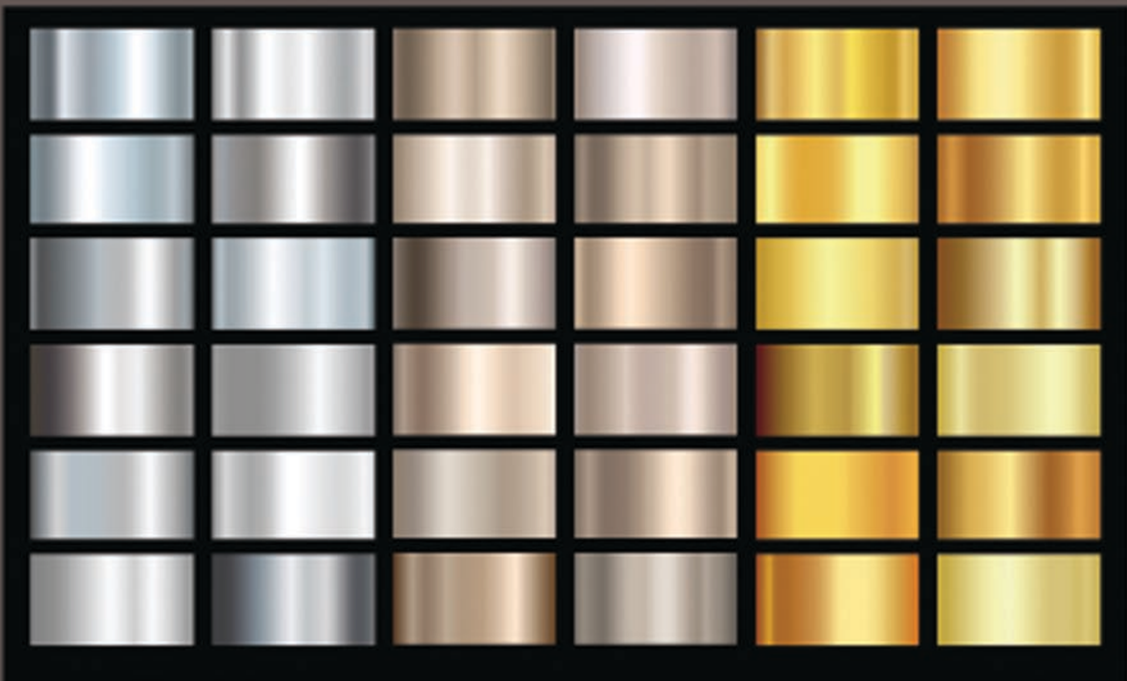
DISCRETE MATHEMATICS AND ITS APPLICATIONS

EXTREMAL FINITE SET THEORY



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Preface

The preface of an academic book should contain information about its topic and its intended audience. Starting with the former, let us say that extremal combinatorics is a huge topic that can be described more easily than that of our book: it is concerned with finding the largest, smallest, or otherwise optimal combinatorial structures with a given property. Here we deal with structures that consist of subsets of a finite set. Most typically, we focus on maximizing the cardinality of a family of sets satisfying some properties.

It is equally important to describe what is *not* in this book. We only mention (or use) results concerning 2-uniform families, i.e. ordinary graphs (luckily, many good books and survey papers are available on extremal graph theory). Many of the results we state in the book have analogous versions in more structured creatures than sets (e.g. permutations, multisets, vector spaces, block designs, etc) or infinite versions - we hardly even mention them. We do not deal with colorings (this book contains hardly any Ramsey-type or anti-Ramsey-type results). We usually focus on the cardinality of a family of sets, and often ignore results about something else, like the minimum degree (co-degree). One particular topic that would really fit in this book, but is completely avoided due to space constraints, is coding theory.

When writing the book we had the following two goals in our minds:

- Presenting as many proof techniques as possible. Chapter 1 is completely devoted to this aim via presenting several proofs of classical results of extremal set theory (apart from the very recent method of Croot, Lev, and Pach) but later chapters also contain brief or more detailed introductions to more novel methods like flag algebras, the container method, etc.

- Giving a detailed and almost complete list of recent developments, results (mostly without proof, but mentioning the proof techniques applied) in the areas covered.

As a consequence, the intended audience consists of

- Graduate students who are eager to to be introduced to the theory of extremal set systems. Note however, that the exercises are not those of a proper textbook. Some of them are rather easy, they check whether the reader understood the definitions, some of them state lemmas needed for theorems with longer proofs, some others are "easier" results from research papers.

- University professors giving courses in (extremal) combinatorics. The above warning about the exercises applies here as well, but we hope that our book could help to design the syllabus of graduate courses in different areas of extremal set theory.

- Researchers interested in recent developments in the field. There has been progress in many of the subtopics we cover since the publication of other books dealing with set systems or extremal combinatorics. In some cases even the latest survey articles are not very new. We are convinced that senior academic people can profit in their research by reading or leafing through some or all of the chapters.

Another way to describe the topic of this book is that we wanted to gather results and proofs that could have been presented at the Extremal Sets Systems Seminar of the Alfréd Rényi Institute of Mathematics. (Some of the theorems in the book were indeed presented there.) This might not be very informative for some of our readers, but we wanted to mention it as both authors gave their first talk at this seminar about fifteen years ago.

We also have Chapter 9 that glances at a different topic. We chose the topic and the results mentioned to illustrate how extremal finite set theory results can be applied in other areas of mathematics.

The topic of Chapter 9 runs under several names in the literature. It is often referred to as combinatorial group testing or pool designs and many others, as most applications of this area are in biology with its own terminology. The reason for which we decided to go with the name combinatorial search theory is again personal: the Combinatorial Search Seminar (formerly known as Combinatorial Search and Communicational Complexity Seminar) is the *other* seminar at the Rényi Institute that the two authors attend (apart from the Combinatorics Seminar, probably the longest running of its sort worldwide, started by Vera Sós and András Hajnal).

A very important final note: We cannot deny that we put more emphasis on topics we know and like. The number of our own results in this book is in no way proportional to their importance.

Notation and Definitions

Here below we gather the basic notions (mostly with definitions, some without) that we will use in the book. We use standard notation.

Sets. We denote by $[n]$ the set $\{1, 2, \ldots, n\}$ of the first n positive integers and for integers $i < j$ let $[i, j] = \{i, i+1, \ldots, j\}$. For a set S we use the notation 2^S for its power set $\{T : T \subseteq S\}$ and $\binom{S}{k}$ for the family of its k-element subsets (k-subsets for short) $\{T \subseteq S : |T| = k\}$. The latter will be referred to as the kth (*full*) *level of* 2^S. We will use the notation $\binom{X}{\le k} := \cup_{i=0}^{k} \binom{X}{i}$ and $\binom{X}{\ge k} := \cup_{i=k}^{|X|} \binom{X}{i}$. The *symmetric difference* $(F \setminus G) \cup (G \setminus F)$ of two sets F and G will be denoted by $F \triangle G$. If a set F is a subset of $[n]$, then its complement $[n] \setminus F$ is denoted by $\overline{F}$. To denote the fact that A is a subset of B we use $A \subseteq B$ and $A \subsetneq B$ with the latter meaning A is a proper subset of B. The Cartesian product of r sets $A_1, \ldots, A_r$ is $A_1 \times A_2 \times \cdots \times A_r = \{(a_1, a_2, \ldots, a_r) : a_i \in A_i \text{ for every } 1 \le i \le r\}$.

Families of sets. We will mostly use the terminology *families* for collections of sets, although *set systems* is also widely used in the literature. If all sets in a family $\mathcal{F}$ are of the same size, we say $\mathcal{F}$ is *uniform*, if this size is k, then we say $\mathcal{F}$ is k-uniform. Sometimes (mostly in the chapters on Turán type problems, and saturation problems) we will use the word hypergaph for k-uniform families.

The *shadow* of a k-uniform family $\mathcal{F} \subseteq \binom{X}{k}$ is $\Delta(\mathcal{F}) = \{G : |G| = k - 1, \exists\, F \in \mathcal{F}\ \ G \subset F\}$, and its *up-shadow* or *shade* is $\nabla(\mathcal{F}) = \{G : |G| = k+1, \exists\, F \in F\ \ F \subset G \subseteq X\}$. For not necessarily uniform families $\mathcal{F} \subseteq 2^X$ we will use $\Delta_m(\mathcal{F}) = \{G : |G| = m, \exists F \in \mathcal{F}\ \ G \subset F\}$ and $\nabla_m(\mathcal{F}) = \{G : |G| = m, \exists F \in F\ \ F \subset G \subseteq X\}$.

Graphs. A graph G is a pair $(V(G), E(G))$ with $V(G)$ its vertex set and $E(G) \subseteq \binom{V(G)}{2}$ its edge set. G is bipartite if there exists a partition $V = A \cup B$ such that every edge of G contains one vertex from A and one from B. In this case, we will write $G = (A, B, E)$. If $U \subseteq V$, the induced subgraph of G on U has vertex set U and edge set $\{e = \{u, v\} \in E(G) : u, v \in U\}$. It is denoted by $G[U]$. In particular, if $V(G)$ is a family of sets, we will write $G[\mathcal{F}]$ for the subgraph induced on the subfamily $\mathcal{F}$ of V. The number of edges in G is denoted by $e(G)$. Similarly the number of edges in $G[U]$ or between two disjoint sets $X, Y \subseteq V(G)$ is denoted by $e(U)$ and $e(X, Y)$, respectively.

A *path of length* l in a graph G between vertices v_0 and v_l is a sequence

$v_0, e_1, v_1, e_2 \ldots, e_l, v_l$ such that v_i is a vertex for all $i = 0, 1, \ldots, l$ and $e_{i+1} = \{v_i, v_{i+1}\}$ is an edge in G for all $i = 0, 1, \ldots, l-1$. We say that a graph G is *connected* if for any two vertices $u, v \in V(G)$ there exists a path between u and v in G. The *(graph) distance* $d_G(u_1, u_2)$ of two vertices u_1, u_2 in G is the length of the shortest path between u_1 and u_2. The *ball* $B_r(v)$ *of radius* r *at vertex* v is the set of all vertices at distance at most r from v. The *open neighborhood* $N(v)$ *of a vertex* v is the set of vertices that are adjacent to v. The closed neighborhood $N[v]$ of v is $N(v) \cup \{v\}$. For a set $U \subseteq V(G)$ we define its open and closed neighborhood by $N(U) = \cup_{u \in U} N(u)$ and $N[U] = \cup_{u \in U} N[u]$. The degree $d_G(v)$ of a vertex $v \in G$ is $|N(v)|$. If all vertices of a graph G have the same degree, then we say that G is *regular*. A set of vertices is *independent* in G if there is no edge between them, and a set of edges is *independent* in G if they do not share any vertices. A *matching* in G is a set of independent edges, and a *perfect matching* is a matching covering all the vertices of G.

We will use the following specific graphs: P_l is a path on l vertices (thus it has $l-1$ edges). If we add an edge between the endpoints of a path, we obtain the *cycle* C_l on l vertices. K_n is the *complete graph* on n vertices, having all the possible $\binom{n}{2}$ edges, while $K_{s,t}$ is the *complete bipartite graph* with one part having size s and the other size t, that has all the possible st edges between the two parts. A graph that does not contain any cycle is a *forest*, and a connected forest is a *tree*, usually denoted by T. A vertex of degree 1 is called a *leaf*, and S_r is the *star* with r leaves, i.e. $S_r = K_{1,r}$. M_n is the *matching* on n vertices, i.e. the graph consisting of $n/2$ independent edges.

Binomial coefficients. The binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the size of $\binom{X}{k}$ for an n-set X. For any n, we have $\binom{n}{k} \le \binom{n}{k+1}$ if and only if $k < n/2$, with equality if n is odd and $k = \lfloor n/2 \rfloor$. For any real x and positive integer k we define $\binom{x}{k} = \prod_{i=0}^{k-1} \frac{x-i}{k-i}$. It is easy to see that for every positive real y and positive integer k there exists exactly one x with $\binom{x}{k} = y$. We will often use the well-known bounds $(\frac{n}{k})^k \le \binom{n}{k} \le (\frac{en}{k})^k$ and the fact that $\binom{x}{k}$ is a convex function.

Posets. A partially ordered set (poset) is a pair $(P, \le)$ where P is a set and $\le$ is a binary relation on P satisfying (i) $p \le p$ for all $p \in P$, (ii) $p \le q, q \le p$ imply $p = q$, (iii) $p \le q$, $q \le r$ imply $p \le r$. If the relation is clear from context, we will denote the poset by P. For any poset $(P, \le_P)$, the *opposite poset* $(P', \le_{P'})$ is defined on the same set of elements with $p \le_{P'} q$ if and only if $q \le_P p$. Two elements $p, q \in P$ are *comparable* if $p \le q$ or $q \le p$ holds, and *incomparable* otherwise. A set $C \subseteq P$ of pairwise comparable elements is called a *chain*, its cardinality is called its *length*. A chain of length k is also called a k-chain and it is denoted by P_k. A set $S \subseteq P$ of pairwise incomparable elements is called an *antichain*.

An element $p \in P$ is *minimal (maximal)* if there does not exist $q \ne p$ in P with $q \le p$ ($p \le q$). Most often we will consider the *Boolean lattice/hypercube* Q_n. Its set of elements is 2^S and $T_1 \le T_2$ if and only if $T_1 \subseteq T_2$ holds. Therefore, a chain of length k (a k-chain) in Q_n is a family $\mathcal{C} = \{C_1, C_2, \dots, C_k\}$ with $C_i \subsetneq C_{i+1}$. A family $\mathcal{F} \subseteq 2^{[n]}$ can be considered as a subposet of the hypercube. We will say that a set $F \in \mathcal{F}$ is minimal/maximal if it is so in this partial ordering, i.e. no proper subset/superset of F is contained in $\mathcal{F}$.

We say that a poset is *totally ordered* if any two elements are comparable. The *initial segment of size s* of a totally ordered set P is the set of the s smallest elements of P. A poset is *ranked* if all maximal chains in it have the same length.

Permutations. A permutation of a set X is a bijective function from X to itself. We will denote permutations by Greek letters σ, π, etc. The set of all permutations of X is denoted by S_X. If X is finite, the number of permutations in S_X is $|X|!$. For any two elements $\sigma, \pi \in S_X$, the composition $\sigma \circ \pi$ also belongs to S_X. For a set $X = \{x_1, x_2, \dots, x_n\}$, let α denote the permutation with $\alpha(x_1) = x_2, \alpha(x_2) = x_3, \dots, \alpha(x_{n-1}) = x_n, \alpha(x_n) = x_1$. Two permutations $\sigma, \pi \in S_x$ are equivalent if $\sigma = \alpha^k \circ \pi$ for some integer k. This defines an equivalence relation on S_X with each equivalence class containing $|X|$ permutations. The equivalence classes are the *cyclic permutations* of X. The number of cyclic permutations of X is $(|X| - 1)!$.

Linear algebra. We will assume that the reader is familiar with the notions of linear independence, vector space dimension, matrix, positive definite and semidefinite matrix. A reader interested in combinatorics but not feeling comfortable with linear algebra might consult Sections 2.1-2.3 of the excellent book [28] by Babai and Frankl. If $v_1, v_2, \dots, v_k$ are vectors in a vector space V, then $\langle v_1, v_2, \dots, v_k \rangle$ denotes the subpace of V spanned by these vectors. The scalar product of two vectors $\underline{u} = (u_1, u_2, \dots, u_n), \underline{v} = (v_1 v_2, \dots, v_n)$ is $\underline{u} \cdot \underline{v} = \sum_{i=1}^n u_i v_i$. $\mathbb{F}_q$ denotes the finite field of q elements, and $\mathbb{F}_q^n$ the vector space of dimension n over $\mathbb{F}_q$. The multiplicative group of nonzero elements of $\mathbb{F}_q$ is denoted by $\mathbb{F}_q^\times$. $\mathbb{F}^n[x]$ denotes the vector space of polynomials of n variables over $\mathbb{F}$. A matrix or vector with only 0 and 1 entries will be called *binary matrix* or *binary vector*.

There are several vectors that can be naturally associated with sets. The *characteristic vector* of a subset $F \subseteq [n]$ is the binary vector $v_F = (v_1, v_2, \dots, v_n)$ with $v_i = 1$ if and only if $i \in F$.

Functions (or mappings). If A is a subset of the domain of a function f, then $f(A)$ will denote the set $\{f(a) : a \in A\}$. The inverse of a function f will be denoted by f^{-1}. To compare the order of magnitude of two functions $f(n)$ and $g(n)$ we will write $f(n) = O(g(n))$ if there exists a positive constant C such that $\left|\frac{f(n)}{g(n)}\right| \le C$ for all $n \in \mathbb{N}$. Similarly, $f(n) = \Omega(g(n))$ means that there exists a positive constant C such that $\left|\frac{f(n)}{g(n)}\right| \ge C$ for all $n \in \mathbb{N}$. If the constant C in the above definitions depends on some other parameter k, then we write $f(n) = O_k(g(n))$ and $f(n) = \Omega_k(g(n))$. If both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ hold, then we write $f(n) = \Theta(n)$. Finally, $f(n) = o(g(n))$, or equivalently $f(n) \ll g(n)$ denotes that $\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| = 0$. Similarly $f(n) = \omega(g(n))$, or equivalently $f(n) \gg g(n)$ denotes that $\lim_{n\to\infty}\left|\frac{f(n)}{g(n)}\right| = \infty$. We say that a real-valued function f is *convex* on an interval $[x_1, x_2]$ if it lies above its tangents. We will use Jensen's inequality, which states that in this case $f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2)$.

Throughout the book, log and ln stand for logarithms of base 2 and e, respectively.

Probability. The probabilities of an event $\mathcal{E}$ and the expected value of a random variable X are denoted by $\mathbb{P}(\mathcal{E})$ and $\mathbb{E}(X)$, respectively. Its standard deviation is $\sigma(X) = \sqrt{\mathbb{E}(X - \mathbb{E}(x))^2} = \sqrt{\mathbb{E}(X^2) - \mathbb{E}(X)^2}$. We say that a sequence $\mathcal{E}_n$ of events holds *with high probability (w.h.p.)* if $\mathbb{P}(\mathcal{E}_n) \to 1$ as n tends to infinity. Standard inequalities concerning probabilities of events will be introduced at the beginning of Chapter 4. For a detailed introduction to applications of probability theory in combinatorics we recommend the book by Alon and Spencer [19]. The binary entropy function $-x\log x - (1-x)\log(1-x)$ is denoted by $h(x)$.

1

Basics

CONTENTS

As its title suggests, this chapter contains some of the very first results of extremal finite set theory (Sperner's theorem and the Erdős-Ko-Rado theorem), and tries to present some of the major techniques used in the field (shifting, the permutation method, the polynomial method). To this end, in many cases we will include several proofs of the same theorem.

1.1 Sperner's theorem, LYM-inequality, Bollobás inequality

In this section we will mostly consider families that form an *antichain* in the hypercube (they are also called *Sperner families*). Recall that it means there are no two different members of the family such that one of them contains the other. We will be interested in the maximum size that an antichain $\mathcal{F} \subseteq 2^{[n]}$ can have. An easy and natural way to create antichains is to collect sets of the same size, i.e. levels. Among full levels of $2^{[n]}$ the largest is (are) the middle one(s). Sperner proved [520] that this is best possible. His result is the first theorem in extremal finite set theory. We will give two proofs of his theorem, then consider a generalization by Erdős and a related notion by Bollobás.

Theorem 1 (Sperner [520]) *If $\mathcal{F} \subseteq 2^{[n]}$ is an antichain, then we have $|\mathcal{F}| \le \binom{n}{\lfloor n/2 \rfloor}$. Moreover $|\mathcal{F}| = \binom{n}{\lfloor n/2 \rfloor}$ holds if and only if $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$ or $\mathcal{F} = \binom{[n]}{\lceil n/2 \rceil}$.*

First proof of Theorem 1. We start with the following simple lemma.

Lemma 2 *Let $G(A,B,E)$ be a connected bipartite graph such that for every vertex $a \in A$ and $b \in B$ the inequality $d_G(a) \geq d_G(b)$ holds. Then for any subset $A' \subseteq A$ the size of its neighborhood $N(A')$ is at least the size of A'. Moreover if $|N(A')| = |A'|$ holds, then $A' = A$, $N(A') = B$ and G is regular.*

Proof of Lemma. The number of edges between A' and $N(A')$ is exactly $\sum_{a\in A'} d_G(a)$ and is at most $\sum_{b\in N(A')} d_G(b)$. By the assumption on the degrees, the number of summands in the latter sum should be at least the number of summands in the former sum; thus we obtain $|N(A')| \geq |A'|$. If the number of summands is the same in both sums, then we must have $d_G(a) = d_G(b)$ for any $a \in A', b \in N(A')$ and all edges incident to $N(A')$ must be incident to A'. By the connectivity of G it follows that $A' = A$ and $N(A') = B$ hold. $\square$

Let $G_{n,k,k+1}$ be the bipartite graph with parts $\binom{[n]}{k}$ and $\binom{[n]}{k+1}$ in which two sets $S \in \binom{[n]}{k}$ and $T \in \binom{[n]}{k+1}$ are joined by an edge if and only if $S \subset T$. We want to apply Lemma 2 to $G_{n,k,k+1}$. It is easy to see that it is connected. The degree of a set $S \in \binom{[n]}{k}$ is $n-k$, while the degree of $T \in \binom{[n]}{k+1}$ is $k+1$, therefore as long as $k \leq \lfloor n/2 \rfloor$ holds, $\binom{[n]}{k}$ can play the role of A, and as soon as $k \geq \lceil n/2 \rceil$ holds, $\binom{[n]}{k+1}$ can play the role of A. Moreover, if $k < \lfloor n/2 \rfloor$ or $k \geq \lceil n/2 \rceil$, then $G_{n,k,k+1}$ is not regular. Note that if $\mathcal{F} \subseteq \binom{[n]}{k}$, then $N(\mathcal{F}) = \nabla(\mathcal{F})$, while if $\mathcal{F} \subseteq \binom{[n]}{k+1}$, then $N(\mathcal{F}) = \Delta(\mathcal{F})$.

To prove Theorem 1 let $\mathcal{F} \subseteq 2^{[n]}$ be an antichain. First we prove that if $\mathcal{F}$ is of maximum size, then it contains sets only of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$. Suppose not and, say, there exists a set of size larger than $\lceil n/2 \rceil$. Then let m be the largest set size in $\mathcal{F}$ and let us consider the graph $G_{n,m-1,m}$. Applying Lemma 2 to $\mathcal{F}_m = \{F \in \mathcal{F} : |F| = m\}$ we obtain that $|\mathcal{F} \setminus \mathcal{F}_m \cup \Delta(\mathcal{F}_m)| > |\mathcal{F}|$ holds. To finish the proof we need to show that $\mathcal{F}' = \mathcal{F} \setminus \mathcal{F}_m \cup \Delta(\mathcal{F}_m)$ is an antichain. Sets of $\Delta(\mathcal{F}_m)$ have largest size in $\mathcal{F}'$, therefore they cannot be contained in other sets of $\mathcal{F}'$. No set $F' \in \Delta(\mathcal{F}_m)$ can contain any other set F from $\mathcal{F}' \cap \mathcal{F}$ as there exists $F'' \in \mathcal{F}_m$ with $F' \subset F''$, thus $F \subset F''$ would follow, and that contradicts the antichain property of $\mathcal{F}$.

We showed that $\mathcal{F} \subseteq \binom{[n]}{\lfloor n/2 \rfloor} \cup \binom{[n]}{\lceil n/2 \rceil}$, which proves the theorem if n is even. If n is odd, then suppose $\mathcal{F}$ contains sets of size both $\lceil n/2 \rceil$ and $\lfloor n/2 \rfloor$. Applying the moreover part of Lemma 2 to $\mathcal{F}_{\lceil n/2 \rceil}$, we again obtain a larger antichain $\mathcal{F} \setminus \mathcal{F}_{\lceil n/2 \rceil} \cup \Delta(\mathcal{F}_{\lceil n/2 \rceil})$ unless $\mathcal{F}_{\lceil n/2 \rceil} = \binom{[n]}{\lceil n/2 \rceil}$. This shows that if $\mathcal{F}$ is a maximum size antichain that contains a set of size $\lceil n/2 \rceil$, then $\mathcal{F} = \binom{[n]}{\lceil n/2 \rceil}$. Similarly, if $\mathcal{F}$ is a maximum size antichain that contains a set of size $\lfloor n/2 \rfloor$, then $\mathcal{F} = \binom{[n]}{\lfloor n/2 \rfloor}$. $\blacksquare$

We give a second proof of Theorem 1 based on the inequality proved independently by Lubell, Yamamoto, and Meshalkin. As Bollobás obtained an even more general inequality that we will prove in Theorem 6, it is sometimes referred to as YBLM-inequality (Miklós Ybl was a famous Hungarian architect in the nineteenth century, who designed, among others, the State

Opera House and St. Stephen's Basilica in Budapest), but we will use the more common acronym.

Lemma 3 (LYM-inequality, [397, 415, 554]) *If $\mathcal{F} \subseteq 2^{[n]}$ is an antichain, then the following inequality holds:*

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1.$$

Moreover, the above sum equals 1 if and only if $\mathcal{F}$ is a full level.

Before the proof let us introduce a further definition. A chain $\mathcal{C} \subseteq 2^{[n]}$ is called a *maximal chain* if it is of length $n+1$, i.e. it contains a set of size i for every $0 \leq i \leq n$.

Proof. Let $\mathcal{F} \subseteq 2^{[n]}$ be an antichain and let us consider the pairs $(F, \mathcal{C})$ such that $\mathcal{C}$ is a maximal chain in $[n]$ and $F \in \mathcal{F} \cap \mathcal{C}$. There are exactly $|F|!(n-|F|)!$ maximal chains containing F, therefore the number of such pairs is $\sum_{F \in \mathcal{F}} |F|!(n-|F|)!$. On the other hand, by the antichain property of $\mathcal{F}$, every maximal chain contains at most one set from $\mathcal{F}$ and thus the number of pairs is at most $n!$. We obtained

$$\sum_{F \in \mathcal{F}} |F|!(n-|F|)! \leq n! \tag{1.1}$$

and dividing by $n!$ yields the LYM-inequality.

Let us now prove the moreover part of the lemma. Clearly, if $\mathcal{F}$ is a full level $\binom{[n]}{k}$ for some k, then $\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} = \binom{n}{k} \cdot \frac{1}{\binom{n}{k}} = 1$ holds. If $\mathcal{F}$ is not a full level, then there exist two sets $F \in \mathcal{F}, G \notin \mathcal{F}$ with $|F| = |G| = |F \cap G| + 1$. We construct maximal chains that do not contain any set from $\mathcal{F}$ and thus (1.1) cannot hold with equality. Consider any maximal chain $\mathcal{C}$ that contains $F \cap G$, G, and $F \cup G$. Any set C of $\mathcal{C}$ with $C \subseteq F \cap G \subset F$ cannot be in $\mathcal{F}$ by the antichain property of $\mathcal{F}$, G is not in $\mathcal{F}$ by definition, finally all sets $C \in \mathcal{C}$ with $C \supseteq F \cup G \supset F$ are not in $\mathcal{F}$ by the antichain property. By the choice of F and G there are no other sets in $\mathcal{C}$ and thus $\mathcal{F} \cap \mathcal{C} = \emptyset$ holds. ■

The function $\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}}$ in the LYM-inequality is called the *Lubell function* of $\mathcal{F}$ and it is often used (for several applications see Chapter 7 on forbidden subposet problems). It is denoted by $\lambda(\mathcal{F}, n)$ and we omit n if it is clear from the context. Having Lemma 3 in hand, the second proof of Theorem 1 is immediate.

Second proof of Theorem 1. Let $\mathcal{F} \subseteq 2^{[n]}$ be an antichain. Then we have $\sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{F \in \mathcal{F}} \frac{1}{\binom{n}{|F|}} \leq 1$ by Lemma 3. Therefore $|\mathcal{F}|$, that equals the number of summands, is at most $\binom{n}{\lfloor n/2 \rfloor}$. The moreover part of the theorem follows from the moreover part of Lemma 3. ■

Let us illustrate the strength of the LYM-inequality by the following generalization of Theorem 1. We say that a family $\mathcal{F}$ of sets is *k-Sperner* if all chains in $\mathcal{F}$ have length at most k. We define $\Sigma(n,k)$ to be the sum of the k largest binomial coefficients of order n, i.e. $\Sigma(n,k)=\sum_{i=1}^{k}\binom{n}{\lfloor\frac{n-k}{2}\rfloor+i}$. Let $\Sigma^*(n,k)$ be the collection of families consisting of the corresponding full levels, i.e. if $n+k$ is odd, then $\Sigma^*(n,k)$ contains one family $\cup_{i=1}^{k}\binom{[n]}{\lfloor\frac{n-k}{2}\rfloor+i}$, while if $n+k$ is even, then $\Sigma^*(n,k)$ contains two families of the same size $\cup_{i=0}^{k-1}\binom{[n]}{\frac{n-k}{2}+i}$ and $\cup_{i=1}^{k}\binom{[n]}{\frac{n-k}{2}+i}$.

Theorem 4 (Erdős, [149]) *If $\mathcal{F}\subseteq 2^{[n]}$ is a k-Sperner family, then $|\mathcal{F}|\le\Sigma(n,k)$ holds. Moreover, if $|\mathcal{F}|=\Sigma(n,k)$, then $\mathcal{F}\in\Sigma^*(n,k)$.*

Proof. We start with the following simple observation.

Lemma 5 *A family $\mathcal{F}$ of sets is k-Sperner if and only if it is the union of k antichains.*

Proof of Lemma. If $\mathcal{F}$ is the union of k antichains, then any chain in $\mathcal{F}$ has length at most k as any chain can contain at most one set from each antichain. Conversely, if $\mathcal{F}$ is k-Sperner, we define the required k antichains recursively. Let $\mathcal{F}_1$ denote the family of all minimal sets in $\mathcal{F}$ and if $\mathcal{F}_j$ is defined for all $1\le j<i$, then let $\mathcal{F}_i$ denote the family of all minimal sets in $\mathcal{F}\setminus\cup_{j=1}^{i-1}\mathcal{F}_j$. The $\mathcal{F}_i$'s are antichains by definition and for every $F\in\mathcal{F}_i$, there exists an $F'\in\mathcal{F}_{i-1}$ with $F'\subset F$. Hence, the existence of a set in $\mathcal{F}_{k+1}$ would imply the existence of a $(k+1)$-chain in $\mathcal{F}$. The partition we obtained is often referred to as the *canonical partition* of $\mathcal{F}$. □

Let $\mathcal{F}\subseteq 2^{[n]}$ be a k-Sperner family. By Lemma 5, $\mathcal{F}$ is the union of k antichains $\mathcal{F}_1,\mathcal{F}_2,\dots,\mathcal{F}_k$. Adding up the LYM-inequalities for all $\mathcal{F}_i$'s we obtain

$$\sum_{F\in\mathcal{F}}\frac{1}{\binom{n}{|F|}}\le k.$$

This immediately yields $|\mathcal{F}|\le\Sigma(n,k)$. If $|\mathcal{F}|=\Sigma(n,k)$, then all these LYM-inequalities must hold with equality. Therefore, by the moreover part of Lemma 3, all $\mathcal{F}_i$'s are full levels and thus $\mathcal{F}\in\Sigma^*(n,k)$. ■

We finish this section by stating and proving an inequality of Bollobás that generalizes the LYM-inequality. We say that a family of pairs of sets $(A_1,B_1),(A_2,B_2),\dots,(A_t,B_t)$ is an *intersecting set pair system* (ISPS) if $A_i\cap B_j\ne\emptyset$ holds if and only if $i\ne j$.

Theorem 6 (Bollobás, [62]) *If $(A_1,B_1),(A_2,B_2),\dots,(A_t,B_t)$ is an ISPS,* *then the following inequality holds:*

$$\sum_{i=1}^{t} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \le 1.$$

In particular, if all A_i's have size at most k and all B_i's have size at most l, then $t \le \binom{k+l}{k}$ holds.

Proof. Let $(A_1, B_1), (A_2, B_2), \ldots, (A_t, B_t)$ be an ISPS and let us define $M = \cup_{i=1}^{t}(A_i \cup B_i)$ and $m = |M|$. Consider all pairs (i, σ) such that σ is a permutation of M and all elements of A_i "come before" the elements of B_i, i.e. $\sigma^{-1}(a) < \sigma^{-1}(b)$ holds for all $a \in A_i, b \in B_i$. We count the number of such pairs in two ways.

If we fix an index i, then the number of permutations σ of M such that (i, σ) has the above property is exactly $|A_i|! \cdot |B_i|!(m - |A_i| - |B_i|)!\binom{m}{|A_i|+|B_i|}$. Indeed, one determines the order of the elements inside A_i, B_i and $M \setminus (A_i \cup B_i)$ independently from each other, and then determines where the elements of $A_i \cup B_i$ are placed in σ.

On the other hand for every permutation σ of M there is at most one pair (i, σ) with the above property. Indeed, if the elements of A_i come before the elements of B_i in σ, then for any $j \ne i$ an element $b \in B_j \cap A_i$ comes before an element $a \in A_j \cap B_i$. As $(A_1, B_1), (A_2, B_2), \ldots, (A_t, B_t)$ is an ISPS, the elements a and b exist. We obtained

$$m! \ge \sum_{i=1}^{t} |A_i|! \cdot |B_i|!(m - |A_i| - |B_i|)!\binom{m}{|A_i| + |B_i|} = \sum_{i=1}^{t} \frac{|A_i|! \cdot |B_i|!}{(|A_i| + |B_i|)!} m!.$$

Dividing by $m!$ yields the statement of the theorem. ■

As we mentioned earlier, Theorem 6 generalizes the LYM-inequality as if $\mathcal{F} = \{F_1, F_2, \ldots, F_m\} \subseteq 2^{[n]}$ is an antichain, then the pairs $A_i = F_i, B_i = \overline{F_i}$ form an ISPS and $\frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} = \frac{1}{\binom{n}{|F_i|}}$.

Exercise 7 *Let $\mathcal{F} \subseteq 2^{[n]}$ be a family such that for any $F, F' \in \mathcal{F}$ we have $|F \setminus F'| \ge l$. Prove that the following inequality holds:*

$$\sum_{F \in \mathcal{F}} \left(\frac{\binom{|F|}{|F|-l}}{\binom{n}{|F|-l}} + \frac{\binom{n-|F|}{l}}{\binom{n}{|F|+l}} \right) \le 2.$$

1.2 The Erdős-Ko-Rado theorem - several proofs

The main notion of this section is the following: a family $\mathcal{F}$ is *intersecting* if for any $F, F' \in \mathcal{F}$ the intersection $F \cap F'$ is non-empty. We will be interested in the maximum possible size that an intersecting family can have. If there are no restrictions on the sizes of sets in $\mathcal{F}$, then we have the following proposition.

Proposition 8 *If $\mathcal{F} \subseteq 2^{[n]}$ is intersecting, then $|\mathcal{F}| \leq 2^{n-1}$.*

Proof. Clearly, an intersecting family cannot contain both F and $\overline{F}$ for any subset F of $[n]$. ■

Note that the bound is sharp as shown by the family $\{F \subseteq [n] : 1 \in F\}$, but there are many other intersecting families of that size (see Exercise 15 at the end of this section). This situation changes if we consider k-uniform families. Again, if all the sets contain the element 1, the family is intersecting (Such families are called *trivially intersecting families*.) The largest trivially intersecting families are the *star* with *center* x $\{F \in \binom{[n]}{k} : x \in F\}$ for some fixed $x \in [n]$. They give constructions of size $\binom{n-1}{k-1}$. The celebrated theorem of Erdős, Ko, and Rado states that this is best possible if $2k \leq n$ holds (the additional condition is necessary, as if $2k > n$, then any pair of k-subsets of $[n]$ intersects). They proved their result in 1938, but they thought that it might not be interesting enough, so they published it only in 1961. They could not have been more wrong about the importance of their theorem: the next chapter is completely devoted to theorems that deal with families satisfying properties defined by intersection conditions.

Theorem 9 (Erdős, Ko, Rado, [163]) *If $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ holds provided $2k \leq n$. Moreover, if $2k < n$, then $|\mathcal{F}| = \binom{n-1}{k-1}$ if and only if $\mathcal{F}$ is a star.*

The first proof we give is more or less the one given in the original paper by Erdős, Ko, and Rado. It uses the very important technique of *shifting* (or compression). Some of the many shifting operations will be defined in this book. For a more detailed introduction see the not very recent survey of Frankl [193].

First proof of Theorem 9. We proceed by induction on k, the base case $k = 1$ being trivial. For $k \geq 2$, let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family. For $x, y \in [n]$, let us define

$$S_{x,y}(F) = \begin{cases} F \setminus \{y\} \cup \{x\} & \text{if } y \in F,\, x \notin F \text{ and } F \setminus \{y\} \cup \{x\} \notin \mathcal{F} \\ F & \text{otherwise} \end{cases} \tag{1.2}$$

and write $S_{x,y}(\mathcal{F}) = \{S_{x,y}(F) : F \in \mathcal{F}\}$.

Lemma 10 *Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family and $x, y \in [n]$. Then $S_{x,y}(\mathcal{F}) \subseteq \binom{[n]}{k}$ is intersecting with $|\mathcal{F}| = |S_{x,y}(\mathcal{F})|$.*

Proof of Lemma. The statements $S_{x,y}(\mathcal{F}) \subseteq \binom{[n]}{k}$ and $|\mathcal{F}| = |S_{x,y}(\mathcal{F})|$ are clear by definition. To prove the intersecting property of $S_{x,y}(\mathcal{F})$ let us call a set $G \in S_{x,y}(\mathcal{F})$ *new* if $G \notin \mathcal{F}$ and *old* if $G \in \mathcal{F}$. Two old sets intersect by the intersecting property of $\mathcal{F}$ and two new sets intersect as by definition they both contain x. Finally, suppose F is an old, and G is a new set of $S_{x,y}(\mathcal{F})$. By definition $x \in G$, so if $x \in F$, then F and G intersect. Suppose $x \notin F$ and consider $F' := G \setminus \{x\} \cup \{y\}$. As G is a new set of $S_{x,y}(\mathcal{F})$ we have $F' \in \mathcal{F}$ and therefore $F \cap F' \neq \emptyset$. If there exists $z \in F \cap F'$ with $z \neq y$, then $z \in F \cap G$ and we are done. If $F \cap F' = \{y\}$, then consider $F'' = F \setminus \{y\} \cup \{x\}$. As F is an old set of $S_{x,y}(\mathcal{F})$, we must have $F'' \in \mathcal{F}$, but then $F' \cap F'' = \emptyset$ contradicting the intersecting property of $\mathcal{F}$. Therefore $F \cap F' \neq \{y\}$ holds. This finishes the proof of the lemma. □

Let us define the weight of a family $\mathcal{G}$ to be $w(\mathcal{G}) = \sum_{G \in \mathcal{G}} \sum_{i \in G} i$. Observe that if $x < y$ and $S_{x,y}(\mathcal{F}) \neq \mathcal{F}$, then $w(S_{x,y}(\mathcal{F})) < w(\mathcal{F})$ holds. Therefore, as the weight is a non-negative integer, after applying a finite number of such shifting operations to $\mathcal{F}$, we obtain a family $\mathcal{F}'$ with the following property: $S_{x,y}(\mathcal{F}') = \mathcal{F}'$ for all $1 \leq x < y \leq n$. We call such a family *left-shifted*. By Lemma 10, to obtain the bound of the theorem it is enough to prove that a left-shifted intersecting family $\mathcal{F}$ has size at most $\binom{n-1}{k-1}$.

Lemma 11 *If $\mathcal{F} \subseteq \binom{[n]}{k}$ is a left-shifted intersecting family, then for any $F_1, F_2 \in \mathcal{F}$ we have $F_1 \cap F_2 \cap [2k-1] \neq \emptyset$.*

Proof of Lemma. Suppose not and let $F_1, F_2 \in \mathcal{F}$ be such that $F_1 \cap F_2 \cap [2k-1] = \emptyset$ holds and $|F_1 \cap F_2|$ is minimal. Let us choose $y \in F_1 \cap F_2$ and $x \in [2k-1] \setminus (F_1 \cup F_2)$. As $\mathcal{F}$ is intersecting, y exists. By the assumption $F_1 \cap F_2 \cap [2k-1] = \emptyset$, we have $y \geq 2k$, and thus $|F_1 \cap [2k-1]|, |F_2 \cap [2k-1]| \leq k-1$, therefore x exists. By definition, we have $x < y$. As $\mathcal{F}$ is left-shifted, we have $F_1' := F_1 \setminus \{y\} \cup \{x\} \in \mathcal{F}$, but then $F_1' \cap F_2 \cap [2k-1] = \emptyset$ and $|F_1' \cap F_2| < |F_1 \cap F_2|$, contradicting the choice of F_1 and F_2. □

We are now ready to prove the bound of the theorem. For $i = 1, 2, \ldots, k$ let us define $\mathcal{F}_i = \{F \in \mathcal{F} : |F \cap [2k]| = i\}$ and $\mathcal{G}_i = \{F \cap [2k] : F \in \mathcal{F}_i\}$. By Lemma 11, $\mathcal{G}_i$ is intersecting for all i and $\mathcal{F} = \cup_{i=1}^k \mathcal{F}_i$. By the inductive hypothesis, we obtain $|\mathcal{G}_i| \leq \binom{2k-1}{i-1}$ for all $i \leq k-1$. By definition, $\mathcal{G}_k = \mathcal{F}_k$ holds, and for every k-subset S of $[2k]$ at most one of S and $[2k] \setminus S$ belongs to $\mathcal{G}_k$, therefore we have $|\mathcal{G}_k| \leq \frac{1}{2}\binom{2k}{k} = \binom{2k-1}{k-1}$. Every set $G \in \mathcal{G}_i$ can be extended to a k-set of $\mathcal{F}_i$ in at most $\binom{n-2k}{k-i}$ ways. We obtain

$$|\mathcal{F}| = \sum_{i=1}^k |\mathcal{F}_i| \leq \sum_{i=1}^k |\mathcal{G}_i| \binom{n-2k}{k-i} \leq \sum_{i=1}^k \binom{2k-1}{i-1}\binom{n-2k}{k-i} = \binom{n-1}{k-1}.$$

We still have to prove that if $2k < n$ holds, then the only intersecting families with size $\binom{n-1}{k-1}$ are stars. First we show this for left-shifted intersecting families, where obviously the center of the star should be 1. If $k = 2$, then there are only two types of maximal intersecting families: the star and the triangle $\{\{1,2\},\{1,3\},\{2,3\}\}$. If $k \geq 3$, then in the inductive argument above, all $\mathcal{F}_i$'s are stars with center 1 and all possible extensions of these sets to k-sets must be present in $\mathcal{F}$, i.e. $\mathcal{F}' := \{F' \in \binom{[n]}{k} : 1 \in F', |F' \cap [2k-1]| \leq k-1\} \subseteq \mathcal{F}$ holds. But for any k-set S with $1 \notin S$ there exists a set $F' \in \mathcal{F}'$ with $S \cap F' = \emptyset$, therefore all sets in $\mathcal{F}$ must contain 1.

For general $\mathcal{F}$, let us start applying shifting operations $S_{x,y}$ with $x < y$. If we obtain a left-shifted family $\mathcal{H}$ that is not a star, then we are done by the above. If not, then at some point we obtain a family $\mathcal{G}$ such that $S_{x,y}(\mathcal{G})$ is a star. Therefore $G \cap \{x, y\} \neq \emptyset$ for all $G \in \mathcal{G}$. Then we apply shifting operations $S_{x',y'}$ with $x' < y'$ and $x', y' \in [n] \setminus \{x, y\}$ until we obtain a family $\mathcal{H}$ with $S_{x',y'}(\mathcal{H}) = \mathcal{H}$ for all $x', y' \in [n] \setminus \{x, y\}$ with $x' < y'$. Clearly, we still have $H \cap \{x, y\} \neq \emptyset$ for all $H \in \mathcal{H}$. The next lemma is the equivalent of Lemma 11.

Lemma 12 *Let A denote the first $2k-2$ elements of $[n] \setminus \{x, y\}$ and let $B = A \cup \{x, y\}$. Then $H \cap H' \cap B \neq \emptyset$ for all $H, H' \in \mathcal{H}$.*

Proof of Lemma. Suppose not and let H, H' be members of $\mathcal{H}$ with $H \cap H' \cap B = \emptyset$ and $|H \cap H'|$ minimal. This means that H and H' meet $\{x, y\}$ in different elements and as $\mathcal{H}$ is intersecting, there exists $y' \in (H \cap H') \setminus B$. Also there exists $x' \in A \setminus (H \cup H')$ as $H \setminus \{x, y\}$ and $H' \setminus \{x, y\}$ are $(k-1)$-sets both containing $y' \notin A$. Then $H \setminus \{y'\} \cup \{x'\} \in \mathcal{H}$ by $S_{x',y'}(\mathcal{H}) = \mathcal{H}$, which contradicts the choice of H and H'. □

Now the exact same calculation and reasoning can be carried out as in the left-shifted case. So $\mathcal{H}$ to have size $\binom{n-1}{k-1}$ we must have that for all $i \leq k-1$ there exists z_i with $z_i \in H$ for all $H \in \mathcal{H}_i = \{H \in \mathcal{H} : |H \cap B| = i\}$ and $|\mathcal{H}_i| = \binom{k-1}{i-1}\binom{n-2k}{k-i}$. Every $H \in \mathcal{H}_k$ must contain z_i for all $i \leq k-1$ as otherwise we would find a $H' \in \mathcal{H}_i$ disjoint from H. So if $z_i \neq z_j$, then $|\mathcal{H}_k| \leq \binom{2k-2}{k-2}$ and thus $|\mathcal{H}| < \binom{n-1}{k-1}$ or z_i is the same for all i and thus $\mathcal{H}$ is a star contradicting our assumption. ■

The next proof we present is due to G.O.H. Katona [337]. This proof is the first application of the *cycle method*. The idea is very similar to that in the proof of the LYM-inequality where one considers maximal chains. One addresses the original problem on a simpler structure $\mathcal{S}$ which is symmetric in the sense that if two sets S_1, S_2 have the same size, then the number of copies of $\mathcal{S}$ containing S_1 equals the number of copies of $\mathcal{S}$ containing S_2. And when this simpler problem is solved, one tries to reduce the original problem to the simpler one. The way to do that is to count the pairs (π, F), where π is a permutation, F is a member of the original family and $\pi(F) \in \mathcal{S}$. For an F it is usually easy to count the number of permutations that form a pair with it, while for every π the solution on the simpler structure gives an upper bound.

In this general form, the method is called the *permutation method* and the cycle method is the following special case: if σ is a cyclic permutation of $[n]$, then a set S is an *interval* of σ if it is a set of consecutive elements. More precisely either $S = \emptyset$ or $S = [n]$ or $S = \{\sigma(i), \sigma(i+1), \ldots, \sigma(i+j)\}$ for some integers $1 \leq i, j \leq n-1$, where addition is considered modulo n. An interval of size k is said to be a *k-interval.* The element $\sigma(i)$ is the *left* *endpoint* of the interval and $\sigma(i+j)$ is the *right endpoint* of the interval. For a fixed permutation σ, there are $n(n-1)+2$ intervals of σ. For a fixed k-set $S \subseteq [n]$ with $1 < k < n$ there are $|F|!(n-|F|)!$ cyclic permutations σ of which S is an interval. Note that $\emptyset$ and $[n]$ are intervals of all the $(n-1)!$ cyclic permutations. Thus in the future calculations they need a separate treatment, but we usually omit that, as it is trivial.

Second proof of Theorem 9. We start with the solution of the problem on the cycle.

Lemma 13 *Let σ be a cyclic permutation of $[n]$ and let $G_1, G_2, \dots, G_r$ be k-intervals of σ that form an intersecting family $\mathcal{G}$. Then $r \leq k$ holds provided $2k \leq n$. Furthermore, if $2k < n$ and $r = k$, then the G_i's are all the k-intervals that contain a fixed element of $[n]$.*

Proof of Lemma. Without loss of generality we may assume that $G_1 = \{\sigma(1), \sigma(2), \dots, \sigma(k)\}$. Then as the G_i's all have the same size and they form an intersecting family, all G_i's have either left endpoint $\sigma(j)$ for some $2 \leq j \leq k$ or right endpoint $\sigma(j)$ for some $1 \leq j \leq k-1$. Furthermore, because of the intersecting property and $2k \leq n$, the k-interval with right endpoint $\sigma(j)$ and the k-interval with left endpoint $\sigma(j+1)$ cannot be both among the G_i's. This proves $r \leq k$.

Let us assume now $2k < n$ and let $\sigma(j)$ be the right endpoint of some G_i with $1 \leq j \leq k-1$. Then k-intervals with left endpoint $\sigma(j+1)$ and $\sigma(j+2)$ cannot be in $\mathcal{G}$. Therefore, as $|\mathcal{G}| = k$, the k-interval with right endpoint $\sigma(j+1)$ must belong to $\mathcal{G}$. Similarly, if for some $2 \leq j \leq k$ the left endpoint of some G_i is $\sigma(j)$, then the k-interval with left endpoint $\sigma(j-1)$ must belong to $\mathcal{G}$. This means that if $|\mathcal{G}| = k$ and $\sigma(1), \sigma(k)$ do not belong to all members of $\mathcal{G}$, then there is exactly one j $(2 \leq j \leq k-1)$ for which $\sigma(j)$ is both a left and a right endpoint and then $\sigma(j)$ is an element of all the intervals in $\mathcal{G}$. $\square$

Let us consider the number of pairs (F, σ), where σ is a cyclic permutation of $[n]$ and $F \in \mathcal{F}$ is an interval of σ. By Lemma 13, for fixed σ the number of such cycles is at most k. On the other hand, every set F in $\mathcal{F}$ is an interval of $k!(n-k)!$ cyclic permutations. We obtained

$$|\mathcal{F}|k!(n-k)! = \sum_{F \in \mathcal{F}} k!(n-k)! \leq k(n-1)!.$$

Dividing by $k!(n-k)!$ yields $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

If $2k < n$ and $|\mathcal{F}| = \binom{n-1}{k-1}$, then by Lemma 13 for every cyclic permutation σ there exists $x_\sigma \in [n]$ such that all intervals of σ containing x_σ belong to $\mathcal{F}$. Suppose toward a contradiction that $\mathcal{F}$ is not a star. Let σ be an arbitrary cyclic permutation and let x_σ be the element that is contained in k intervals of σ all in $\mathcal{F}$. Let $F_1 \in \mathcal{F}$ be the interval that has right endpoint x_σ in σ and let $F_2 \in \mathcal{F}$ be the interval that has left endpoint x_σ in σ. Finally let $F \in \mathcal{F}$ be a set with $x_\sigma \notin F$ (if no such F exists, then $\mathcal{F}$ is a star with center x_σ). Writing

$A = F \cap F_1, B = F \cap F_2$, and $C = F \setminus (A \cup B)$ let π be a cyclic permutation such that for some $1 \le j_1 < j_2 \le j_3 < j_4 < j_5$ we have $(F_1 \setminus \{x_\sigma\}) \setminus F = \{\pi(1), \dots, \pi(j_1)\}$, $A = \{\pi(j_1+1), \dots, \pi(j_2)\}$, $C = \{\pi(j_2+1), \dots, \pi(j_3)\}$, $B = \{\pi(j_3+1), \dots, \pi(j_4)\}$, and $(F_2 \setminus \{x_\sigma\}) \setminus F = \{\pi(j_4+1), \dots, \pi(j_5)\}$. In words, F is an interval of π such that elements of A are at one end of the interval, elements of B are at the other end, and elements of C (if any) are in the middle. Furthermore, all remaining elements of F_1 except x_σ are placed next to A and all remaining elements of F_2 except x_σ are placed next to B. Note that we have not decided about the position of x_σ in π. We distinguish two cases:

Case I: $C \neq \emptyset$.

Then let $\pi(j_5+1) = x_\sigma$ and therefore F_2 is an interval of π. We obtain that $x_\pi \in B$. As C is non-empty, the size of $(C \cup F_2) \setminus \{x_\sigma\}$ is at least k. Therefore any of its k-subintervals containing x_π should belong to $\mathcal{F}$. But by definition, it (at least one of them) is disjoint from F_1. This contradiction finishes the proof in Case I.

Case II: $C = \emptyset$ and thus $F \subset (F_1 \cup F_2) \setminus \{x_\sigma\}$ holds and $(F_1 \cup F_2) \setminus \{x_\sigma\}$ has size $2k-2$.

Then let $\pi(2k) = x_\sigma$. As F is an interval of π, we have $x_\pi \in F$. If $x_\pi \in F \cap F_1 = A$, then $(F_1 \cup \pi(n)) \setminus \{x_\sigma\} = \{\pi(n), \pi(1), \dots, \pi(k-1)\}$ is a k-interval of π that contains x_π, therefore it must belong to $\mathcal{F}$. But as $n > 2k$, we have $\pi(n) \neq x_\sigma$ and therefore $(F_1 \cup \pi(n)) \setminus \{x_\sigma\}$ is disjoint from $F_2 \in \mathcal{F}$, a contradiction to the intersecting property of $\mathcal{F}$. If $x_\pi \in F \cap F_2$, then $(F_2 \cup \pi(2k-1)) \setminus \{x_\sigma\} = \{\pi(k), \pi(k+1), \dots, \pi(2k-1)\}$ is a k-interval of π containing x_π, therefore it belongs to $\mathcal{F}$. But it is disjoint from $F_1 \in \mathcal{F}$ contradicting the intersecting property of $\mathcal{F}$. This finishes the proof of Case II. ■

We present a third proof of the Erdős-Ko-Rado theorem that uses the polynomial method. The main idea of this method is to assign polynomials $p_F(x)$ to every set $F \in \mathcal{F}$ and show that these polynomials are linearly independent in the appropriate vector space V. If this is so, then $|\mathcal{F}| \le \dim(V)$ follows. Let us start with a general lemma giving necessary conditions for polynomials to be independent.

Lemma 14 *Let $p_1(x), p_2(x), \dots, p_m(x) \in \mathbb{F}^n[x]$ be polynomials and $v_1, v_2, \dots, v_m \in \mathbb{F}^n$ be vectors such that $p_i(v_i) \neq 0$ and $p_i(v_j) = 0$ holds for all $1 \le j < i \le m$. Then the polynomials are linearly independent.*

Proof. Suppose that $\sum_{i=1}^m c_i p_i(x) = 0$. As $p_i(v_1) = 0$ for all $1 < i$ we obtain $c_1 p_1(v_1) = 0$ and therefore $c_1 = 0$ holds. We proceed by induction on j. If $c_h = 0$ holds for all $h < j$, then using this and $p_i(v_j) = 0$ for all $i > j$, we obtain $c_j p_j(v_j) = 0$ and therefore $c_j = 0$. ■

Recall that the characteristic vector v_F of a set $F \subseteq [n]$ is the binary vector of length n with ith entry 1 if and only if $i \in F$. By definition, we have $v_F \cdot v_G = |F \cap G|$ for any pair of sets $F, G \subseteq [n]$. Therefore if $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family, then the polynomials $p_F(x) = \prod_{i=1}^{k-1}(x \cdot v_F - i)$ and the characteristic vectors v_F satisfy the conditions of Lemma 14 (independently of the order of sets of $\mathcal{F}$). What is the dimension of the smallest vector space that contains these polynomials? As characteristic vectors have only 0 and 1 entries, we can replace all powers x_j^h by x_j in the p_F's and the polynomials p'_F obtained will still satisfy the conditions of Lemma 14. The vector space generated by these polynomials is a subspace of $V = \langle\{x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_h}, 1 \leq i_1 < i_2 < \ldots < i_h, 0 \leq h \leq k-1\}\rangle$ and clearly, we have $\dim(V) = \sum_{i=0}^{k-1}\binom{n}{i}$ holds. This yields the weak bound $|\mathcal{F}| \leq \sum_{i=0}^{k-1}\binom{n}{i}$. The next proof that improves this and attains the bound of Theorem 9 is due to Füredi, Hwang, and Weichsel [238] and uses an idea that first appeared in a paper of Blokhuis [55].

Third proof of Theorem 9. Our plan is to add $\sum_{i=0}^{k-1}\binom{n}{i} - \binom{n-1}{k-1} = 2\sum_{j=0}^{k-2}\binom{n-1}{j}$ many sets to $\mathcal{F}$ and define corresponding polynomials and vectors such that the conditions of Lemma 14 are still satisfied. More precisely, we will also change some of the polynomials corresponding to sets in $\mathcal{F}$, and replace the characteristic vectors by other vectors. Let $a \in [n]$ be arbitrary and let us define $\mathcal{A} = \mathcal{F}_0 \cup \mathcal{H} \cup \mathcal{F}_1 \cup \mathcal{G}$ where $\mathcal{F}_0 = \{F \in \mathcal{F} : a \notin F\}$, $\mathcal{H} = \{H \subset [n] : a \notin H, 0 \leq |H| \leq k-2\}$, $\mathcal{F}_1 = \{F \in \mathcal{F} : a \in F\}$, $\mathcal{G} = \{G \subset [n] : a \in G : 1 \leq |G| \leq k-1\}$. Clearly, $|\mathcal{H}| = |\mathcal{G}| = \sum_{j=0}^{k-2}\binom{n-1}{j}$ holds. We define the following polynomials:

- For $F \in \mathcal{F}_0$ let $p_F(x) = \prod_{i=1}^{k-1}(v_F \cdot x - i)$, therefore for a set $S \subseteq [n]$ we have $p_F(v_S) = 0$ if and only if $1 \leq |(F \cap S| \leq k-1$. As opposed to what we had before, let $u_F = v_{\overline{F} \setminus \{a\}}$.
- For $H \in \mathcal{H}$ we define $p_H(x) = (\mathbf{1} \cdot x - (n-k-1))\prod_{h \in H} x_h$, where $\mathbf{1}$ denotes the vector of length n with all 1 entries. For a set $S \subseteq [n]$ we have $p_H(v_S) = 0$ if and only if $|S| = n-k-1$ or $H \not\subseteq S$. We let $u_H = v_H$.
- For $F \in \mathcal{F}_1$ we consider $p_F(x) = \prod_{i=0}^{k-2}(v_{F\setminus\{a\}} \cdot x - i)$; therefore for a set $S \subseteq [n]$ we have $p_F(v_S) = 0$ if and only if $0 \leq |(F \setminus \{a\}) \cap S| \leq k-2$. Let $u_F = v_{F\setminus\{a\}}$.
- For $G \in \mathcal{G}$ we define $p_G(x) = \prod_{g \in G} x_g$. For a set $S \subseteq [n]$ we have $p_G(v_S) = 0$ if and only if $G \not\subset S$. We let $u_G = v_G$.

Again, as all vectors are binary, we can change every power x_i^s to x_i in these polynomials, so they are contained in the vector space of polynomials with n real variables that have degree at most 1 in each variable. Therefore $|\mathcal{F}| + |\mathcal{H}| + |\mathcal{G}| = |\mathcal{A}| \leq \sum_{i=0}^{k-1}\binom{n}{i}$ and thus $|\mathcal{F}| \leq \binom{n-1}{k-1}$ provided these polynomials are independent. To check that the conditions of Lemma 14 hold, we still need to define an order of these sets and polynomials. First we enumerate the polynomials p_F belonging to sets in $\mathcal{F}_0$ in arbitrary order, then the polynomials p_H belonging to sets in $\mathcal{H}$ such that if $|H| < |H'|$ then H comes before H'. Then come polynomials p_F belonging to sets in $\mathcal{F}_1$ in arbitrary order, finally polynomials p_G belonging to sets in $\mathcal{G}$ such that if $|G| < |G'|$ then G comes before G'. We verify that the conditions of Lemma 14 hold by a simple case analysis.

- For $F, F' \in F_0$ we have $p_F(u_F) \neq 0$ as $F \cap \overline{F} = \emptyset$ and $p_F(u'_F) = 0$ as $1 \leq |F' \cap (\overline{F} \setminus \{a\})| \leq k-1$ if $F \neq F'$. Indeed, $1 \leq |F' \cap \overline{F}| \leq k-1$ as $\mathcal{F}$ is intersecting, and also we have $a \notin F'$. For $H \in \mathcal{H}$ we have $p_H(u_F) = 0$ as $|\overline{F} \setminus \{a\}| = n-k-1$. For $F_1 \in \mathcal{F}_1$ we have $p_{F_1}(u_F) = 0$ as $|(F_1 \setminus \{a\}) \cap (\overline{F} \setminus \{a\})| \leq k-2$. Indeed, $|F_1 \cap \overline{F}| \leq k-1$ as $\mathcal{F}$ is intersecting, and a is contained in both. For $G \in \mathcal{G}$ we have $p_G(u_F) = 0$ as $G \not\subset \overline{F} \setminus \{a\}$ since $a \in G$ for all $G \in \mathcal{G}$.

- For $H, H' \in \mathcal{H}$ with $|H| \leq |H'|$ we have $p_H(u_H) \neq 0$ as $H \subseteq H$ and $|H| \leq k-2 < n-k-1$ (this is the only time we use the assumption $2k \leq n$). We also have $p_{H'}(u_H) = 0$ as $H' \not\subset H$. For $F \in \mathcal{F}_1$ we have $p_F(u_H) = 0$ as $|H| \leq k-2$ and for $G \in \mathcal{G}$ we have $p_G(u_H) = 0$ as $G \not\subset H$ since $a \in G, a \notin H$.

- Trivially, for $F, F' \in \mathcal{F}_1$ we have $p_F(u_F) \neq 0$ and $p_{F'}(u_F) = 0$. For $G \in \mathcal{G}$ we have $p_G(u_F) = 0$ as $G \not\subset F \setminus \{a\}$ since $a \in G$ for all $G \in \mathcal{G}$.

- For $G, G' \in \mathcal{H}$ with $|G| \leq |G'|$ we have $p_G(u_G) \neq 0$ as $G \subseteq G$, and $p_{G'}(u_G) = 0$ as $G' \not\subset G$.

■

Exercise 15 *Show that any maximal intersecting family $\mathcal{F} \subseteq 2^{[n]}$ of sets has size 2^{n-1}.*

1.3 Intersecting Sperner families

This short section is devoted to antichains that also possess the intersecting property. They are called *intersecting Sperner families*. They are interesting in their own right, but we would also like to give more examples of easy applications of the cycle method. First we prove the result of Milner that determines the maximum possible size of an intersecting Sperner family $\mathcal{F} \subseteq 2^{[n]}$. Obviously, if n is odd, then $\binom{[n]}{\lceil n/2 \rceil}$ is intersecting Sperner and by Theorem 1 it has maximum size even among all antichains. Therefore the important part of the following theorem is when n is even. The proof we present here is due to Katona [340].

Theorem 16 (Milner, [419]) *If $\mathcal{F} \subseteq 2^{[n]}$ is an intersecting Sperner family, then $|\mathcal{F}| \leq \binom{n}{\lceil \frac{n+1}{2} \rceil}$.*

Proof. First we prove the following lemma.

Lemma 17 *Let σ be a cyclic permutation of $[n]$ and let $G_1, G_2, \ldots, G_r$ be intervals of σ that form an intersecting Sperner family. Then the following inequality holds:*

$$\sum_{i=1}^{r} \binom{n}{|G_i|} \leq n \binom{n}{\lceil \frac{n+1}{2} \rceil}.$$

Proof of Lemma. By being an antichain, we see that all the G_i's have distinct left endpoints and therefore $r \leq n$ holds. This finishes the proof if n is odd (as in that case $\binom{n}{\lceil \frac{n+1}{2} \rceil}$ is the largest binomial coefficient). If n is even, we distinguish two cases.

Case I: $r = n$.

We can assume that the left endpoint of G_i is $\sigma(i)$. Then by the Sperner property, we must have $|G_i| \leq |G_{i+1}|$. Consequently, we obtain $|G_1| \leq |G_2| \leq \ldots |G_n| \leq |G_1|$ and therefore all G_i's must have the same size. By the intersecting property this size is at least $\lceil \frac{n+1}{2} \rceil$.

Case II: $r < n$.

By the intersecting property and Lemma 13, at most $n/2$ intervals have size $n/2$. Therefore we have

$$\sum_{i=1}^{r} \binom{n}{|G_i|} \leq \frac{n}{2}\binom{n}{\frac{n}{2}} + \left(\frac{n}{2} - 1\right)\binom{n}{\frac{n}{2}+1} = n\binom{n}{\frac{n}{2}+1} = n\binom{n}{\lceil \frac{n+1}{2} \rceil}.$$

□

Let $\mathcal{F} \subseteq 2^{[n]}$ be an intersecting Sperner family and let us consider the sum

$$\sum_{\sigma, F} \binom{n}{|F|},$$

where the summation is over all cyclic permutations σ of $[n]$ and sets $F \in \mathcal{F}$ that are intervals of σ. For a fixed set F, the number of cyclic permutations of which F is an interval is $|F|!(n-|F|)!$, therefore the above sum equals $|\mathcal{F}| \cdot n!$. On the other hand, by Lemma 17, the sum is at most $(n-1)! \cdot n\binom{n}{\lceil \frac{n+1}{2} \rceil}$. This yields $|\mathcal{F}| \cdot n! \leq (n-1)! \cdot n\binom{n}{\lceil \frac{n+1}{2} \rceil}$ and the theorem follows. ■

Theorem 18 (Bollobás, [64]) *If $\mathcal{F} \subseteq 2^{[n]}$ is an intersecting Sperner family such that all sets of $\mathcal{F}$ have size at most $n/2$, then the following inequality holds:*

$$\sum_{F \in \mathcal{F}} \frac{1}{\binom{n-1}{|F|-1}} \leq 1.$$

Note that this inequality is a strengthening of Theorem 9 as any family containing sets of the same size is an antichain.

Proof. We start with a generalization of Lemma 13.

Lemma 19 *Let σ be a cyclic permutation of $[n]$ and let $G_1, G_2, \ldots, G_r$ be intervals of σ of size at most $n/2$ that form an intersecting Sperner family. Then $r \leq \min\{|G_i| : 1 \leq i \leq r\}$ holds.*

Proof of Lemma. By symmetry, it is enough to show that $r \leq |G_1| =: j$. If $G_1 = \{\sigma(i), \sigma(i+1), \ldots, \sigma(i+j-1)\}$, then by the intersecting Sperner property, all G_k's must have one of their endpoints in G_1 and $\sigma(i)$ cannot be a left endpoint and $\sigma(i+j-1)$ cannot be a right endpoint. This would give $2j-1$ possible other intervals in the family. Notice that as all G_k's have size at most $n/2$, if $\sigma(i+h)$ is a right endpoint, then $\sigma(i+h+1)$ cannot be a left endpoint and vice versa. This leaves at most $j-1$ possible other G_k's. $\square$

Let us consider the sum

$$\sum_{\sigma,F} \frac{1}{|F|},$$

where the summation is over all cyclic permutations σ of $[n]$ and sets $F \in \mathcal{F}$ that are intervals of σ. For fixed σ, by Lemma 19, we have that if $F_1, F_2, \ldots, F_r$ are the intervals of σ belonging to $\mathcal{F}$, then $\sum_{i=1}^{r} \frac{1}{|F_i|} \leq 1$. For fixed $F \in \mathcal{F}$ the number of cyclic permutations of which F is an interval is $|F|!(n-|F|)!$, therefore $\sum_{\sigma} \frac{1}{|F|} = (|F|-1)!(n-|F|)!$. We obtained

$$\sum_{F \in \mathcal{F}} (|F|-1)!(n-|F|)! \leq (n-1)!,$$

and dividing by $(n-1)!$ finishes the proof. ■

Theorem 20 (Greene, Katona, Kleitman, [273]) *If $\mathcal{F} \subseteq 2^{[n]}$ is an intersecting Sperner family, then the following inequality holds:*

$$\sum_{F \in \mathcal{F}, |F| \leq n/2} \frac{1}{\binom{n}{|F|-1}} + \sum_{F \in \mathcal{F}, |F| > n/2} \frac{1}{\binom{n}{|F|}} \leq 1.$$

Proof. The core of the proof is again an inequality for intersecting Sperner families of intervals equipped with an appropriate weight function.

Lemma 21 *Let σ be a cyclic permutation of $[n]$ and let $G_1, G_2, \ldots, G_r$, $G'_1, G'_2, \ldots, G'_s$ be intervals of σ that form an intersecting Sperner family such that $|G_i| \leq n/2$ for all $1 \leq i \leq r$ and $|G'_j| > n/2$ for all $1 \leq j \leq s$. Then the following inequality holds:*

$$\sum_{i=1}^{r} \frac{n-|G_i|+1}{|G_i|} + \sum_{j=1}^{s} 1 \leq n.$$

Proof. We distinguish two cases.

Case I: $r = 0$.

Then the first sum of the left hand side of the inequality is empty and all we have to prove is $s \leq n$. This follows from the Sperner property as the left endpoints of the intervals must be distinct.

CASE II: $r > 0$.

We may assume that $k = |G_1|$ is the smallest size among all G_i's and that $G_1 = \{\sigma(1), \sigma(2), \ldots, \sigma(k)\}$. Note that the weight $\frac{n-w+1}{w}$ is monotone decreasing in w, therefore all weights are at most $\frac{n-k+1}{k}$. Because of the intersecting Sperner property every G_i and G'_j has either left endpoint $\sigma(u)$ for some $u = 2, 3, \ldots, k$ or right endpoint $\sigma(u')$ for some $u' = 1, 2, \ldots, k-1$. Therefore apart from G_1 the possible remaining sets can be partitioned into at most $k-1$ pairs (not all pairs and not both sets of a pair are necessarily present): the $(u-1)$st such pair consists of the set with left endpoint $\sigma(u)$ and the set with right endpoint $\sigma(u-1)$. Observe that because of the intersecting property every pair contains at most one set with size smaller than $n/2$. Therefore the sum of all weights is at most

$$\frac{n-k+1}{k} + (k-1)\Big(\frac{n-k+1}{k} + 1\Big) = n.$$

□

Having Lemma 21 in hand we prove the theorem by considering the sum

$$\sum_{\sigma, F, |F| \leq n/2}^{r} \frac{n-|F|+1}{|F|} + \sum_{\sigma, F, |F| > n/2} 1,$$

where the summation in both sums is over all cyclic permutations σ and all sets F of $\mathcal{F}$ that are intervals of σ. By Lemma 21, for fixed σ the sum is at most n. Recall that for fixed F the number of cyclic permutations of which F is an interval is $|F|!(n-|F|)!$. We obtain

$$\sum_{F \in \mathcal{F}, |F| \leq n/2}^{r} \frac{n-|F|+1}{|F|}|F|!(n-|F|)! + \sum_{F \in \mathcal{F} |F| > n/2} |F|!(n-|F|)! \leq$$
$$\leq (n-1)! \cdot n = n!.$$

Note that $\frac{n-|F|+1}{|F|}|F|!(n-|F|)! = (|F|-1)!(n-|F|+1)!$, therefore dividing by $n!$ finishes the proof of the theorem. ■

Exercise 22 (Scott [501]) *Prove Theorem 16 for even n in the following three steps:*

- *By an argument similar to that in the first proof of Theorem 1, show that a maximum size intersecting Sperner family $\mathcal{F} \subseteq \binom{[n]}{\frac{n}{2}} \cup \binom{[n]}{\frac{n}{2}+1}$.*
- *Observe that $\nabla(\mathcal{F} \cap \binom{[n]}{\frac{n}{2}})$ is disjoint from $\mathcal{F} \cap \binom{[n]}{\frac{n}{2}+1}$.*
- *Using the cycle method show $|\mathcal{F} \cap \binom{[n]}{\frac{n}{2}}| \leq |\nabla(\mathcal{F} \cap \binom{[n]}{\frac{n}{2}})|$ holds.*

1.4 Isoperimetric inequalities: the Kruskal-Katona theorem and Harper's theorem

The original geometric isoperimetric problem dates back to antiquity and states that if R is a region in the plane of area 1 and the boundary of R is a simple curve C, then the length of C is minimized when C is a cycle. There are mainly two discrete versions of this problem. For a graph G and a subset U of the vertex $V(G)$, the *vertex boundary of* U is $\partial U = \{v \in V(G) \setminus U : \exists u \in U$ such that u is adjacent to $v\}$ and the *exterior of* U is $ext(U) = V(G) \setminus (U \cup \partial U)$. For fixed integer m and a graph G the *vertex isoperimetric problem* is to minimize $|\partial U|$ over all subsets $U \subseteq V(G)$ with $|U| = m$. The *edge isoperimetric problem* is to minimize the number of edges between U and ∂U over all subsets $U \subseteq V(G)$ with $|U| = m$. Note that the edge isoperimetric problem is equivalent for values m and $|V(G)| - m$. Discrete isoperimetric appear frequently in the literature, we only consider some examples for graphs the vertex set of which are associated with set families.

Our first example is the bipartite graph $G = G_{n,k-1,k}$ that we defined in the first proof of Theorem 1. Let us remind the reader that G is bipartite with parts $\binom{[n]}{k}$ and $\binom{[n]}{k-1}$ and the $(k-1)$-set F is adjacent to the k-set F' if and only if $F \subset F'$. Note that if $\mathcal{F} \subseteq \binom{[n]}{k}$, then $\partial\mathcal{F} = N(\mathcal{F}) = \Delta(\mathcal{F})$ holds.

The *lexicographic ordering* of sets is defined by A being smaller than B if and only if $\min A \setminus B < \min B \setminus A$ holds (i.e. the first difference matters). However, the following theorem uses an ordering where the last different element matters. Let us define the *colex ordering* $\prec_k$ on $\binom{\mathbb{Z}^+}{k}$ by setting $A \prec_k B$ if and only if $\max A \setminus B < \max B \setminus A$ holds. The smallest element of this ordering is the set $[k]$ and for every $n \geq k$ the family $\binom{[n]}{k}$ is an initial segment of $(\prec_k, \binom{\mathbb{Z}^+}{k})$. For general m, we denote its initial segment of size m by $\mathcal{I}^k_m$.

Theorem 23 (Kruskal, Katona, [335, 385]) *Let $\mathcal{F}$ be a k-uniform family of size m. Then the inequality $|\Delta(\mathcal{F})| \geq |\Delta(\mathcal{I}^k_m)|$ holds.*

The Kruskal-Katona theorem is not a real vertex isoperimetric inequality as it states a sharp lower bound on the size of ∂U only for some sets $U \subseteq V(G)$: those that do not contain vertices from the class corresponding to sets of size $k-1$.

Proof. The proof we present here is due to Bollobás and Leader [70]. It uses the following generalization of the shifting operation introduced in the first proof of Theorem 9. For disjoint sets X, Y with $|X| = |Y|$ we define

$$S_{X,Y}(F) = \begin{cases} F \setminus Y \cup X & \text{if } Y \subseteq F, X \cap F = \emptyset \text{ and } F \setminus Y \cup X \notin \mathcal{F} \\ F & \text{otherwise,} \end{cases} \tag{1.3}$$

and write $S_{X,Y}(\mathcal{F}) = \{S_{X,Y}(F) : F \in \mathcal{F}\}$. We say that $\mathcal{F}$ is (X, Y)-*shifted* if $S_{X,Y}(\mathcal{F}) = \mathcal{F}$, and $\mathcal{F}$ is *s-shifted* if it is (X,Y)-shifted for all disjoint pairs X, Y with $|X| = |Y| = r$, $X \prec_r Y$ and $r \le s$. Finally, we say that $\mathcal{F}$ is *shifted* if it is shifted for all $s \le k$. Note that by definition, every family $\mathcal{F}$ is $(\emptyset, \emptyset)$-shifted and thus 0-shifted. Also, being 1-shifted and left-shifted are equivalent.

First we prove a lemma that characterizes initial segments of the colex ordering with the above compression operation.

Lemma 24 *A family $\mathcal{F} \subseteq \binom{\mathbb{Z}^+}{k}$ is an initial segment of the colex ordering if and only if $\mathcal{F}$ is shifted.*

Proof of Lemma. Suppose first that $\mathcal{F}$ is shifted. If $\mathcal{F}$ is not an initial segment, then there exist $F \in \mathcal{F}, G \notin \mathcal{F}$ with $G \prec_k F$. Writing $X = G \setminus F, Y = F \setminus G$ and $r = |F \cap G|$, we have $X \cap Y = \emptyset$, $X \prec_r Y$ and $(F \setminus Y) \cup X = G$. As $\mathcal{F}$ is shifted, we must have $G \in \mathcal{F}$, a contradiction.

Conversely, if $\mathcal{F}$ is an initial segment of the colex ordering, then for any F, X, Y with $Y \subseteq F, X \cap F = \emptyset, X \prec_r Y$ we have $(F \setminus Y) \cup X \prec_k F$ and thus $(F \setminus Y) \cup X \in \mathcal{F}$. This shows $S_{X,Y}(\mathcal{F}) = \mathcal{F}$ and $\mathcal{F}$ is shifted. $\square$

Our strategy to prove the theorem is to apply shifting operations S_{X_i,Y_i} to $\mathcal{F}$ until we obtain a shifted family $\mathcal{F}'$ and show that the size of the shadow never increased when applying a shifting operation. Unfortunately, it would not be true for an arbitrary sequence of operations; however, the following lemma holds.

Lemma 25 *Let $\mathcal{F}$ be a family of k-subsets of $\mathbb{Z}^+$ and X, Y be disjoint sets with $|X| = |Y|$ and suppose that for every $x \in X$ there exists $y \in Y$ such that $\mathcal{F}$ is $(X \setminus \{x\}, Y \setminus \{y\})$-shifted. Then $\Delta(S_{X,Y}(\mathcal{F})) \le \Delta(\mathcal{F})$ holds.*

Proof of Lemma. We define an injective mapping from $\Delta(S_{X,Y}(\mathcal{F})) \setminus \Delta(\mathcal{F})$ to $\Delta(\mathcal{F}) \setminus \Delta(S_{X,Y}(\mathcal{F}))$. If A is in $\Delta(S_{X,Y}(\mathcal{F})) \setminus \Delta(\mathcal{F})$, then there exists $x \in [n]$ such that $A \cup \{x\} \in S_{X,Y}(\mathcal{F}) \setminus \mathcal{F}$. Therefore $X \subseteq A \cup \{x\}$, $Y \cap (A \cup \{x\}) = \emptyset$ and $S^{-1}_{X,Y}(A \cup \{x\}) = A \cup \{x\} \setminus X \cup Y \in \mathcal{F}$. If $x \in X$ held, then by the condition of the lemma there would exist $y \in Y$ for which $S_{X \setminus \{x\}, Y \setminus \{y\}}(A \cup \{x\} \setminus X \cup Y) = A \cup \{y\}$ would be in $\mathcal{F}$ and thus A would belong to $\Delta(\mathcal{F})$, a contradiction. Therefore we have $x \notin X$ and so $A \cup Y \setminus X \subsetneq A \cup \{x\} \setminus X \cup Y$ and $A \cup Y \setminus X \in \Delta(\mathcal{F})$.

We show that $A \cup Y \setminus X \notin \Delta(S_{X,Y}(\mathcal{F}))$. Suppose towards a contradiction that there exists u such that $(A \cup Y \setminus X) \cup \{u\} \in S_{X,Y}(\mathcal{F})$ holds. As $Y \subset (A \cup Y \setminus X) \cup \{u\}$ we have $(A \cup Y \setminus X) \cup \{u\} \in \mathcal{F}$. If $u \in X$, then by the condition of the lemma, there exists a $v \in Y$ such that $S_{X \setminus \{u\}, Y \setminus \{v\}}(\mathcal{F}) = \mathcal{F}$. It follows that $(A \cup Y \setminus X) \cup \{u\} \setminus (Y \setminus \{v\}) \cup (X \setminus \{u\}) = A \cup \{v\}$ belongs to $\mathcal{F}$, in particular $A \in \Delta(\mathcal{F})$, a contradiction. If $u \notin X$, then $S_{X,Y}((A \cup Y \setminus X) \cup \{u\}) = A \cup \{u\} \in S_{X,Y}(\mathcal{F})$. Recall that we indirectly assumed $(A \cup Y \setminus X) \cup \{u\} \in S_{X,Y}(\mathcal{F})$ holds also. But then both these sets must belong to $\mathcal{F}$, too, which contradicts $A \notin \Delta(\mathcal{F})$.

This means that the images of the mapping $i : A \mapsto A \cup Y \setminus X$ are indeed in $\Delta(\mathcal{F}) \setminus \Delta(S_{X,Y}(\mathcal{F}))$. Moreover, i is injective as for all $A \in \Delta(S_{X,Y}(\mathcal{F})) \setminus \Delta(\mathcal{F})$ we showed that $X \subset A$ and it is obvious that $Y \cap A = \emptyset$. $\square$

All that is left to prove is that starting with any family $\mathcal{F} \subseteq \binom{\mathbb{Z}^+}{k}$ using only shifting operations allowed in Lemma 25 one can arrive at a shifted family $\mathcal{F}'$. The next claim finishes the proof of the theorem.

Claim 26 *For every $1 \leq s \leq k$ and $\mathcal{F} \subseteq \binom{\mathbb{Z}^+}{k}$, using only shifting operations $S_{X,Y}$ with $X \prec_r Y$, and $r \leq s$ that satisfy the conditions of Lemma 25, one can turn $\mathcal{F}$ into an s-shifted family $\mathcal{F}'$.*

Proof of Claim. We proceed by induction on s. As every family is 0-shifted, all shifting operations $S_{X,Y}$ are allowed when X and Y are singletons; thus, as seen in the first proof of Theorem 9, one can achieve a left-shifted family. Assume that for some $s \geq 2$ the statement of the claim is proved for $s-1$ and let $\mathcal{F} \subseteq \binom{\mathbb{Z}^+}{k}$. Suppose there is an integer r with $2 \leq r \leq s$, and a disjoint pair of r-sets X, Y with $X \prec_r Y$ and $S_{X,Y}(\mathcal{F}) \neq \mathcal{F}$. By induction, we can achieve an $(r-1)$-shifted family $\mathcal{F}^*$ using only allowed operations. Now we show that $S_{X,Y}$ is an allowed operation, i.e. for every $x \in X$ there exists $y \in Y$ such that $\mathcal{F}^*$ is $(X \setminus \{x\}, Y \setminus \{y\})$-shifted. Take any $y \in Y$ with $y \neq \max Y$, then we have $X \setminus \{x\} \prec_{r-1} Y \setminus \{y\}$, and by the $(r-1)$-shifted property of $\mathcal{F}^*$ we have $S_{X \setminus \{x\}, Y \setminus \{y\}}(\mathcal{F}^*) = \mathcal{F}^*$.

Finally, this process ends, as whenever a shifting operation $S_{X,Y}$ with $X \prec_r Y$ changes a family $\mathcal{F}$, then $\sum_{G \in S_{X,Y}(\mathcal{F})} |G| < \sum_{F \in \mathcal{F}} |F|$ holds. $\square\blacksquare$

It is easy to see that for every pair of positive integers m and k there exists a sequence $a_k > a_{k-1} > \cdots > a_l > 0$ with $a_i \geq i$ such that $m = \sum_{i=l}^{k} \binom{a_i}{i}$ holds. If so, then again it is easy to see that $\mathcal{I}_m^k$ is the union of $\binom{[a_k]}{k}$, $\{S \cup \{a_k + 1\} : S \in \binom{[a_{k-1}]}{k-1}\}$, $\{S \cup \{a_k + 1, a_{k-1} + 1\} : S \in \binom{[a_{k-2}]}{k-2}\}$, etc. It follows that $|\Delta(\mathcal{I}_m^k)| = \sum_{i=l}^{k} \binom{a_i}{i-1}$. Unfortunately, it is very inconvenient to calculate using this expression, therefore in most applications of the shadow theorem, one uses the following version of Theorem 23 due to Lovász.

Theorem 27 (Lovász, [392]) *Let $\mathcal{F}$ be a k-uniform family with $|\mathcal{F}| = \binom{x}{k}$ for some real number $x \geq k$. Then $|\Delta(\mathcal{F})| \geq \binom{x}{k-1}$ holds. Moreover if $\Delta(\mathcal{F}) = \binom{x}{k-1}$, then x is an integer and $\mathcal{F} = \binom{X}{k}$ for a set X of size x.*

We present a proof by Keevash [347].

Proof. We start with stating an equivalent form of the theorem. Let $\mathcal{K}_{r+1}^r$ be the family consisting of all the $r+1$ r-sets on an underlying set of size $r+1$. For an r-uniform family $\mathcal{G}$ we denote by $\mathcal{K}_{r+1}^r(\mathcal{G})$ the set of copies of $\mathcal{K}_{r+1}^r$ in $\mathcal{G}$ and by $\mathcal{K}_{r+1}^r(v) = \mathcal{K}_{r+1}^r(\mathcal{G}, v)$ the set of copies of K_{r+1}^r in $\mathcal{G}$ containing v.

Lemma 28 *If $\mathcal{G}$ is a $(k-1)$-uniform family with $|\mathcal{G}| = \binom{x}{k-1}$, then*

$|\mathcal{K}_k^{k-1}(\mathcal{G})| \leq \binom{x}{k}$ *and equality holds if and only if* x *is an integer and* $\mathcal{G} = \binom{X}{k-1}$ *for some set* X *of size* x.

Before proving the lemma we show that it is indeed equivalent to the theorem. We prove first that the theorem implies the lemma. If $\mathcal{G}$ satisfies the conditions of the lemma, then let us define $\mathcal{F} = \mathcal{K}_k^{k-1}(\mathcal{G})$. If $|\mathcal{F}| > \binom{x}{k}$ held, then by the theorem we would have $\binom{x}{k-1} < |\Delta(\mathcal{F})| \leq |\mathcal{G}|$, a contradiction. Assume now the lemma holds and let $\mathcal{F}$ satisfy the conditions of the theorem. Suppose $|\Delta(\mathcal{F})| = \binom{y}{k-1}$ with $y < x$ and apply the lemma to $\mathcal{G} = \Delta(\mathcal{F})$. We obtain $|\mathcal{F}| \leq |\mathcal{K}_k^{k-1}(\mathcal{G})| \leq \binom{y}{k} < \binom{x}{k}$, a contradiction.

The equivalence of the cases of equality follows similarly.

Proof of Lemma. We proceed by induction on k with the case $k-1 = 1$ being trivial. We will need the following definition. If $\mathcal{G}$ is a k-uniform family and v is an element of the underlying set, then the *link* $\mathcal{L}(v)$ is a $(k-1)$-uniform family with $S \in \mathcal{L}(v)$ if and only if $S \cup \{v\} \in \mathcal{G}$.

Claim 29 ***(i)*** $|\mathcal{K}_k^{k-1}(v)| \leq |\mathcal{G}| - d_{\mathcal{G}}(v)$,

(ii) $|\mathcal{K}_k^{k-1}(v)| \leq |K_{k-1}^{k-2}(\mathcal{L}(v))|$,

(iii) $|\mathcal{K}_k^{k-1}(v)| \leq (\frac{x}{k-1} - 1)d_{\mathcal{G}}(v)$ *for every vertex* v *and equality holds only if* $d_{\mathcal{G}}(v) = \binom{x-1}{k-2}$.

Proof of Claim. Observe that for a k-set S not containing v, $S \cup \{v\}$ spans a copy of $\mathcal{K}_k^{k-1}$ in $\mathcal{G}$ if and only if $S \in \mathcal{G}$ and S spans a copy of $\mathcal{K}_{k-1}^{k-2}$ in $\mathcal{L}(v)$. The first condition implies **(i)**, the second implies **(ii)**.

To see **(iii)**, suppose first that $d_{\mathcal{G}}(v) \geq \binom{x-1}{k-2}$. Then by **(i)**, we have $|\mathcal{K}_k^{k-1}(v)| \leq \binom{x}{k-1} - d_{\mathcal{G}}(v) \leq (\frac{x}{k-1} - 1)d_{\mathcal{G}}(v)$. If $d_{\mathcal{G}}(v) \leq \binom{x-1}{k-2}$, then let $x_v \leq x$ be the real number with $d_{\mathcal{G}}(v) = \binom{x_v - 1}{k-2}$. Using induction and **(ii)**, we obtain $|\mathcal{K}_k^{k-1}(v)| \leq |\mathcal{K}_{k-1}^{k-2}(\mathcal{L}(v))| \leq \binom{x_v-1}{k-1} = (\frac{x_v}{k-1} - 1)d_{\mathcal{G}}(v) \leq (\frac{x}{k-1} - 1)d_{\mathcal{G}}(v)$. In both cases, $d_{\mathcal{G}}(v) = \binom{x-1}{k-2}$ is necessary for all inequalities to hold with equality. □

Using **(iii)** of Claim 29 we have

$$k|\mathcal{K}_k^{k-1}(\mathcal{G})| = \sum_v |\mathcal{K}_k^{k-1}(v)| \leq \sum_v (\frac{x}{k-1} - 1)d_{\mathcal{G}}(v)$$
$$= (\frac{x}{k-1} - 1)|\mathcal{G}|(k-1) = k\binom{x}{k}.$$

This finishes the proof of the inductive step for the inequality, and equality holds if all the degrees are equal to $\binom{x-1}{k-2}$. But then writing $n = |\cup_{G \in \mathcal{G}} G|$ we have $n\binom{x-1}{k-2} = \sum_v d_{\mathcal{G}}(v) = (k-1)|\mathcal{G}| = (k-1)\binom{x}{k-1} = x\binom{x-1}{k-2}$. Therefore x equals n and $\mathcal{G} = \binom{\cup_{G \in \mathcal{G}} G}{k-1}$. □

■