer. Although written contemporaneously, these two parts were published some twenty-three years apart, because, as the authors' note in their preface to Volume II (dealing with superfluid Bose liquids) explains, the volume had been long delayed in the unrealized, and probably unreasonable, hope that they might find the time to include in it a discussion of superconductivity at the same level of presentation as that found in Volume I and the sections of the projected Volume II that dealt with the superfluid Bose liquid. The explosion of research results on superconductivity in the mid-sixties rendered this plan moot, and so the material dealing with the superfluid Bose liquid was first published over twenty years after the time it had been written.

During the past decade, interest in quantum liquids has continued to increase. With the discovery of strongly correlated electron systems such as the heavy electron metals and superconductors, the Kondo insulators, and especially the low temperature organic superconductors and the high temperature cuprate superconductors, we now have before us a broad spectrum of materials for which

Landau's description of normal state behavior fails, while their superconducting behavior is unconventional in that the pairing state is not the s-wave state found in conventional low temperature superconductors. This same decade has witnessed a revival of interest in Bose liquids, now that Bose condensation of atoms other than Helium can be achieved by ingenious optical techniques. The theory of quantum liquids thus continues to be a work-in-progress, and the present volume may be regarded as providing the backdrop against which these new discoveries are played out.

David Pines  
Tesuque, NM  
August 1999

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# **The Theory of Quantum Liquids**

**Normal Fermi Liquids**

**Volume I**



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# Vita

## **David Pines**

Professor of Physics at the University of Illinois at Urbana-Champaign, he is affiliated with the Center for Advanced Study. He has made pioneering contributions to an understanding of many-body problems in condensed matter and nuclear physics, and to theoretical astrophysics. Editor of Addison-Wesley's *Frontiers in Physics* series and the American Physical Society's *Reviews of Modern Physics*, Dr. Pines is a member of the National Academy of Sciences and is a Fellow of the American Academy of Arts and Sciences and the American Association for the Advancement of Science. He is a past Board Chairman of the Santa Fe Institute and currently serves as Co-Chairman of the Institute's Science Board. Dr. Pines received the Eugene Feenberg Memorial Medal for Contributions to Many-Body Theory in 1985, the P.A.M. Dirac Silver Medal for the Advancement of Theoretical Physics in 1984, and the Friemann Prize in Condensed Matter Physics in 1983.

## **Philippe Nozières**

Professor of Physics at the Collège de France, Paris, he studied at the Ecole Normale Supérieure in Paris and conducted research at Princeton University. Dr. Nozières has served as a Professor at the University of Paris and at the University of Grenoble. His research is currently based at the Laue Langevin Institute in Grenoble. A member of the Académie des Sciences, he has been awarded the Holweck Award of the French Physical Society and the Institute of Physics. Dr. Nozières' work has been concerned with various facets of the many-body problem, and his work currently focuses on crystal growth and surface physics.



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## Special Preface

We began writing this book at a time when field theoretical methods in statistical mechanics were expanding rapidly. Our aim was to focus on the physics which lies behind such sophisticated techniques, to describe simple physical facts in a simple language. Hence our deliberate choice of “elementary” methods in explaining such fundamental concepts as elementary excitations, their interactions and collisions, etc.... Rather than elaborating on calculations, we tried to explain *qualitative* and *unifying* aspects of an extremely broad and diversified field. Such a limited scope—albeit ambitious—probably explains why our book has retained popularity throughout the years. It is a comforting thought to evolve from a “frontier” level to a “classic” status. We hope it is not only a matter of age!

The book was originally organized in two volumes. Volume I dealt with “normal” Fermi fluids, *i.e.*, those which display no order of any type. Typical examples are liquid  $^3\text{He}$  or electron liquids at temperatures above a possible superfluid transition. We discussed at length the nature of elementary excitations, the central concept of response functions, the new features brought about by the long range of Coulomb interactions in charged systems. Volume II was supposed to deal with superfluid systems, both bosons ( $^4\text{He}$ ) and fermions (metallic superconductors); it was never completed. The main reason was a matter of timing. The year, 1965, marked an explosive growth of the work on superconductors, with such new concepts as phase coherence, the Josephson effect, etc. Things were moving fast, while our ambition was to provide a carefully thought out picture, in which concepts and methods were put in perspective. It was definitely not the appropriate time, and consequently Volume II fell into oblivion. We nevertheless had completed a long chapter on Bose condensation and liquid  $^4\text{He}$ , which has been widely circulated in the community. After some hesitation, we have decided to take the opportunity of this “classic” series to publish as Volume II our text, written in 1964, as it

stands. We do this partly because it contains physical concepts that have perhaps not been pursued in the detail they deserve (e.g., the interaction of elementary excitations), partly because we hope our early work will provide a perspective on the field of  $^4\text{He}$  which will help the reader appreciate the subsequent evolution of ideas.

Altogether, the present volume is centered around a mean field approach, appropriately generalized in order to cope with strong coupling situations. Subsequent developments involved in a number of fluctuation dominated problems, such as critical phenomena, or the Kondo problem in magnetic alloys; these were in a process of development in 1965, and we did not consider them. We did, however, emphasize the importance of interactions between elementary excitations, "ancestor" of mode-mode coupling.

During the past twenty years, the subject of Fermi liquids has developed significantly. New Fermi liquids have been discovered (superfluid  $^3\text{He}$ , dilute mixtures of  $^3\text{He}$  in  $^4\text{He}$ , spin polarized  $^3\text{He}$ , superfluid neutron matter in neutron stars) while understanding of  $^3\text{He}$ , the "archetypical" normal Fermi liquid, has been increased substantially by careful measurements of its "Landau" properties, such as transport coefficients, specific heat, and zero sound, and by neutron scattering experiments which probe the density and spin excitation spectra in the "non-Landau" regime.

Review articles which deal with these topics at a level comparable to that of this book include:

- P.W. Anderson and W.F. Brinkman, Theory of Anisotropic Superfluidity in  $^3\text{He}$ , in the *Physics of Liquid and Solid Helium, Part II*, ed. by K. H. Bennemann and J. B. Ketterson, J. Wiley Pub., pp. 177-286 (1978).
- A. J. Leggett, A Theoretical Description of the New Phases of Liquid  $^3\text{He}$ , in the *Rev. Mod. Phys.* 47, pp. 331 (1975).
- G. Baym and C. J. Pethick, Landau Fermi Liquid Theory and Low Temperature Properties of Normal Liquid  $^3\text{He}$ , and Low Temperature Properties of Dilute Solutions of  $^3\text{He}$  in Superfluid  $^4\text{He}$ , in *The Physics of Liquid and Solid Helium, Part II*, ed. K. H. Bennemann and J. B. Ketterson, J. Wiley Pub., pp. 1-175 (1978).
- D. Pines, Excitations and Transport in Quantum Liquids, in *Highlights of Condensed Matter Theory, Soc. Italiano di Fisica*, pp. 580 (1985); and *Can. Jour. Phys.* 65, pp. 1357 (1987).

The subject of broken symmetry in condensed matter physics represents another significant concept which has been developed considerably since the appearance of our book. A theoretical discussion of magnetic instabilities, Stoner's theory of ferromagnetism, and spin density waves may be found in the review articles in "Magnetism," ed. G. Rado and H. Suhl, Academic Press, 1963, with a general discussion of broken symmetry may be found inter alia, in "Basic Notions of Condensed Matter Physics," P. W. Anderson, Addison Wesley, 1983, as well as in the review articles on superfluid  $^3\text{He}$  listed above.

Looking over the past twenty-five years since we began work on this book, the scene has changed in the field of many-body physics. What was new has become standard wisdom, while new phenomena have taken center stage. Nevertheless, the feeling of

excitement remains the same—as witnessed by the upsurge of work on localizations, narrow band materials, the quantum Hall effect, heavy electron systems, high  $T_c$  superconductors, etc. All of that must be anchored on solid ground—a ground which we tried to lay down in the original edition of this book. We hope that our 1965 baby has aged well, and it gives us pleasure to “launch her” again.

David Pines  
Philippe Nozières



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# Preface

Our aim in writing this book has been to provide a unified, yet elementary, account of the theory of quantum liquids. Strictly speaking, a quantum liquid is a spatially homogeneous system of strongly interacting particles at temperatures sufficiently low that the effects of quantum statistics are important. In this category fall liquid  $^3\text{He}$  and  $^4\text{He}$ . In practice, the term is used more broadly, to include those aspects of the behavior of conduction electrons in metals and degenerate semiconductors which are not sensitive to the periodic nature of the ionic potential. The conduction electrons in a metal may thus be regarded as a normal Fermi liquid, or a superfluid Fermi liquid, depending on whether the metal in question is normal or superconducting.

While the theory of quantum liquids may be said to have had its origin some twenty-five years ago in the classic work of Landau on  $^4\text{He}$ , it is only within the past decade that it has emerged as a well-defined subfield of physics. Thanks to the work of many people, we possess today a unified point of view and a language appropriate for the description of many-particle systems. We understand where elementary excitations afford an apt description and where they do not; we appreciate the relationship between quasiparticle excitations and collective modes, and how both derive from the basic interactions of the system particle. There now exists a number of model solutions for the many-body problem, solutions which can be shown to be valid for a given class of particle interactions and system densities: examples are an electron system at high densities and low temperatures, and a dilute boson system at low temperatures. In addition, there is a semiphenomenological theory, due to Landau, which describes the macroscopic behavior of an arbitrary normal Fermi liquid at low temperatures. Finally, and most important, has been the development of a successful microscopic theory of superconductivity by Bardeen, Cooper, and Schrieffer.

These developments have profoundly altered the main lines of research in quantum statistical mechanics: it has changed from the study of dilute, weakly interacting gases to an investigation of quantum liquids in which the interaction between particles plays an essential role. The resulting body of theory has developed to the point that it should be possible to present a coherent account of quantum liquids for the non-specialist, and such is our aim.

In writing this book, we have had three sorts of readers in mind:

- (i) Students who have completed the equivalent of an undergraduate physics major, and have taken one year of a graduate course in quantum mechanics.
- (ii) Experimental physicists working in the fields of low-temperature or solid-state physics.
- (iii) Theoretical physicists or chemists who have not specialized in many-particle problems.

Our book is intended both as a text for a graduate course in quantum statistical mechanics or low temperature theory and as a monograph for reference and self-study. The reader may be surprised by its designation as a text for a course on statistical mechanics, since a perusal of the table of contents shows few topics that are presently included in most such courses. In fact, we believe it is time for extension of our teaching of statistical mechanics, to take into account all that we have learned in the past decade. We hope that this book may prove helpful in that regard and that it may also prove useful as a supplementary reference for an advanced course in solid-state physics.

We have attempted to introduce the essential physical concepts with a minimum of mathematical complexity; therefore, we have not made use of either Green's functions or Feynman diagrams. We hope that their absence is compensated for by our book being more accessible to the experimentalist and the nonspecialist. Accounts of field-theoretic methods in many-particle problems may be found in early books by the authors [D. Pines, *The Many-Body Problem*, Benjamin, New York (1962), P. Nozières, *The Theory of Interacting Fermi Systems*, Benjamin, New York (1963)] and in L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics*, Benjamin, New York (1962), and by A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics*, Prentice-Hall, New York (1964), to mention but a few reference works.

The decision to publish the book in two volumes stems, in part, from its length, and in part, from the natural division of quantum liquids into two classes, normal and superfluid. A third and perhaps controlling factor has been that a single volume would have meant a delay in publication of the present material of well over a year.

Although our book is a large one, we have not found it possible to describe all quantum liquids, or every aspect of the behavior of a given liquid. For example, we have not included a description of nuclear matter, of phase transitions, or of variational calculations of the ground state of various many-particle systems. On the other hand, we have compared theory with experiment in a number of places and, where appropriate, have

compared and contrasted the behavior of different quantum liquids.

We have chosen to begin the book with the Landau theory of a neutral Fermi liquid in order to illustrate, in comparatively elementary fashion, the way both quantum statistics and particle interaction determine system behavior. We next consider the description of an arbitrary quantum liquid; we discuss the mathematical theory of linear response and correlations, which establishes the language appropriate for that description. We then go on to discuss, in Volume I, charged Fermi liquids, and in Volume II, the superfluid Bose liquid and superconductors.

The authors began work on this book in Paris, at the Laboratoire de Physique of the Ecole Normale Supérieure, in the fall of 1962, when one of us was on leave from the University of Illinois. Since then we worked on the book both in Paris and at the University of Illinois. We should like to thank Professor Yves Rocard, of the Université de Paris, and Professor G. M. Almy, head of the Physics Department at the University of Illinois, for their support and encouragement. One of us (D.P.) would also like to thank the John Simon Guggenheim Memorial Foundation for their support during 1962 and 1963, and the Army Research Office (Durham) for their support during 1963 and 1964.

We should like to express our gratitude to the many friends and colleagues to whom we have turned for advice and discussion, and particularly to Professor John Bardeen for his advice and encouragement. We are deeply indebted to Dr. Conyers Herring for his careful reading of a preliminary version of Chapter 1, and to Professor Gordon Baym, who read carefully the entire manuscript and whose comments have improved both the accuracy and the clarity of our presentation. We owe an especial debt of gratitude to Dr. Odile Betbeder-Matibet, who has been of substantial assistance in the correction of the proof, and to Mme. M. Audouin, who has helped in the preparation of the index.

David Pines  
Philippe Nozières

December 1966



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## INTRODUCTION

Let us consider a gas of neutral atoms interacting through a short-range binary potential. At high enough temperatures and low enough pressures, the gas is dilute. Each atom moves as if it were essentially free, apart from infrequent collisions with other atoms or with the container walls. The system is well described by the elementary kinetic theory of gases. It displays the usual properties of a classical gas; the specific heat  $C_v$  is temperature independent; in the case of fermions (particles with spin) the spin susceptibility varies inversely with the temperature, according to Curie's law.

As the pressure is increased and the temperature is lowered, the above picture tends to break down for two distinct reasons. On the one hand, because of the increase in density, the interaction between particles becomes far more efficient. On the other hand, the decrease in the temperature weakens the kinetic energy compared to the particle "interaction" energy. At some stage, the gas undergoes a first-order transition to a liquid state. This state is characterized by strong particle correlations, which insure the cohesion of the liquid. The transition is essentially dynamic in character, since it arises from the particle interaction. It corresponds to a purely classical effect, giving rise to a *classical liquid*.

As the temperature is lowered still further, the kinetic energy of the liquid is further decreased, while the interaction between the particles plays a correspondingly more important role. As a result, in almost all cases one observes a first-order phase transition from the liquid state to a solid state. The only exceptions are the isotopes of helium,  $^3\text{He}$  and  $^4\text{He}$ , which remain liquid down to the lowest attainable temperatures. Helium is anomalous because the forces between the atoms are relatively weak, while, because of its low mass, the zero-point oscillations of the individual atoms are large.

Helium thus remains liquid through a temperature regime in which

quantum effects must be taken into account. These become important when the thermal de Broglie wavelength of a particle,  $(\hbar^2/2M\kappa T)^{1/2}$ , becomes comparable to the average spacing between particles; this occurs at about 3–4°K for helium. Quantum effects may be viewed as deriving from the symmetry properties of the many-body wave function, and are essentially *statistical* in their nature. One expects (and finds) that at sufficiently low temperatures, where the quantum nature of the liquid has become manifest,  $^3\text{He}$ , which obeys Fermi–Dirac statistics, and  $^4\text{He}$ , which obeys Bose–Einstein statistics, will behave quite differently. We are thus led to consider the theory of *quantum liquids*, in which an important role is played by both the degeneracy characteristic of a quantum many-particle system, and the interaction between the particles.

$^3\text{He}$  (a Fermi liquid) and  $^4\text{He}$  (a Bose liquid) are the only “real” quantum liquids found in nature. However, one can also regard the conduction electrons in metals, semimetals, and degenerate semiconductors as a quantum liquid; this electron “liquid” is not homogeneous, since the electrons in a solid move in the periodic field of the ion cores. For many purposes, however, one can neglect the influence of this periodic potential. For conduction electrons in metals the degeneracy temperature (at which quantum effects become of importance) is of the order of 50,000°K; it is  $\sim 100^\circ\text{K}$  for a semimetal, and  $\sim 3^\circ\text{K}$  for a typical semiconductor with a conduction electron density of  $10^{16}$ .

Strangely enough, despite the often quite sizable particle interaction and despite the fact that one is dealing with a quantum-mechanical many-particle system, quantum liquids at sufficiently low temperatures are better understood than their classical counterparts. The explanation lies in the concept of *elementary excitations*, which under suitable circumstances provide a complete description of the low-lying excited states of the quantum liquid. At very low temperatures, only a few such excitations are present; the excitations are long-lived and interact weakly with one another; most properties of the system can be explained in terms of them.

In Volume I of this book we shall be concerned with a single group of quantum liquids, *normal Fermi liquids*. A normal Fermi liquid may be roughly defined as a degenerate Fermi liquid in which the properties of the system are *not* drastically modified by the particle interactions, no matter how strong they might be. In other words, the liquid retains the essential properties of the noninteracting fermion system. (It has a well-defined Fermi surface, its specific heat varies linearly with the temperature, etc.) Examples of normal Fermi liquids are  $^3\text{He}$  above 4 millidegrees and conduction electrons in metals which are not superconducting. In 1956 Landau constructed an elegant, semiphenomeno-

logical theory of the macroscopic behavior of normal Fermi liquids in the low-temperature limit. We present the Landau theory for neutral Fermi liquids in Chapter 1, and apply it to  $^3\text{He}$ .

There are shortcomings to the Landau theory. It is not applicable to microscopic phenomena, those which involve distances of the order of the interparticle spacing, or energies comparable to that of a particle on the Fermi surface. Moreover, it is, in a certain sense, too complete in that it provides far more information than any experiment will ever sample. It is therefore of interest to develop a direct description of experimental measurements on many-particle systems. An exact formalism can be developed so long as the system responds *linearly* to the measuring apparatus. The general theory of linear response, applicable to both microscopic and macroscopic phenomena, is presented in Chapter 2. It establishes the connections between response and correlation functions and the extent to which these may be related to the spectrum of elementary excitations. The theory provides a number of exact results of great practical importance. More important, it establishes the *language* one should use in discussing the properties of quantum liquids, in both the microscopic and macroscopic regimes, and thus enables one to appreciate the physical features which are common to all quantum liquids; these unifying aspects are too often obscured by a diversity of mathematical descriptions.

Chapters 3 and 4 are devoted to charged Fermi liquids. Because of the long range of the Coulomb interaction, a charged Fermi liquid differs appreciably from its neutral counterpart. In Chapter 3 the new physical features introduced by the Coulomb interaction, screening and plasma oscillation, are introduced, and described in detail. The Landau theory is then generalized and applied to the description of a number of macroscopic phenomena encountered in electron liquids. Chapter 3 is analogous to Chapter 1 in that important physical concepts are introduced and described in the macroscopic limit. Chapter 4 resembles Chapter 2; it contains a formal description of measurements on charged particle systems and is concerned with microscopic as well as macroscopic phenomena; the various dielectric functions of interest are defined, and applied to a number of problems of physical interest.

In Chapters 1 through 4, we deal with certain exact relationships between various physical quantities, or with macroscopic theories, whose validity is limited to phenomena in the macroscopic regime. Detailed microscopic theories are, by contrast, subject to certain limitations: either they represent solutions to model problems (the description of a physical system in a limited range of densities or interaction strength, one which rarely corresponds to physical reality); or they provide an approximate account of real physical systems, to an extent which is difficult to estimate with precision. Both kinds of microscopic

theories are considered in Chapter 5. The random phase approximation is developed and applied to the high-density electron gas, a model problem for which it provides an accurate description. The structure of the generalized random phase approximation for both neutral and charged particle systems is described, and connection is made, in the macroscopic limit, to the Landau theory. Approximate microscopic theories, which are intended for an electron liquid at metallic densities, are developed and applied to the description of simple metals.

Volume II of this book is devoted to the theory of superfluid quantum liquids. Mathematically, a superfluid is characterized by macroscopic occupation of a single quantum state, the "condensate." Physically, the most dramatic manifestation of superfluid behavior is the resistance-free motion associated with that single state. Bose liquids, such as  $^4\text{He}$ , may be regarded as the simplest superfluid systems, since the existence of a condensate is already evident in the lowest-order perturbation-theoretic treatment. Superfluidity in Fermi liquids is, on the other hand, a far more subtle matter. As a result, many years passed between the experimental discovery of superconductivity by Kammerlingh Onnes in 1911, and the successful microscopic theory of Bardeen, Cooper, and Schrieffer in 1957.

The theory of Bose liquids is developed and applied to  $^4\text{He}$  in Chapter 6, while the theory of superfluid Fermi liquids is developed and applied to superconductivity in Chapter 7. The essential features of superfluid behavior are described in Chapters 8 to 14. We attempt to present the treatments in parallel as much as possible, in order to emphasize the similarities in behavior of the two kinds of superfluids. There is, however, one important difference. For the superfluid Fermi liquid, there exists an excellent microscopic theory, that of Bardeen, Cooper, and Schrieffer, which is in close agreement with essentially all experiments on superconductivity. On the other hand, no satisfactory microscopic theory exists for the only Bose liquid found in nature,  $^4\text{He}$ . As a result, our primary emphasis in developing the Bose liquid theory is on phenomenological considerations, and on the macroscopic theory to which Landau and his collaborators have made so many important contributions.

One final comment: although the title of our book indicates that we shall confine our attention to quantum liquids, this is not quite the case in practice. Since the interacting fermion system of principal physical interest is that formed by conduction electrons in a metal, we have attempted to indicate, at the appropriate places, the generalizations required to take into account the effects of the ionic periodic potential. These "solid-state" effects are discussed in Sections 1.3, 3.8, 4.5, 5.6, and 7.6.

## CHAPTER 1

# NEUTRAL FERMI LIQUIDS

Let us consider a noninteracting Fermi gas in equilibrium at a temperature  $T$ . The probability that a single particle has energy  $\epsilon$  is given by the well-known expression

$$f(\epsilon) = \frac{1}{1 + \exp [(\epsilon - \mu)/\kappa T]}$$

where  $\kappa$  is Boltzmann's constant. The constant  $\mu$ , known as the chemical potential, is adjusted in such a way as to give the correct total number of particles. At high temperatures  $\mu$  is negative and very much smaller than  $-\kappa T$ ;  $f(\epsilon)$  reduces to the usual Maxwell-Boltzmann expression; the gas is "classical." In the opposite limit,  $T \rightarrow 0$ ,  $f(\epsilon)$  becomes a Fermi-Dirac step function, which jumps from 1 to zero at the positive chemical potential  $\mu_0$ : the gas is said to be *fully degenerate*. The transition from one regime to the other occurs around the "degeneracy temperature,"

$$T_F = \frac{\mu_0}{\kappa}.$$

In the degenerate region, the number of excited states available to the system is very much reduced by the exclusion principle, which acts to "freeze" the distribution; at a temperature  $T$ , only those particles whose energy is within  $\kappa T$  of the Fermi energy are affected by a change in temperature. This reduction has striking physical consequences: the specific heat becomes proportional to  $T$ , instead of being constant; the spin susceptibility becomes temperature independent, instead of varying as  $1/T$ .

For real fermion systems, the particle interaction and the exclusion principle act simultaneously; we are thus led to study *degenerate Fermi liquids*, in which both effects are important. In some systems, the nature of the degenerate gas is drastically modified by the particle interactions. Such is the case, for instance, in a superconducting electron gas. Frequently, the interacting liquid retains many properties of the gas: it is then said to be *normal* (a definition which will be made more explicit in the course of this chapter). A normal Fermi liquid at  $T = 0$  has a sharply defined Fermi surface  $S_F$ ; its elementary excitations may be pictured as *quasiparticles* outside  $S_F$  and *quasiholes* inside  $S_F$ , in close analogy with the single-particle excitations of a noninteracting Fermi gas. Such a resemblance explains why so many properties of the liquid can be interpreted in terms of a "one-particle approximation." To consider another example, the one-electron theory of solids provides a correct account of a large number of "sophisticated" phenomena in metals (de Haas-van Alphen effect, transport properties, etc.) even though it ignores the not-inappreciable particle interaction. Again, the explanation of this success is found in the concept of quasiparticle excitations. Let us emphasize that such single-particle theories are not complete; there exist "many-body" effects which arise as a consequence of particle interaction, and which are characteristic of the liquid state.

At the present time, we do not possess a theory that completely describes the properties of an interacting Fermi liquid at an arbitrary temperature. The problem may be formulated by means of sophisticated field-theoretic techniques. However, the general solutions that have been obtained (see, for example, Balian and de Dominicis) are of a somewhat formal character, and have not led, as yet, to explicit results which may be compared with experiment. Fortunately, one may obtain a number of simple results in the limit of *low temperatures* ( $T \ll T_F$ ), for phenomena occurring on a *macroscopic* scale. The relevant theory was constructed by Landau (1956) on a semiphenomenological basis; Landau's assumptions have since been substantiated by detailed microscopic analysis [Pitaevskii (1959), Luttinger and Nozières (1962)]. In this chapter, we shall adopt Landau's semiphenomenological point of view, and explore the applications of the theory to neutral systems such as degenerate liquid  $^3\text{He}$ .

As we have pointed out in the introduction,  $^3\text{He}$  is the only degenerate Fermi liquid found in nature. With minor modifications the theory of Fermi liquids may also be applied to electrons in metals or semimetals (and to nuclear matter). The extension of Landau's theory to charged systems (such as electrons in metals) involves certain difficulties which arise from the long-range character of the Coulomb interaction. For

that reason we postpone a detailed discussion of charged Fermi liquids until Chapter 3.

In Section 1.1, we shall introduce the notion of a quasiparticle by means of a careful study of the relation between interacting and non-interacting systems. In a real Fermi liquid, the quasiparticles are not quite independent; there remains a certain *interaction energy* between excited quasiparticles. This new physical feature is the key to the Landau theory, the fundamentals of which are presented in Section 1.2. The theory is applied to various equilibrium properties in Section 1.3; its extension to electrons in metals is also briefly discussed. We then proceed to study nonequilibrium properties. The transport equation for quasiparticles is set up in Section 1.4; it is applied to a calculation of the current density in Section 1.5. Section 1.6 is devoted to an extensive discussion of "localized" quasiparticle excitations, and of their interaction with the surrounding medium; we are led to develop a formal solution of the transport equation, which proves useful in other respects. The important concept of collective modes is introduced in Section 1.7; the damping of the collective modes is discussed in some detail, and the stability of the ground state against collective excitations is studied from both a static and a dynamic point of view. Section 1.8 is concerned with the consequences of real collisions between quasiparticles; physical phenomena that are discussed include the lifetime of quasiparticles, the usual transport coefficients, and the collision-induced damping of collective modes. A comparison between zero sound and first sound is carried out in the following section. Finally, the theory is applied to degenerate  ${}^3\text{He}$  in Section 1.10.

This whole chapter is based on the Landau theory of Fermi liquids, which shows clearly all the important features brought about by the interaction. This theory is rigorous only in well-defined limits. We shall stress the conditions of applicability throughout our discussion of the theory.

## 1.1. THE QUASIPARTICLE CONCEPT

### ELEMENTARY EXCITATIONS OF A NONINTERACTING FERMI GAS

Let us first consider a system of  $N$  noninteracting free fermions, each of mass  $m$ , enclosed in a volume  $\Omega$ . The eigenstates of the total system are antisymmetrized combinations of  $N$  different single-particle states. Each single particle is characterized by two quantum numbers, its momentum  $\mathbf{p}$  and its spin,  $\sigma = \pm \frac{1}{2}$ ; its normalized wave function in

configuration space is a simple plane wave:

$$\psi_{\mathbf{p}}(\mathbf{r}) = \frac{1}{\sqrt{\Omega}} e^{i\mathbf{p}\cdot\mathbf{r}}. \quad (1.1)$$

The total wave function is a Slater determinant made up of  $N$  such plane waves. All the eigenstates of the system can be characterized by the distribution function  $n_{\mathbf{p}\sigma}$ , which is equal to 1 if the state  $\mathbf{p}, \sigma$  is occupied, to zero otherwise. (In what follows, we shall omit the spin index  $\sigma$ , and include it in  $\mathbf{p}$ , unless specified otherwise.)

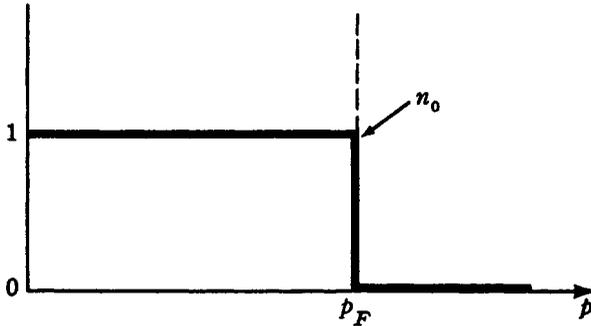


FIGURE 1.1. The ground state distribution of quasiparticles.

A particle with momentum  $\mathbf{p}$  possesses a kinetic energy  $p^2/2m$ . In the absence of interaction, the energies of the particles are simply additive: the total energy  $E$  of the system is given by

$$E = \sum_{\mathbf{p}} n_{\mathbf{p}} \frac{p^2}{2m}. \quad (1.2)$$

The ground state is obtained by filling the  $N$  plane wave states of lowest energy. The ground state distribution is shown in Fig. 1.1, the Fermi momentum  $p_F$  being given by

$$\frac{N}{\Omega} = \frac{1}{3\pi^2} \left( \frac{p_F}{\hbar} \right)^3. \quad (1.3)$$

All the plane wave states lying inside the *Fermi surface*  $S_F$  (here the sphere of radius  $p_F$ ) are filled in the ground state; those lying outside  $S_F$  are empty.

Let us add a single particle to the system. The ground state of the  $(N + 1)$ -particle system is obtained if the additional particle is added in the lowest available momentum state, one on the Fermi surface. The

chemical potential  $\mu$ , defined as

$$\mu = E_0(N + 1) - E_0(N) = \frac{\partial E_0}{\partial N}, \quad (1.4a)$$

is thus given by

$$\mu = p_F^2/2m. \quad (1.4b)$$

The chemical potential is equal to the energy of a particle *on the Fermi surface*. This result will be seen to apply equally well in the presence of particle interaction.

Excited states of the system are best specified with reference to the ground state. A given excited state is obtained by "exciting" a certain number of particles across the Fermi surface. Such a procedure is equivalent to creating an equal number of *particles* outside  $S_F$  and of *holes* inside  $S_F$ . Particles and holes thus appear as "elementary excitations," whose configurations give rise to all excited states. The amount of "excitation" is characterized by the departure of the distribution function from its value in the ground state

$$\delta n_{\mathbf{p}} = n_{\mathbf{p}} - n_{\mathbf{p}}^0. \quad (1.5)$$

A particle excitation of momentum  $\mathbf{p}' > p_F$  corresponds to  $\delta n_{\mathbf{p}} = \delta_{\mathbf{p}\mathbf{p}'}$ , while a hole excitation of momentum  $\mathbf{p}' < p_F$  corresponds to  $\delta n_{\mathbf{p}} = -\delta_{\mathbf{p}\mathbf{p}'}$ . For the noninteracting system, the excitation energy is simply

$$E - E_0 = \sum_{\mathbf{p}} \frac{p^2}{2m} \delta n_{\mathbf{p}}. \quad (1.6)$$

At low temperatures, particles and holes will only be excited near the Fermi surface;  $\delta n_{\mathbf{p}}$  will typically be of order 1 in a small region surrounding  $S_F$ , and will otherwise be negligible.

In an isolated system, the total number of particles is conserved: the number of excited particles must therefore be equal to that of excited holes. This restriction is sometimes inconvenient. It is then preferable to work with what is equivalent to the grand canonical ensemble of statistical mechanics, a system which is characterized by its chemical potential  $\mu$  rather than by its number of particles  $N$ . Such a situation may be realized by imagining the system to be in contact with a reservoir of particles. In such cases, the quantity of interest is not the energy  $E$ , but rather the *free energy*, given by  $F = E - \mu N$  at zero temperature. It follows from Eq. (1.6) that the excitation free energy associated with the distribution  $\delta n_{\mathbf{p}}$  is given by

$$F - F_0 = \sum_{\mathbf{p}} \left( \frac{p^2}{2m} - \mu \right) \delta n_{\mathbf{p}}. \quad (1.7)$$

Equation (1.7) obviously reduces to (1.6) when the number of particles is conserved, i.e., when  $\sum_{\mathbf{p}} \delta n_{\mathbf{p}} = 0$ .

According to (1.7), the free energy of a particle with momentum  $\mathbf{p}$  is  $(p^2/2m - \mu)$ ; it corresponds to the free energy of an elementary excitation *outside* the Fermi surface. Inside  $S_F$ , the excitations are *holes*, for which  $\delta n_{\mathbf{p}} = -1$ . The free energy associated with these excitations is then  $(\mu - p^2/2m)$ , rather than  $(p^2/2m - \mu)$ . Since  $\mu$  is equal to  $p^2/2m$  on the Fermi surface, the free energy of an *elementary excitation* of momentum  $\mathbf{p}$  (not to be confused with that of a particle) may be written as  $|p^2/2m - \mu|$ , a result valid both inside and outside  $S_F$ . Note that the excitation free energy is always positive: this is necessary to ensure the stability of the ground state.

#### DEFINITION OF QUASIPARTICLES AND QUASIHOLES

Let us now turn to the case of an interacting Fermi liquid. We are interested in the nature of its elementary excitations. A "frontal" attack on the problem involves the introduction of Green's functions, and the mathematical apparatus of many-body perturbation theory, an approach which lies far beyond the scope of this book. We therefore adopt an alternative approach, which consists in comparing the interacting "real" liquid with the noninteracting "ideal" gas; we establish a one-to-one correspondence between the eigenstates of the two systems. Such an approach will provide us with a qualitative understanding of the excitation spectrum of an interacting system.

Consider an eigenstate of the ideal system, characterized by a distribution function  $n_{\mathbf{p}}$ . In order to establish a connection with the real system, we imagine that the interaction between the particles is switched on infinitely slowly. Under such "adiabatic" conditions, the ideal eigenstates will progressively transform into certain eigenstates of the real interacting system. However, there is no *a priori* reason why such a procedure should generate *all* real eigenstates. For instance, it may well happen that the real ground state may not be obtained in that way; superconductors furnish us with a specific example of such a failure. We therefore must *assume* that the real ground state may be adiabatically generated starting from some ideal eigenstate with a distribution  $n_{\mathbf{p}}^{\circ}$ . This statement may be considered as the definition of a *normal* fermion system.

For reasons of symmetry, the distribution  $n_{\mathbf{p}}^{\circ}$  of an isotropic system is spherical. As a result, the spherical Fermi surface is not changed when the interaction between particles is switched on: the real ground state is generated adiabatically from the ideal ground state. Matters are otherwise when the Fermi surface for the noninteracting system is

anisotropic (as is the case for metals). Under these circumstances, the Fermi surface will certainly be deformed when the interaction is switched on. In such cases, the real ground state may be shown to follow adiabatically from some *excited* state of the noninteracting system. This situation will not cause any major difficulty with the theory.

Let us now add a particle with momentum  $\mathbf{p}$  to the ideal distribution  $n_{\mathbf{p}}^0$  and, again, turn on the interaction between the particles adiabatically. We generate an excited state of the real liquid, which likewise has momentum  $\mathbf{p}$ , since momentum is conserved in particle collisions. As the interaction is increased we may picture the bare particle as slowly perturbing the particles in its vicinity; if the change in interaction proceeds sufficiently slowly, the entire system of  $N + 1$  particles will remain in equilibrium. Once the interaction is completely turned on, we find that our particle moves together with the surrounding particle distortion brought about by the interaction. In the language of field theory, we would say that the particle is "dressed" with a self-energy cloud. We shall consider the "dressed" particle as an independent entity, which we call a *quasiparticle*. The above excited state corresponds to the real ground state plus a quasiparticle of momentum  $\mathbf{p}$ .

Let  $S_F$  be the Fermi surface characterizing the unperturbed distribution  $n_{\mathbf{p}}^0$  from which the real ground state is built up. Because of the exclusion principle, quasiparticle excitations can be generated only if their momentum  $\mathbf{p}$  lies *outside*  $S_F$ . The quasiparticle distribution in  $\mathbf{p}$  space is thus sharply bounded by the Fermi surface  $S_F$ .

Using the same adiabatic switching procedure, we can define a *quasihole*, with a momentum  $\mathbf{p}$  lying *inside* the Fermi surface  $S_F$ ; we may do likewise for higher configurations involving several excited quasiparticles and quasiholes. The quasiparticles and quasiholes thus appear as *elementary excitations* of the real system which, when combined, give rise to a large class of excited states. We have established our desired one-to-one correspondence between ideal and real eigenstates.

Actually, our adiabatic switching method is likely to run into difficulties when the real state under study is damped as a consequence of particle interaction. If the time over which the interaction is turned on is longer than the lifetime of the state that we wish to generate, the switching on of the interaction is no longer *reversible*, since the state has decayed long before the physical value of the interaction is reached. On the other hand, if the interaction is turned on too fast, the process is no longer *adiabatic*, and we do not generate eigenstates of the interacting system. This limitation is not due to mathematical clumsiness: it follows from the physical uncertainty arising from the finite lifetime of the state under consideration.

Such difficulties do not arise for the ground state, which, being stable, can be precisely specified. On the other hand, since quasiparticles and quasiholes undergo real collisions, which lead to damping, any definition of the elementary excitations is somewhat imprecise. Fortunately, the quasiparticle lifetime becomes sufficiently long in the immediate vicinity of the Fermi surface, that the quasiparticle concept makes sense in that region. In pure systems at zero temperature, the lifetime varies as the inverse square of the energy separation from  $S_F$ . Since the quasiparticle is better and better defined as one gets closer and closer to the Fermi surface,  $S_F$  remains sharply defined. It should be kept in mind that a quasiparticle is only strictly defined if it is right on the Fermi surface. In order for our theory to make sense, we must be careful to introduce quasiparticles only in the neighborhood of  $S_F$ .

Let us consider an eigenstate of the noninteracting system, characterized by some distribution function  $n_p$  for the usual "bare" particles. By switching on the interaction adiabatically, we obtain an eigenstate of the real system, which may be labeled by the same function  $n_p$ . In the interacting system,  $n_p$  describes the *distribution of quasiparticles*. The "excitation" of the system is measured by the departure  $\delta n_p$  from the ground state distribution

$$\delta n_p = n_p - n_p^\circ. \quad (1.8)$$

At low temperatures, one only samples low-lying excited states, for which  $\delta n_p$  is restricted to the immediate vicinity of the Fermi surface. Under such conditions, quasiparticle damping is negligible, and our overall picture becomes meaningful.

Let us emphasize that the physically meaningful quantity is  $\delta n_p$ , rather than  $n_p$ . (It does not make much sense to define an equilibrium distribution  $n_p^\circ$  in a range where quasiparticles are unstable.) We shall thus be careful to formulate our theory in terms of the departure from equilibrium,  $\delta n_p$ ;  $n_p^\circ$  will be used only as an intermediate step. In fact, our results will always involve the gradient  $\nabla n_p^\circ$ , a quantity which is localized on the Fermi surface.

#### ENERGY OF QUASIPARTICLES

For the ideal system, there exists a simple linear relation between the energy of a given state and the corresponding distribution function. When particle interaction is taken into account, the relation between the state energy,  $E$ , and the quasiparticle distribution function,  $n_p$ , becomes much more complicated. It may be expressed in a functional form,  $E[n_p]$ , which one cannot in general specify explicitly. If, however,  $n_p$  is sufficiently close to the ground state distribution  $n_p^\circ$ , we can

carry out a Taylor expansion of this functional. On writing  $n_{\mathbf{p}}$  in the form (1.8), and taking  $\delta n_{\mathbf{p}}$  to be small, or to extend over a small region in momentum space, we obtain

$$E[n_{\mathbf{p}}] = E_0 + \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} \delta n_{\mathbf{p}} + O(\delta n^2), \quad (1.9)$$

where  $\epsilon_{\mathbf{p}}$  is the first functional derivative of  $E$ . Since each summation over  $\mathbf{p}$  carries a factor  $\Omega$ ,  $\epsilon_{\mathbf{p}}$  is of order  $\Omega^0 = 1$ .

If  $\delta n_{\mathbf{p}}$  describes a state with one extra quasiparticle  $\mathbf{p}$ , the energy of the state is  $(E_0 + \epsilon_{\mathbf{p}})$ :  $\epsilon_{\mathbf{p}}$  is the energy of the quasiparticle. According to Eq. (1.9), the energies of several excited quasiparticles are simply additive, within corrections of order  $(\delta n)^2$ . We shall assume that  $\epsilon_{\mathbf{p}}$  is continuous when  $\mathbf{p}$  crosses the Fermi surface. This statement is not obvious and, again, should be considered as a characteristic of "normal" systems.

On the Fermi surface,  $\epsilon_{\mathbf{p}}$  must equal a constant  $\epsilon_F$  (the Fermi energy): otherwise, we could lower the ground state energy by transferring a particle from a state inside the Fermi surface to one of lower energy outside  $S_F$ . Since the ground state for  $(N + 1)$  particles is obtained by adding a quasiparticle on the Fermi surface,  $\epsilon_F$  is simply the *chemical potential*,  $\mu = \partial E_0 / \partial N$ , at zero temperature. This very important property was first established by Hugenholtz and Van Hove (1958).

In practice, we need only values of  $\epsilon_{\mathbf{p}}$  in the vicinity of the Fermi surface, where we can use a series expansion. The gradient of  $\epsilon_{\mathbf{p}}$ ,

$$\mathbf{v}_{\mathbf{p}} = \nabla_{\mathbf{p}} \epsilon_{\mathbf{p}}, \quad (1.10)$$

plays the role of a "group velocity" of the quasiparticle. We shall see later that  $\mathbf{v}_{\mathbf{p}}$  is the velocity of a quasiparticle wave packet. In the absence of a magnetic field, and for a system which is reflection invariant,  $\epsilon_{\mathbf{p}}$  and  $\mathbf{v}_{\mathbf{p}}$  do not depend on the spin  $\sigma$ . For an isotropic system,  $\epsilon_{\mathbf{p}}$  depends only on  $p$ , and may be denoted as  $\epsilon_p$ . The velocity  $\mathbf{v}_{\mathbf{p}}$  is then parallel to  $\mathbf{p}$ , so that we can write (on the Fermi surface)

$$v_{p_F} = p_F / m^*, \quad (1.11)$$

where  $m^*$  is called the *effective mass* of the quasiparticle. We remark that the present definition of the effective mass differs from that adopted to describe the motion of independent particles in a periodic lattice; in the latter case,  $1/m^*$  is defined as the second derivative of  $\epsilon_p$  with respect to  $p$ .

In an anisotropic system, the velocity  $|\mathbf{v}_{\mathbf{p}}|$  varies over the Fermi surface; the notion of an effective mass is somewhat artificial. It is then convenient to introduce the density  $\nu(\epsilon)$  of quasiparticle states having

an energy  $(\mu + \epsilon)$ :

$$\nu(\epsilon) = \sum_{\mathbf{p}} \delta(\epsilon_{\mathbf{p}} - \mu - \epsilon). \quad (1.12)$$

At low temperatures, all physical properties will depend on the density of states on the Fermi surface,  $\nu(0)$ .

This discussion shows the similarity between bare particles in a Fermi gas and quasiparticles in a Fermi liquid. They have the same distribution in momentum space, bounded at  $T = 0$  by a sharp Fermi surface. Both follow Fermi statistics. The main difference that we have pointed out is the change of energy and velocity brought about by the interaction with the surrounding medium. Actually, quasiparticles have physical features that do not appear in noninteracting systems: these are discussed in the next section.

### 1.2. INTERACTION BETWEEN QUASIPARTICLES: LANDAU'S THEORY OF FERMILIQUIDS

#### EXPANSION OF THE FREE ENERGY

As we pointed out earlier, the quantity of physical interest is the free energy  $F = E - \mu N$ , rather than the energy  $E$ . The "excitation" free energy, as measured from the ground state value  $F_0$ , is given by

$$F - F_0 = E - E_0 - \mu(N - N_0), \quad (1.13)$$

where  $N_0$  is the number of particles in the ground state. In order to generalize the expansion (1.9), we need to calculate  $(N - N_0)$ . For that purpose, we note that by adding one quasiparticle to the ground state, we add *exactly one* bare particle to the system as a whole. This follows at once from our "adiabatic" definition of quasiparticles: the state with one extra quasiparticle is derived from an ideal state containing  $(N + 1)$  particles, and the total number of particles is conserved when the interaction is switched on adiabatically. The difference  $(N - N_0)$  may thus be written as

$$N - N_0 = \sum_{\mathbf{p}} \delta n_{\mathbf{p}}. \quad (1.14)$$

Using Eqs. (1.9) and (1.14), we obtain, within corrections of order  $(\delta n)^2$ ,

$$F - F_0 = \sum_{\mathbf{p}} (\epsilon_{\mathbf{p}} - \mu) \delta n_{\mathbf{p}}. \quad (1.15)$$

If the number of particles of the system is kept constant, the difference

(1.14) vanishes:  $(E - E_0)$  is then equal to  $(F - F_0)$ , and may be written in the form (1.15), as well as (1.9).

Most of the properties that we shall consider will involve a displacement of the Fermi surface by a small amount  $\delta$ . The corresponding value of  $\delta n_p$  is equal to  $\pm 1$  in a thin sheet of width  $\delta$  centered on the Fermi surface; it vanishes outside this sheet. Where  $\delta n_p$  is nonzero,  $(\epsilon_p - \mu)$  is of order  $\delta$ . The difference  $(F - F_0)$ , given by Eq. (1.15), is thus of order  $\delta^2$ . The expansion (1.15), which looks like a first-order expansion, is actually a second-order one. Our approach is therefore consistent only if we push the Taylor expansion of the functional  $(F - F_0)$  one step further, to include all terms of second order in the displacement of the Fermi surface. We are thus led to write

$$F - F_0 = \sum_{\mathbf{p}} (\epsilon_{\mathbf{p}} - \mu) \delta n_{\mathbf{p}} + \frac{1}{2} \sum_{\mathbf{p}\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'} \delta n_{\mathbf{p}} \delta n_{\mathbf{p}'} + O(\delta n^3). \quad (1.16)$$

Equation (1.16) is the heart of the phenomenological theory of Fermi liquids proposed by Landau (1956). Its most important feature is the new quadratic term, which describes the interaction between quasiparticles. Such a term is not present in the approximate theory of Sommerfeld; we shall see later that it considerably modifies a number of physical properties of the system.

Equation (1.16) furnishes the leading terms of an expansion of  $(F - F_0)$  in powers of the relative number of excited quasiparticles. The latter is measured by the ratio

$$\alpha = \frac{\sum_{\mathbf{p}} |\delta n_{\mathbf{p}}|}{N}. \quad (1.17)$$

Landau's approximation is valid whenever  $\alpha$  is small. We must be careful to maintain consistent approximations throughout any calculation, and to keep in our results only the leading terms with respect to the parameter  $\alpha$ .

The coefficient  $f_{\mathbf{p}\mathbf{p}'}$  is the second variational derivative of  $E$  (or  $F$ ) with respect to  $n_{\mathbf{p}}$ . It is accordingly invariant under permutation of  $\mathbf{p}$  and  $\mathbf{p}'$ . Since each summation over a momentum index carries a factor  $\Omega$ ,  $f_{\mathbf{p}\mathbf{p}'}$  is of order  $1/\Omega$ . This is easily understood if we realize that  $f_{\mathbf{p}\mathbf{p}'}$  is the *interaction energy* of the excited quasiparticles  $\mathbf{p}$  and  $\mathbf{p}'$ . Each of the latter is spread out over the whole volume  $\Omega$ ; the probability that they interact with one another is of order  $a^3/\Omega$ , where  $a$  is the range of the interaction.

We shall assume that  $f_{\mathbf{p}\mathbf{p}'}$  is continuous when  $\mathbf{p}$  or  $\mathbf{p}'$  crosses the Fermi surface. Once again, this may be considered as characteristic of a "normal system." In practice, we shall only need values of  $f$  on the

Fermi surface, at points such that  $\epsilon_{\mathbf{p}} = \epsilon_{\mathbf{p}'} = \mu$ . Then  $f_{\mathbf{p}\mathbf{p}'}$  depends only on the direction of  $\mathbf{p}$  and  $\mathbf{p}'$ , and on the spins  $\sigma$  and  $\sigma'$ .

If there is no applied magnetic field, the system is invariant under time reversal, which implies

$$f_{\mathbf{p}\sigma, \mathbf{p}'\sigma'} = f_{-\mathbf{p}-\sigma, -\mathbf{p}'-\sigma'} \quad (1.18)$$

(we have explicitly introduced the spin indices). If furthermore the Fermi surface is invariant under reflection  $\mathbf{p} \rightarrow -\mathbf{p}$ , Eq. (1.18) becomes

$$f_{\mathbf{p}\sigma, \mathbf{p}'\sigma'} = f_{\mathbf{p}-\sigma, \mathbf{p}'-\sigma'}. \quad (1.19)$$

In that case,  $f_{\mathbf{p}\sigma, \mathbf{p}'\sigma'}$  depends only on the relative orientation of the spins  $\sigma$  and  $\sigma'$ ; there are only two independent components, corresponding respectively to parallel spins and antiparallel spins. It is convenient to write these in the form

$$\begin{aligned} f_{\mathbf{p}\mathbf{p}'}^{\uparrow\uparrow} &= f_{\mathbf{p}\mathbf{p}'}^s + f_{\mathbf{p}\mathbf{p}'}^a, \\ f_{\mathbf{p}\mathbf{p}'}^{\uparrow\downarrow} &= f_{\mathbf{p}\mathbf{p}'}^s - f_{\mathbf{p}\mathbf{p}'}^a, \end{aligned} \quad (1.20)$$

where  $f_{\mathbf{p}\mathbf{p}'}^s$  and  $f_{\mathbf{p}\mathbf{p}'}^a$  are the spin symmetric and spin antisymmetric parts of the quasiparticle interaction. The antisymmetric term  $f_{\mathbf{p}\mathbf{p}'}^a$  may be interpreted as due to an *exchange* interaction energy  $2f_{\mathbf{p}\mathbf{p}'}^a$ , which appears only when the spins are parallel. In the Russian literature, Eq. (1.20) is usually replaced by

$$f_{\mathbf{p}\sigma, \mathbf{p}'\sigma'} = \varphi_{\mathbf{p}\mathbf{p}'} + \delta \cdot \delta' \psi_{\mathbf{p}\mathbf{p}'}, \quad (1.21)$$

where  $\delta$  and  $\delta'$  are the spin matrices. The coefficient  $\psi$  is four times bigger than our  $f^a$ . We prefer to use the more symmetric decomposition (1.20).

If the system is isotropic, (1.20) may be further simplified. In that case, for  $\mathbf{p}$  and  $\mathbf{p}'$  on the Fermi surface,  $f_{\mathbf{p}\mathbf{p}'}^s$  and  $f_{\mathbf{p}\mathbf{p}'}^a$  depend only on the angle  $\xi$  between the directions of  $\mathbf{p}$  and  $\mathbf{p}'$ . They may be expanded in a series of Legendre polynomials

$$f_{\mathbf{p}\mathbf{p}'}^{s(a)} = \sum_{\ell=0}^{\infty} f_{\ell}^{s(a)} P_{\ell}(\cos \xi). \quad (1.22)$$

$f$  is completely determined by the set of coefficients  $f_{\ell}^s$  and  $f_{\ell}^a$ . It is convenient to express the latter in "reduced" units by setting

$$\nu(0) f_{\ell}^{s(a)} = \frac{\Omega m^* p_F}{\pi^2 \hbar^3} f_{\ell}^{s(a)} = F_{\ell}^{s(a)}. \quad (1.23)$$

The dimensionless quantities  $F_{\ell}^s$  and  $F_{\ell}^a$  measure the strength of the interaction as compared to the kinetic energy.

## LOCAL ENERGY OF A QUASIPARTICLE

Let us consider a state of the system with a certain distribution of excited quasiparticles  $\delta n_{\mathbf{p}'}$ . To this system we now add an extra quasiparticle,  $\mathbf{p}$ . According to Eq. (1.16), the free energy of the additional quasiparticle is equal to

$$\bar{\epsilon}_{\mathbf{p}} - \mu = (\epsilon_{\mathbf{p}} - \mu) + \sum_{\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'} \delta n_{\mathbf{p}'}. \quad (1.24)$$

If  $\mathbf{p}$  is close enough to the Fermi surface, both terms of Eq. (1.24) have the same order of magnitude. The free energy of a quasiparticle thus depends on the state of the system through its interaction energy with other excited quasiparticles. This physical effect, which does not exist in noninteracting systems, arises from the quadratic term of (1.16).

The quantity  $\bar{\epsilon}_{\mathbf{p}}$  plays a central role in the development of the Landau theory. In order to clarify its physical meaning, let us suppose that the system is slightly inhomogeneous. The departure from equilibrium may be characterized by a position-dependent distribution of excited quasiparticles,  $\delta n_{\mathbf{p}'}(\mathbf{r})$  (see Section 1.4). As a result,  $\bar{\epsilon}_{\mathbf{p}}$  depends on  $\mathbf{r}$ : it may be regarded as a *local energy*, perturbed by the surrounding distortion of the medium. We shall see that the gradient of  $\bar{\epsilon}_{\mathbf{p}}$  in ordinary space

$$\nabla_{\mathbf{r}} \bar{\epsilon}_{\mathbf{p}} = \nabla_{\mathbf{r}} \left\{ \sum_{\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'} \delta n_{\mathbf{p}'} \right\} \quad (1.25)$$

may be interpreted as an average *force* exerted by the surrounding medium on the quasiparticle  $\mathbf{p}$ .

It is mathematically convenient to introduce the distribution function

$$\bar{n}_{\mathbf{p}}^{\circ} = n^{\circ}(\bar{\epsilon}_{\mathbf{p}} - \mu), \quad (1.26)$$

where  $n^{\circ}$  is the usual Fermi-Dirac step function defined by

$$n^{\circ}(x) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases} \quad (1.27)$$

It will be seen that for the slightly inhomogeneous system, the distribution  $\bar{n}_{\mathbf{p}}^{\circ}$  corresponds to a "local equilibrium" of quasiparticles, in much the same way as  $n_{\mathbf{p}}^{\circ} = n^{\circ}(\epsilon_{\mathbf{p}} - \mu)$  describes the true equilibrium.

The departure from this state of local equilibrium is measured by the difference

$$\delta \bar{n}_{\mathbf{p}} = n_{\mathbf{p}} - \bar{n}_{\mathbf{p}}^{\circ}. \quad (1.28)$$

On comparing the definitions of  $\delta n_{\mathbf{p}}$  and  $\delta \bar{n}_{\mathbf{p}}$ , we see that

$$\delta n_{\mathbf{p}} = \delta \bar{n}_{\mathbf{p}} + \frac{\partial n_{\mathbf{p}}^{\circ}(\epsilon_{\mathbf{p}} - \mu)}{\partial \epsilon_{\mathbf{p}}} (\bar{\epsilon}_{\mathbf{p}} - \epsilon_{\mathbf{p}}). \quad (1.29)$$

Using Eq. (1.24), we obtain

$$\delta\bar{n}_{\mathbf{p}} = \delta n_{\mathbf{p}} - \frac{\partial n^0}{\partial \epsilon_{\mathbf{p}}} \sum_{\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'} \delta n_{\mathbf{p}'}. \quad (1.30)$$

In practice,  $\delta n_{\mathbf{p}}$  will always contain a factor  $\partial n^0/\partial \epsilon_{\mathbf{p}}$ . At zero temperature, we have

$$\frac{\partial n^0}{\partial \epsilon_{\mathbf{p}}} = -\delta(\epsilon_{\mathbf{p}} - \mu); \quad (1.31)$$

it follows that both  $\delta n$  and  $\delta\bar{n}$  are restricted to the Fermi surface  $S_F$ .

For an isotropic system at zero temperature, the relation between  $\delta n_{\mathbf{p}}$  and  $\delta\bar{n}_{\mathbf{p}}$  may be greatly simplified. We split  $\delta n_{\mathbf{p}}$  and  $\delta\bar{n}_{\mathbf{p}}$  into a spin symmetric and a spin antisymmetric part, by setting

$$\delta n_{\mathbf{p},\pm} = \delta n_{\mathbf{p}}^s \pm \delta n_{\mathbf{p}}^a \quad (1.32)$$

(with a similar expression for  $\delta\bar{n}_{\mathbf{p}}$ ). We further expand these quantities in a series of normalized spherical harmonics, such as

$$\delta n_{\mathbf{p}}^s = \sum_{\ell m} \delta(\epsilon_{\mathbf{p}} - \mu) \delta n_{\ell m}^s Y_{\ell m}(\theta, \varphi), \quad (1.33)$$

where  $(\theta, \varphi)$  are the polar angles of  $\mathbf{p}$ . By putting these expanded forms of  $\delta n_{\mathbf{p}}$  and  $\delta\bar{n}_{\mathbf{p}}$  back into Eq. (1.30), together with the expansion (1.22) of  $f_{\mathbf{p}\mathbf{p}'}$ , and by making use of the addition theorem for spherical harmonics, we find

$$\begin{aligned} \delta\bar{n}_{\ell m}^s &= \left(1 + \frac{F_{\ell}^s}{2\ell + 1}\right) \delta n_{\ell m}^s, \\ \delta\bar{n}_{\ell m}^a &= \left(1 + \frac{F_{\ell}^a}{2\ell + 1}\right) \delta n_{\ell m}^a, \end{aligned} \quad (1.34)$$

where the coefficients  $F_{\ell}^s$  and  $F_{\ell}^a$  are given by (1.23). The passage from  $\delta n_{\mathbf{p}}$  to  $\delta\bar{n}_{\mathbf{p}}$  is thus straightforward.

The quantity  $\delta\bar{n}_{\mathbf{p}}$  will prove to be extremely useful in the following sections. Many properties are worked out much more easily in terms of  $\delta\bar{n}_{\mathbf{p}}$  than of  $\delta n_{\mathbf{p}}$ . The interplay of the two quantities is important and Eqs. (1.30) and (1.34) will be used very often.

#### EQUILIBRIUM DISTRIBUTION OF QUASIPARTICLES AT A FINITE TEMPERATURE

Let us consider a system at temperature  $T$ , with a chemical potential  $\mu$ : the latter may be defined as the adiabatic derivative

$$\mu = (\partial E/\partial N)_S. \quad (1.35)$$

Using elementary thermodynamics, we may show that

$$\mu = (\partial E / \partial N)_T - T(\partial S / \partial N)_T. \quad (1.36)$$

At very low temperatures, the second term of Eq. (1.36) is negligible compared to the first; the chemical potential is then equal to the isothermal derivative of  $E$  with respect to  $N$ , which is easily calculated.

We obtain the equilibrium distribution of quasiparticles,  $n_p^\circ(T, \mu)$ , by a well-known procedure: given a distribution  $n_p$ , one counts the number of ways  $W$  in which the quasiparticles may be distributed among the various occupied states; one then chooses  $n_p$  in such a way as to maximize  $W$ , while keeping the total free energy constant. We find

$$n_p^\circ(T, \mu) = \frac{1}{1 + \exp [(\bar{\epsilon}_p - \mu) / \kappa T]}, \quad (1.37)$$

where  $\bar{\epsilon}_p$  is the quasiparticle *local* energy appropriate to the distribution  $n_p^\circ(T, \mu)$ . From the maximum value  $W_{\max}$ , we may obtain the entropy  $S = \kappa \log W_{\max}$ .

The "local" energy  $\bar{\epsilon}_p$  is given by Eq. (1.24),  $\delta n_p$  being here equal to

$$\delta n_p = n_p^\circ(T, \mu) - n_p^\circ(0, \mu). \quad (1.38)$$

At very low temperatures, the integral

$$\int \delta n_p p^2 dp$$

is of order  $T^2$  in any direction of momentum space. The interaction energy between excited quasiparticles vanishes to the same order, and may therefore be neglected in comparison to  $(\epsilon_p - \mu)$ , which is of order  $T$ . Hence in the low temperature limit, we may replace  $\bar{\epsilon}_p$  by  $\epsilon_p$  in the equilibrium distribution (1.37). By the same token, we prove that the number of particles at constant  $\mu$  (or conversely the chemical potential at constant  $N$ ) remains constant up to order  $T^2$ .

The thermal motion will only excite quasiparticles within a distance  $\kappa T$  from the Fermi surface. The percentage of excited quasiparticles is  $\kappa T / \mu$ , which is essentially the expansion parameter of (1.16). In neglecting the terms of order  $\delta n^3$ , we make an error in the energy of the order of  $(\kappa T / \mu)^3$ . In view of this uncertainty, it would be meaningless to evaluate to this order the temperature dependence of any other factor entering the theory. To keep in the spirit of Landau's theory, we should retain only the leading terms with respect to  $T$  (usually of order 0 or 1). There is no need to consider higher-order corrections, which are of the same order as the terms of (1.16) that are regarded as negligible.

### 1.3. EQUILIBRIUM PROPERTIES

We shall now apply the Landau theory to the study of a number of macroscopic properties, characteristic of the system at equilibrium. We shall find that some of these properties are affected by the interaction between quasiparticles, while others are not.

#### SPECIFIC HEAT

Let us first consider the specific heat, defined as

$$C_v = (\partial E / \partial T)_N. \quad (1.39)$$

It is left as a problem to the reader to show that at low temperature  $C_v$  may be put in the more convenient form

$$C_v = (\partial F / \partial T)_\mu. \quad (1.40)$$

In order to calculate  $C_v$ , we write the free energy in the form (1.16),  $\delta n_p$  being given by (1.38). We have seen that the integral of  $\delta n_p$  in any direction of momentum space vanishes up to order  $T^2$ . The interaction energy in Eq. (1.16) is therefore of order  $T^4$ , and is thus negligible with respect to the main term of order  $T^2$ . The "thermal" free energy  $[F(T) - F_0]$  is given by an expression of the same form as for a non-interacting system. A straightforward calculation leads to

$$F(T) - F_0 = \frac{\pi^2}{6} \nu(0) (\kappa T)^2, \quad (1.41)$$

where the density of states  $\nu(0)$  is defined by (1.12). For an isotropic system,  $\nu(0)$  is given by

$$\nu(0) = \frac{\Omega m^* p_F}{\pi^2 \hbar^3}, \quad (1.42)$$

from which we obtain the specific heat

$$C_v = \frac{m^* p_F}{3 \hbar^3} \kappa^2 T. \quad (1.43)$$

A measurement of the slope of the linear specific heat therefore yields the *effective mass* of the quasiparticles. We remark that  $C_v$  is not affected by the interaction between quasiparticles.

#### COMPRESSIBILITY AND SOUND VELOCITY

Let  $E_0(\Omega)$  be the ground state energy of the system, regarded as a function of the volume  $\Omega$ . The pressure  $P$  may be defined as

$$P = -\partial E_0 / \partial \Omega. \quad (1.44)$$

(At an arbitrary temperature, Eq. (1.44) should be an adiabatic derivative: we have seen that near  $T = 0$  there is no difference between adiabatic and isothermal processes). The compressibility  $\kappa$  is then given by

$$\frac{1}{\kappa} = -\Omega \frac{\partial P}{\partial \Omega}. \quad (1.45)$$

For a large system,  $E_o$  is an "extensive" quantity, proportional to the volume when the density  $\rho = N/\Omega$  is kept constant. We may thus write

$$E_o = \Omega f(\rho). \quad (1.46)$$

One finds directly from Eqs. (1.44) and (1.46) that

$$\frac{1}{\kappa} = \rho^2 f''(\rho). \quad (1.47)$$

These quantities may be related to the chemical potential  $\mu = \partial E_o / \partial N$ . Indeed, it is straightforward to establish that

$$\frac{1}{\kappa} = N \rho (\partial \mu / \partial N)_\Omega. \quad (1.48)$$

The compressibility is related to the velocity of sound,  $s$ , in the usual way:

$$s^2 = \frac{1}{\kappa m \rho} = \frac{N}{m} \frac{\partial \mu}{\partial N}. \quad (1.49)$$

We now calculate  $\partial \mu / \partial N$  or, rather, its inverse  $\partial N / \partial \mu$ . Let us increase  $\mu$  by an infinitesimal amount  $d\mu$ . The Fermi surface swells slightly. (We assume reflection invariance, in order to have the same Fermi surface for both spin directions.) An arbitrary point  $A$  of the original Fermi surface undergoes a normal displacement

$$dp_F = (\partial p_F / \partial \mu) d\mu \quad (1.50)$$

which brings it to the position  $B$  (Fig. 1.2). On the new Fermi surface, the quasiparticle energy must be equal to  $(\mu + d\mu)$ : this fixes the displacement  $dp_F$ . More precisely,  $B$  must be such that

$$\epsilon_B(\mu + d\mu) - \epsilon_A(\mu) = d\mu. \quad (1.51)$$

The left-hand side of Eq. (1.51) contains two terms:

(i) When passing from  $A$  to  $B$ , the momentum  $\mathbf{p}$  changes, giving rise to an energy shift  $v_p dp_F$ ;

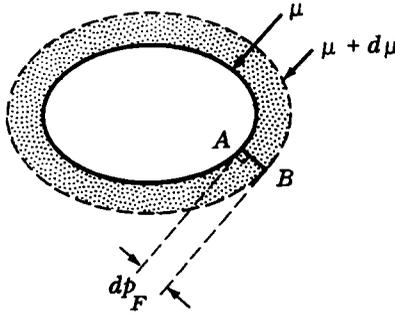
(ii) Increasing  $\mu$  by an amount  $d\mu$  means adding quasiparticles in the shaded area of Fig. 1.2.  $\epsilon_p$  is then increased by the interaction energy with these extra quasiparticles.

Let  $\delta n_{\mathbf{p}}$  be the change of the distribution function resulting from the displacement of the Fermi surface.  $\delta n_{\mathbf{p}}$  is equal to +1 in the shaded area of Fig. 1.2 and to 0 outside. Collecting the two contributions to Eq. (1.51), we obtain

$$v_{\mathbf{p}} dp_F + \sum_{\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'} \delta n_{\mathbf{p}'} = d\mu. \quad (1.52)$$

Since  $f_{\mathbf{p}\mathbf{p}'}$  is a smooth function of  $\mathbf{p}$  and  $\mathbf{p}'$ , we may replace the above  $\delta n_{\mathbf{p}}$  by a suitably normalized  $\delta$ -function. The correct choice is easily verified to be (at zero temperature)

$$\delta n_{\mathbf{p}} = -\frac{dn^0}{d\epsilon_{\mathbf{p}}} \frac{\partial \epsilon_{\mathbf{p}}}{\partial p} dp_F = \delta(\epsilon_{\mathbf{p}} - \mu) v_{\mathbf{p}} dp_F. \quad (1.53)$$



**FIGURE 1.2.** Deformation of the Fermi surface when the chemical potential is increased from  $\mu$  to  $(\mu + d\mu)$ .

Inserting this into Eq. (1.52), and dividing by  $d\mu$ , we obtain

$$v_{\mathbf{p}} (\partial p_F / \partial \mu) + \sum_{\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'} v_{\mathbf{p}'} (\partial p'_F / \partial \mu) \delta(\epsilon_{\mathbf{p}'} - \mu) = 1. \quad (1.54)$$

This integral equation gives  $\partial p_F / \partial \mu$  in every direction: we have thus derived the change of the Fermi surface as  $\mu$  varies. In order to get  $\partial N / \partial \mu$ , we remark that

$$dN = \sum_{\mathbf{p}} \delta n_{\mathbf{p}} = \sum_{\mathbf{p}} \delta(\epsilon_{\mathbf{p}} - \mu) v_{\mathbf{p}} dp_F. \quad (1.55)$$

The sound velocity is then given by

$$N/ms^2 = \partial N / \partial \mu = \sum_{\mathbf{p}} \delta(\epsilon_{\mathbf{p}} - \mu) v_{\mathbf{p}} (\partial p_F / \partial \mu). \quad (1.56)$$

For an isotropic system,  $\partial p_F / \partial \mu$  is independent of direction. In that case, only the  $\ell = 0$ , spin-independent term of  $f_{\mathbf{p}\mathbf{p}'}$  contributes to Eq. (1.54). Using the definition (1.23), we find that

$$v_{\mathbf{p}}(\partial p_F / \partial \mu) = \frac{1}{1 + F_0^s}. \quad (1.57)$$

Inserting this into Eq. (1.56), we finally obtain

$$\frac{N}{ms^2} = \frac{\partial N}{\partial \mu} = \frac{\nu(0)}{1 + F_0^s}. \quad (1.58)$$

In Eq. (1.58) we have the usual result for independent particles (of mass  $m^*$ ) modified by the factor  $(1 + F_0^s)^{-1}$ ; the latter term is a direct consequence of the interaction between the quasiparticles.

We have gone at length through this demonstration to show clearly how the new features brought about by the particle interaction came into the problem. Actually, the result (1.58) may be readily obtained if we note that the *local energy* of a quasiparticle on the Fermi surface is always equal to the chemical potential,  $\mu$ . When  $\mu$  increases by an amount  $d\mu$ , the local Fermi energy is increased by the same amount. The new distribution may thus be written as  $n^\circ(\bar{\epsilon}_{\mathbf{p}} - \mu - d\mu)$ . The departure from the local equilibrium distribution  $n^\circ(\bar{\epsilon}_{\mathbf{p}} - \mu)$  is equal to

$$\delta \bar{n}_{\mathbf{p}} = -d\mu(\partial n^\circ / \partial \epsilon_{\mathbf{p}}). \quad (1.59)$$

For an isotropic system,  $\delta \bar{n}_{\mathbf{p}}$  is isotropic and spin independent. Making use of (1.34), we obtain

$$\delta n_{\mathbf{p}} = \frac{\delta \bar{n}_{\mathbf{p}}}{1 + F_0^s} = -\frac{\partial n^\circ / \partial \epsilon_{\mathbf{p}}}{1 + F_0^s} d\mu, \quad (1.60)$$

from which Eqs. (1.57) and (1.58) follow at once. This derivation stresses the usefulness of  $\delta \bar{n}_{\mathbf{p}}$  for practical calculations.

Let us now return to the result (1.58). Replacing  $\nu(0)$  by its expression (1.42), we obtain the sound velocity

$$s^2 = (p_F^2 / 3mm^*)(1 + F_0^s). \quad (1.61)$$

Since  $m^*$  is known from the specific heat, a measurement of  $s$  yields  $F_0^s$ . In the limit of weak interactions,  $m^* \rightarrow m$ , while  $F_0^s \rightarrow 0$ . The velocity of sound then tends toward  $v_F / \sqrt{3}$ . We note that Eq. (1.61) only makes sense if  $(1 + F_0^s) > 0$ . Otherwise, the system is unstable: an imaginary value of sound velocity signifies an exponential buildup of density fluctuations.

## SPIN SUSCEPTIBILITY

Let us apply to our Fermi liquid a uniform dc magnetic field  $\mathcal{H}$ . As usual, the system will develop a paramagnetic spin moment and a diamagnetic orbital moment. We shall postpone until Sections 4.7 and 5.4 consideration of the orbital effects which, although important, are much more difficult to treat.

In the field  $\mathcal{H}$ , a particle of spin  $\sigma = \pm \frac{1}{2}$  acquires a magnetic energy  $-g\beta\sigma\mathcal{H}$ , where  $g = 2$  is the Landé  $g$ -factor and  $\beta$  the Bohr magneton,  $e\hbar/2mc$ . The two spin orientations are no longer in equilibrium, having different chemical potentials ( $\mu \pm g\beta\mathcal{H}/2$ ). In order to restore equilibrium, the Fermi surfaces for spins  $\pm \frac{1}{2}$  must split, in such a way as to have the same chemical potential. In weak fields, the two Fermi surfaces are displaced by opposite amounts, the final chemical potential remaining equal to  $\mu$  (within corrections of order  $\mathcal{H}^2$ ).

The displacement of the Fermi surface with spin  $\sigma$  must be such as to bring the corresponding chemical potential from  $(\mu - g\beta\sigma\mathcal{H})$  back to  $\mu$ . Since the chemical potential is equal to the *local* energy of a quasi-particle on the Fermi surface, the equilibrium distribution in the presence of the field may be written as

$$n_{p\sigma}(\mathcal{H}) = n^0(\tilde{\epsilon}_p - \mu - g\beta\sigma\mathcal{H}). \quad (1.62)$$

By repeating the argument which led to Eq. (1.59), we obtain the departure from local equilibrium

$$\delta\bar{n}_{p\sigma} = -g\beta\sigma\mathcal{H} \frac{\partial n^0}{\partial \epsilon_p}. \quad (1.63)$$

In order to calculate the total magnetization

$$\mathfrak{M} = \sum_{\mathbf{p}} g\beta\sigma \delta n_{p\sigma}, \quad (1.64)$$

we must pass from  $\delta\bar{n}_{p\sigma}$  to  $\delta n_{p\sigma}$ . This is easily done if we assume the system to be isotropic.  $\delta\bar{n}_{p\sigma}$  is then isotropic and spin *antisymmetric*. Making use of Eq. (1.34), we find

$$\delta n_{p\sigma} = \frac{\delta\bar{n}_{p\sigma}}{1 + F_o^a} = - \frac{g\beta\sigma\mathcal{H}}{1 + F_o^a} \frac{\partial n^0}{\partial \epsilon_p}. \quad (1.65)$$

The calculation of  $\mathfrak{M}$  is then straightforward. One finds (putting  $g = 2$ )

$$\mathfrak{M} = \beta^2 \frac{\nu(0)}{1 + F_o^a} \mathcal{H}. \quad (1.66)$$

The spin susceptibility  $\chi_P$  is given by

$$\chi_P = \frac{\mathfrak{M}}{\Omega\mathcal{C}} = \frac{m^*p_F}{\pi^2\hbar^3} \frac{\beta^2}{1 + F_0^a}. \quad (1.67)$$

We see in (1.67) that the spin susceptibility is modified by the exchange interaction  $F_0^a$ . We cannot derive  $\chi_P$  from the simple knowledge of the density of states, as is the case for a noninteracting system. Indeed, by comparing experimental values of the specific heat (i.e., of  $m^*$ ) and of the susceptibility, one can "measure" the coefficient  $F_0^a$ . Again, Eq. (1.67) is meaningful only when

$$1 + F_0^a > 0. \quad (1.68)$$

Otherwise, the long wavelength fluctuations of the magnetic moment become unstable: the system becomes ferromagnetic.

#### EXTENSION TO ELECTRONS IN METALS

To what extent can the Landau theory be applied to electrons in metals? In the present section, we shall answer this question in a rather general way, putting aside until Chapter 3 the new features brought about by the long range of the Coulomb interaction.

Let us first summarize the main properties of a system of *noninteracting* electrons which move in the periodic potential of the crystalline lattice of a solid. The single-particle eigenstates are Bloch waves, with a wave function

$$\psi_{pn}(\mathbf{r}) = e^{i\mathbf{p}\cdot\mathbf{r}}u_{pn}(\mathbf{r}), \quad (1.69)$$

where  $u_{pn}$  has the periodicity of the lattice. The eigenstates are characterized by two quantum numbers, a "band index"  $n$ , and a wave vector  $\mathbf{p}$ , the latter lying in the first Brillouin zone of the crystal. The Bloch wave (1.69) may be considered as a mixture of plane waves, each having a wave vector  $(\mathbf{p} + \mathbf{K})$  where  $\mathbf{K}$  is a vector of the reciprocal lattice. If, as above, the wave vectors are restricted to the first Brillouin zone, we may conclude that the wave vector (or the momentum) is still a good quantum number, despite the presence of a periodic potential.

The ground state of an  $N$ -electron system is obtained by filling the  $N$  lowest Bloch wave states. The solid is a metal whenever there remain unfilled bands. The ground state distribution then levels off on some surface in momentum space, lying in the first Brillouin zone, which is called the Fermi surface  $S_F$ . We note that several bands may remain unfilled: the Fermi surface then possesses several sheets, one per unfilled band.

The Fermi surface has the symmetry of the crystalline lattice; it will certainly be anisotropic. Due to invariance under time reversal, the Fermi surface is invariant under combined spin inversion and spatial reflection with respect to the origin. If the crystal does not have a center of symmetry, the Fermi surfaces corresponding to the two directions of spin will be different.

Having surveyed very quickly the fundamentals of the one-electron approximation in metals, we now ask how Landau's theory can be used to take account of the Coulomb interaction between electrons. By using the adiabatic switching procedure introduced in Section 1.2, we can again establish a one-to-one correspondence between the eigenstates of the real system and those of the noninteracting system. For a *normal* metal, the ground state will be adiabatically generated starting from some "ideal" eigenstate characterized by a Fermi surface  $S_F$  (as we have mentioned  $S_F$  will most likely correspond to an excited state of the noninteracting system). In the same way, we can define a quasiparticle by adding one particle with quantum numbers  $(n, \mathbf{p})$  to the noninteracting system, and then switching on the interaction very slowly. Since the total momentum is conserved in the interaction, the final quasiparticle will possess the same quantum numbers as the original Bloch wave, namely a momentum  $\mathbf{p}$  in the first Brillouin zone and a band index. The quasiparticle thus acquires the same characteristics as a Bloch wave, and we may define a Fermi surface, etc.

Because of the damping due to particle collisions, the concept of a quasiparticle is only valid in the immediate vicinity of the Fermi surface, where damping effects are negligible. This represents a major departure from the one-electron approximation, in which Bloch waves were defined over the entire Brillouin zone. However, for most practical purposes this restriction is unimportant. Most physical effects, such as transport properties, cyclotron resonance, etc., involve thermally excited electrons, lying within a distance  $\kappa T$  from the Fermi surface.  $\kappa T$  is typically  $10^{-4}$  to  $10^{-2}$  eV, far smaller than the Fermi energy, of the order of 10 eV. The percentage of excited quasiparticles will always remain very small, so that we can neglect their damping.

In the absence of interaction, a Bloch wave possesses an energy  $\epsilon_{n\mathbf{p}}^0$ , which depends on  $\mathbf{p}$  in a rather complicated way. Everywhere on the Fermi surface, we may define a velocity  $\mathbf{v}_{\mathbf{p}}^0$ , equal to the gradient of  $\epsilon_{n\mathbf{p}}^0$ . As a result of the periodic potential  $\mathbf{v}_{\mathbf{p}}^0$  will be different from the velocity  $\hbar\mathbf{p}/m$  of a free particle. Let us now turn on the interaction. The quasiparticles have an energy  $\epsilon_{n\mathbf{p}}$ , equal to the first derivative of the energy functional.  $\epsilon_{n\mathbf{p}}$  is constant over the Fermi surface, and is equal to the chemical potential  $\mu$ . We can again define at every point

of  $S_F$  a velocity  $\mathbf{v}_p$ , which will be different from  $\mathbf{v}_p^0$ . The difference between  $\mathbf{v}_p$  and  $\hbar\mathbf{p}/m$  is now due to two distinct effects:

- (a) The influence of the periodic lattice on each electron;
- (b) The "many-body" effect arising from the Coulomb interaction.

In the case of a nearly isotropic Fermi surface, such as one finds in the alkali metals, we can write

$$v_{p_F} = \frac{\hbar p_F}{m^*}, \quad (1.69a)$$

where  $m^*$ , the effective mass, represents the combined influence of the periodic field and electron interactions. Usually these two "influences" are so deeply intermingled that it is not possible to disentangle them. We return to this question in Sections 3.8 and 5.6.

Just as for the Fermi liquid, by considering the second variational derivative of the energy with respect to  $n_p$ , we can define an interaction energy between quasiparticles,  $f_{pp'}$ . However, since the Fermi surface of a metal is not, in general, isotropic,  $f_{pp'}$  depends on the directions of both  $\mathbf{p}$  and  $\mathbf{p}'$ . As a result, the calculations of quasiparticle properties are very much more complicated. Consider, for example, the preceding calculation of the paramagnetic susceptibility, which is, in principle, still valid for metals. Where there is anisotropy, the displacement of the Fermi surface,  $dp_F/d\mathcal{H}$ , varies over  $S_F$ , and satisfies an integral equation which must be solved to obtain  $\chi_P$ . Gone, then, is the simplicity of a result like Eq. (1.67); indeed, there may appear so many parameters that a comparison between theory and experiment is not very fruitful. We conclude that comparison between theory and experiment for equilibrium properties is likely to be fruitful only in cases in which the Fermi surface is essentially isotropic.

Another complication in dealing with electrons in metals arises from the long range of the Coulomb interaction. We shall discuss ways of dealing with this in Chapter 3; here we merely remark that the above considerations go through formally provided one regards  $f_{pp'}$  as the *screened* quasiparticle interaction. Thus when due account is taken of screening it is possible to define an electronic compressibility which, for a metal with an isotropic Fermi surface, is related to the state density and  $F_0^s$  through Eqs. (1.48) and (1.58),

$$N_{\rho\kappa} = \frac{\nu(0)}{1 + F_0^s}. \quad (1.69b)$$

Again, let us emphasize that a simple formula like (1.69b) applies only to the case of an essentially isotropic electronic Fermi surface.

Because of the difficulties with anisotropy, one of the most interesting applications of the Landau theory to metal physics is the determination of those effects which are *not* influenced by particle interaction. Phenomena in which the interaction,  $f_{pp}$ , plays no role may then be safely used to determine the Fermi surface and other properties. We return to this question in our consideration of electron transport phenomena in Chapter 3.

We mention one final complication for electrons in metals. Not only do the electrons interact with each other, they interact as well with the lattice vibrations or *phonons* of the crystal. One must therefore inquire in detail as to which phenomena are influenced by this further electron-phonon interaction. Of the equilibrium properties considered here, it has recently been shown [Simkin (1963), Kadanoff and Prange (1964)] that the spin susceptibility and compressibility are *not* influenced by the electron-phonon interaction, while the specific heat is altered. Thus Eqs. (1.67) and (1.69b) may be applied directly to the case of an isotropic electron system, while the  $m^*$  one measures in a specific heat experiment will differ from that one would estimate via Eq. (1.69a) on the basis of electron-electron interactions alone. A comparison between theory and experiment for the equilibrium properties of some simple isotropic metals is given in Section 5.6.

#### 1.4. TRANSPORT EQUATION FOR QUASIPARTICLES

##### DEFINITION OF NONHOMOGENEOUS DISTRIBUTION FUNCTIONS

Until now, we have considered stable, homogeneous distributions, for which the function  $n_p$  was independent of both position and time. We shall now consider a more general problem, in which a weak time-dependent inhomogeneous perturbation is applied to the ground state. Such a perturbation may arise either from an external field, or as a consequence of an internal fluctuation of the system. We assume the perturbation is sufficiently weak that the response of the system is linear. (The formal theory of linear response functions is developed in Chapter 2.)

Let us consider a distribution function  $n_p(\mathbf{r}, t)$ , which depends explicitly on both the position  $\mathbf{r}$  and the time  $t$ . Such a description obviously violates the uncertainty principle: it is not possible to specify simultaneously both the momentum  $\mathbf{p}$  of a quasiparticle and its position  $\mathbf{r}$ , or its energy  $\epsilon_p$  and the time  $t$ . This is a major difficulty, which fortunately disappears if we restrict ourselves to *macroscopic* phe-

nomena, such that the typical wave vectors and frequencies remain much smaller than the corresponding atomic parameters.

Let us consider the Fourier transform of  $n_p(\mathbf{r}, t)$  with respect to space and time. Within our assumption of linear response, each Fourier component may be treated independently. Thus, it suffices to consider a particular plane wave perturbation, of wave vector  $\mathbf{q}$  and frequency  $\omega$ . We shall write the distribution function as

$$n_p(\mathbf{r}, t) = n_p^0 + \delta n_p(\mathbf{q}, \omega) e^{i(\mathbf{q} \cdot \mathbf{r} - \omega t)}. \quad (1.70)$$

The uncertainty principle gives rise to an uncertainty  $\hbar \mathbf{q}$  in the momentum  $\mathbf{p}$ , and  $\hbar \omega$  in the energy  $\epsilon_p$ . This will be relatively unimportant if the distribution function  $n_p$  is smooth enough over that range of  $\mathbf{p}$  and  $\epsilon_p$ . At a temperature  $T$ , the characteristic "width" of the Fermi surface is  $\kappa T$  for the energy,  $\kappa T/v_F$  for the momentum. We might thus expect the distribution function  $\delta n_p(\mathbf{q}, \omega)$  to make sense if the following conditions are met:

$$\begin{aligned} \hbar q v_F &\ll \kappa T, \\ \hbar \omega &\ll \kappa T. \end{aligned} \quad (1.71)$$

Under such conditions, the concept of a distribution function is meaningful: we are essentially in a *classical* regime, at least as far as the uncertainty principle is concerned.

The conditions (1.71) are rather restrictive. Actually, the Landau theory applies in the much broader range

$$\begin{aligned} \hbar q v_F &\ll \mu, \\ \hbar \omega &\ll \mu, \end{aligned} \quad (1.72)$$

with, however, a different interpretation of the distribution function  $\delta n_p(\mathbf{q}, \omega)$ . Thus far, we have defined  $\delta n_p$  as the *probability* for finding a quasiparticle with wave vector  $\mathbf{p}$ . In order to extend the Landau theory to the range (1.72), we must consider  $\delta n_p$  as the *probability amplitude* for finding a "pair," consisting in a quasiparticle of momentum  $(\mathbf{p} + \hbar \mathbf{q}/2)$  and a quasihole with momentum  $(\mathbf{p} - \hbar \mathbf{q}/2)$ . (Obviously, this no longer violates the uncertainty principle, since the momentum is not exactly equal to  $\mathbf{p}$ .) Such a description is equivalent to Wigner's semiclassical approach to statistical mechanics. We shall discuss this point further in Chapter 5. Here we simply assume that the uncertainty principle can be ignored when the conditions (1.72) are met.

According to (1.72), Landau's theory may be applied to a *macroscopic* perturbation, the scale of which is very large compared to the atomic scale. Thus the wavelength of the perturbation must be much larger

than the interparticle distance, the frequency much smaller than typical atomic frequencies. Put another way, the Landau result represents the leading term of an expansion of the system response in powers of  $q/k_F$  and  $\omega/\mu$ . This limitation of the theory must be kept in mind.

#### EXPANSION OF THE ENERGY

Let us consider a state characterized by a distribution function

$$n_{\mathbf{p}}(\mathbf{r}, t) = n_{\mathbf{p}}^{\circ} + \delta n_{\mathbf{p}}(\mathbf{r}, t). \quad (1.73)$$

The departure  $\delta n_{\mathbf{p}}$  from the ground state is supposed to be small. We assume  $\delta n_{\mathbf{p}}$  to contain only long wavelength fluctuations, and to be restricted to the vicinity of the Fermi surface, where quasiparticles are well defined. Note that  $n_{\mathbf{p}}$  gives the distribution in a *unit* volume centered at point  $\mathbf{r}$ : the momentum  $\mathbf{p}$  is accordingly quantized for  $\Omega = 1$ .

As before, the total energy is a functional  $E[n_{\mathbf{p}}(\mathbf{r}, t)]$  of the distribution function. If  $\delta n_{\mathbf{p}}$  is small, we may perform a Taylor expansion of this functional:

$$E = E_0 + \sum_{\mathbf{p}} \int d^3\mathbf{r} \epsilon(\mathbf{p}, \mathbf{r}) \delta n_{\mathbf{p}}(\mathbf{r}) + \frac{1}{2} \sum_{\mathbf{p}\mathbf{p}'} \iint d^3\mathbf{r} d^3\mathbf{r}' f(\mathbf{p}\mathbf{r}, \mathbf{p}'\mathbf{r}') \delta n_{\mathbf{p}}(\mathbf{r}) \delta n_{\mathbf{p}'}(\mathbf{r}') + \dots \quad (1.74)$$

Let us assume our system is invariant under spatial translation. It follows that  $\epsilon(\mathbf{p}, \mathbf{r})$  does not depend on  $\mathbf{r}$ , and is thus equal to  $\epsilon_{\mathbf{p}}$ . Furthermore  $f(\mathbf{p}\mathbf{r}, \mathbf{p}'\mathbf{r}')$  then depends only on the difference  $(\mathbf{r} - \mathbf{r}')$ .

At this stage, we must take account of the nature of the interaction. In this chapter we are only interested in *short range* forces, which are substantial only over distances of atomic size: this is the case for  ${}^3\text{He}$ . If the perturbation corresponds to a macroscopic variation in space and time,  $\delta n_{\mathbf{p}}$  may be considered as constant over the range of the interaction: in (1.74) we may replace  $\delta n_{\mathbf{p}'}(\mathbf{r}')$  by  $\delta n_{\mathbf{p}'}(\mathbf{r})$ . The energy can then be written as

$$E = E_0 + \int d^3\mathbf{r} \delta E(\mathbf{r}), \quad (1.75)$$

$$\delta E(\mathbf{r}) = \sum_{\mathbf{p}} \epsilon_{\mathbf{p}} \delta n_{\mathbf{p}}(\mathbf{r}) + \frac{1}{2} \sum_{\mathbf{p}\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'} \delta n_{\mathbf{p}}(\mathbf{r}) \delta n_{\mathbf{p}'}(\mathbf{r}),$$

where the interaction energy  $f_{\mathbf{p}\mathbf{p}'}$  is defined by

$$f_{\mathbf{p}\mathbf{p}'} = \int d^3\mathbf{r}' f(\mathbf{p}\mathbf{r}, \mathbf{p}'\mathbf{r}'). \quad (1.76)$$

The energy is thus a *local* function of the distribution  $\delta n_{\mathbf{p}}$ : at every point, we find the same relation as for an homogeneous system. The  $f_{\mathbf{p}\mathbf{p}'}$  defined in (1.76) is the same as that of the preceding sections, calculated for  $\Omega = 1$ . (Let us emphasize again that this conclusion holds only for *macroscopic* perturbations.) It is physically clear that the *local* character of the energy is a direct consequence of the *short range* of the interaction.

#### TRANSPORT EQUATION FOR QUASIPARTICLES

According to Eq. (1.75), the local excitation energy of a quasiparticle  $\mathbf{p}$  is equal to

$$\bar{\epsilon}_{\mathbf{p}}(\mathbf{r}) = \epsilon_{\mathbf{p}} + \sum_{\mathbf{p}'} f_{\mathbf{p}\mathbf{p}'} \delta n_{\mathbf{p}'}(\mathbf{r}). \quad (1.77)$$

Note that this energy depends on  $\mathbf{p}$  and  $\mathbf{r}$ . The gradient  $\nabla_{\mathbf{p}} \bar{\epsilon}$  gives the velocity of the quasiparticle, as usual. On the other hand,  $(-\nabla_{\mathbf{r}} \bar{\epsilon})$  is equivalent to a kind of "diffusion" force, which tends to push quasiparticles toward regions of minimum energy.

In order to describe the transport properties, Landau considered the quasiparticles as independent, described by a classical Hamiltonian  $\bar{\epsilon}_{\mathbf{p}}(\mathbf{r})$ . The problem is thus reduced to the development of the appropriate kinetic theory for a gas of quasiparticles. We shall discuss the physical meaning of this assumption in a moment: we first establish the transport equation.

We shall use a well-known procedure of kinetic theory: we consider a small volume element in phase space  $d^3\mathbf{p} d^3\mathbf{r}$ , and we calculate the flow of quasiparticles through each side of this element. By establishing the balance of the flow inward and outward, we obtain the equation

$$\frac{\partial n_{\mathbf{p}}}{\partial t} + \nabla_{\mathbf{r}} n_{\mathbf{p}} \cdot \nabla_{\mathbf{p}} \bar{\epsilon}_{\mathbf{p}} - \nabla_{\mathbf{p}} n_{\mathbf{p}} \cdot \nabla_{\mathbf{r}} \bar{\epsilon}_{\mathbf{p}} = 0. \quad (1.78)$$

This well-known result governs the flow of quasiparticles in phase space, in the absence of collisions and external forces.

Equation (1.78) refers to the total distribution function for all quasiparticles. But we know that  $n_{\mathbf{p}}$  is only defined in the vicinity of the Fermi surface. Furthermore, the concept of independent quasiparticles assumed by Landau cannot be true for all values of the momentum  $\mathbf{p}$ : it is physically obvious that only *excited* quasiparticles in the immediate vicinity of the Fermi surface can be considered as independent. We must therefore extract from (1.78) a transport equation describing the flow of only excited quasiparticles. This is easily done if we write

$n_{\mathbf{p}}$  in the form (1.73). If we keep only terms of first order in  $\delta n_{\mathbf{p}}$ , and use (1.77), we obtain

$$\frac{\partial \delta n_{\mathbf{p}}(\mathbf{r}t)}{\partial t} + \nabla_{\mathbf{r}} \delta n_{\mathbf{p}}(\mathbf{r}t) \cdot \mathbf{v}_{\mathbf{p}} - \nabla_{\mathbf{p}} n_{\mathbf{p}}^{\circ} \cdot \sum_{\mathbf{p}'} \int_{\mathbf{p}\mathbf{p}'} \nabla_{\mathbf{r}} \delta n_{\mathbf{p}'}(\mathbf{r}t) = 0. \quad (1.79)$$

This linearized transport equation involves only values of  $\mathbf{p}$  close to the Fermi surface, thanks to the factor  $\nabla_{\mathbf{p}} n_{\mathbf{p}}^{\circ} = -\mathbf{v}_{\mathbf{p}} \delta(\epsilon_{\mathbf{p}} - \mu)$ .

Let us analyze in some detail the physical meaning of Eq. (1.79). The first two terms (which provide the usual transport equation for a noninteracting system), describe a flow of totally independent excited quasiparticles. The last term, which arises from their interaction, may be interpreted as a flow of the ground state particles dragged by the inhomogeneities of the excitation distribution. We are thus led to the following picture: the elementary excitations, few in number, are completely independent. In homogeneous systems, they do not interact with the "ground state" particles. However, if  $\delta n_{\mathbf{p}}$  is not homogeneous, the excited quasiparticles create a force field which acts on the ground state distribution, and distorts it; this effect is of first order in  $\delta n_{\mathbf{p}}$ , and thus quite important.

The neglect of the interaction between two *excited* quasiparticles is quite reasonable at low temperatures, where their density is small, except for very low-frequency phenomena (see Section 1.9). On the other hand, reducing the interaction between the excited particles and the ground state particles to an *average* macroscopic force is a rather bold assumption. One might expect microscopic correlations to be of importance. According to Landau, they do not play any role: this assumption is indeed true, and may be proved within the framework of perturbation theory. One may think of this result as arising from the exclusion principle, which renders the ground state distribution essentially rigid.

The linearized transport equation may be written very simply in terms of the departure from local equilibrium  $\delta \bar{n}_{\mathbf{p}}$ . On comparing the definition (1.30) with (1.79), we may write the latter in the form

$$\frac{\partial \delta n_{\mathbf{p}}(\mathbf{r}, t)}{\partial t} + \mathbf{v}_{\mathbf{p}} \cdot \nabla_{\mathbf{r}} \delta \bar{n}_{\mathbf{p}}(\mathbf{r}, t) = 0. \quad (1.80)$$

This compact result may also be obtained directly from Eq. (1.78), by writing  $n_{\mathbf{p}}$  in the form (1.28), i.e.,

$$n_{\mathbf{p}}(\mathbf{r}, t) = n^{\circ}(\epsilon_{\mathbf{p}}) + \delta \bar{n}_{\mathbf{p}}(\mathbf{r}, t). \quad (1.81)$$

The function  $n^\circ(\tilde{\epsilon}_p)$  gives a contribution to the gradient terms which may be written as

$$\left( \frac{\partial n^\circ}{\partial \epsilon_p} \nabla_r \tilde{\epsilon}_p \right) \cdot \nabla_p \tilde{\epsilon}_p - \left( \frac{\partial n^\circ}{\partial \epsilon_p} \nabla_p \tilde{\epsilon}_p \right) \cdot \nabla_r \tilde{\epsilon}_p$$

and which therefore vanishes. The only contribution to these gradients arises from  $\delta \tilde{n}_p$ ; if we keep only the first-order terms, we arrive at Eq. (1.80).

According to (1.80), the transport equation involves the time derivative of  $\delta n_p$ , and the spatial derivative of  $\delta \tilde{n}_p$ . This difference may be understood by noting that the gradient terms describe the *diffusion* of quasiparticles, which is certainly governed by the local energy. In this respect, the departure from equilibrium is measured by  $\delta \tilde{n}_p$ , not by  $\delta n_p$ .

In its form (1.80), our transport equation misses two important effects: the influence of external forces and the collisions between excited quasiparticles. Let us first consider collisions, which represent the "dissipative," irreversible part of the interaction between excited quasiparticles (the interaction energy  $f_{pp'}$  being, by contrast, a reactive, reversible effect of that interaction). The collisions are similar in nature to those between molecules in the usual kinetic theory of gases. Their importance is qualitatively measured by some collision frequency  $\nu$ .

At the low temperatures in which we are interested, collisions are inhibited by the exclusion principle, and are thus comparatively infrequent:  $\nu$  is small. Collisions will therefore play a role only in very low-frequency phenomena ( $\omega \lesssim \nu$ ) (viscosity, thermal conduction, etc.) where they limit the response to an external force. We shall discuss these problems in some detail in Section 1.8, where we shall see that one can take account of collisions by adding to the right-hand side of the transport equation a "collision integral"  $I(\delta n_p)$ , which measures the rate of change of  $\delta n_p$  due to collisions. If we work at a frequency  $\omega$ , much larger than the collision frequency  $\nu$ , we expect the collisions to play no role: we can then drop the collision integral.

Finally, let us apply a force  $\mathfrak{F}_p$  to the quasiparticle  $p$ : the distribution drifts in momentum space. This gives rise to an additional term on the left-hand side of Eq. (1.80), equal to

$$\mathfrak{F}_p \cdot \nabla_p n_p(\mathbf{r}, t). \quad (1.82)$$

Usually, the excitation  $\delta n_p$  of the system is proportional to the applied force  $\mathfrak{F}$ ; in expression (1.82), we may then replace  $n_p$  by the equilibrium distribution  $n_p^\circ$ , since we are concerned only with first-order changes in  $n$ .

In most problems, we know from first principles the force exerted on a *bare* particle. In order to deduce the force  $\mathfrak{F}_p$  felt by a *quasiparticle*, we