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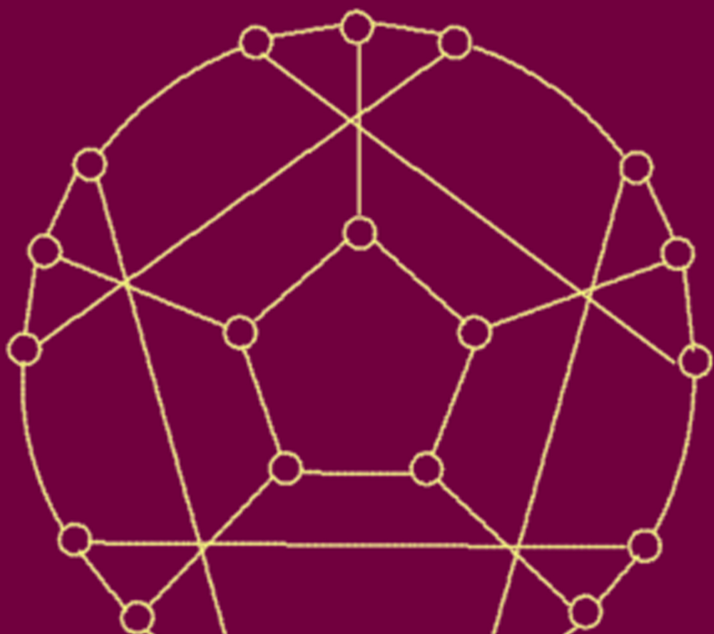
CHROMATIC GRAPH THEORY

SECOND EDITION



Gary Chartrand
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Chromatic Graph Theory

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Chromatic Graph Theory

Second Edition

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To Ralph Stanton (1922-2010)

whose contributions to discrete mathematics and enthusiastic support of mathematicians are acknowledged and greatly appreciated by all who knew him; he will long be remembered.

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PREFACE TO THE SECOND EDITION

Graph coloring is one of the most popular areas of graph theory, likely due to its many fascinating problems and applications as well as the sheer beauty of the subject. Because of the numerous interesting problems that have been introduced in this area over many decades, there are many options available as to what might be included in a book on graph colorings. Surely, the topics that are considered fundamental and historically important in this area must be included. However, there are several topics on graph colorings that have appeared in recent years and which have attracted the attention and interest of many mathematicians but have not previously appeared in books. We decided to include a number of these as well.

In this second edition of *Chromatic Graph Theory*, we begin with a discussion of the origin of the Four Color Problem, which is the beginning of graph colorings. The primary goal of this book, however, is to introduce graph theory with a coloring theme and emphasis – to explore connections between major topics in graph theory and graph colorings and to look at graph colorings in a variety of ways. This book has been written with the intention of using it for one or more of the following purposes:

- for a course in graph theory with an emphasis on graph colorings, where this course could be either a beginning course in graph theory or a follow-up course to an elementary graph theory course;
- for a reading course on graph colorings;
- for a seminar on graph colorings;
- as a reference book for individuals interested in graph colorings.

To accomplish this, it has been our goal to write this book in an engaging, student-friendly style so that it contains carefully explained proofs and examples as well as many exercises of varying difficulty. Both the text and exercises contain problems that suggest research topics that can be explored further.

MATERIAL NEW TO THE SECOND EDITION

Among the material new to this second edition are the following:

(1) additional material on Ramsey theory, including topics dealing with bipartite, proper, and monochromatic Ramsey numbers and Gallai-Ramsey numbers;

(2) in addition to rainbow-connected graphs, there are discussions dealing with proper connected and rainbow Hamiltonian-connected graphs;

(3) rainbow disconnection in connected graphs;

(4) additional material that looks at domination through coloring;

(5) majestic, royal, and regal colorings, which are edge colorings that induce vertex coloring in a set theoretic manner;

(6) rainbow mean colorings, which are edge colorings that induce a vertex coloring defined in an arithmetic manner;

(7) zonal labelings of planar graphs and its connection with the Four Color Problem.

The second edition of *Chromatic Graph Theory* contains over 150 new exercises, some of which suggest new areas for research in graph colorings.

OUTLINE OF CONTENTS

The second edition of *Chromatic Graph Theory* consists of 18 chapters (Chapters 0–17). Chapter 0 sets the stage for the remainder of the book. This chapter describes the origin of graph colorings and, in a sense, how coloring contributed to graph theory becoming an area of mathematics. The origin of the Four Color Problem, some of its history, and attempts to solve the problem are discussed. This laid the groundwork for numerous types of colorings in graphs, many of which are presented in Chapters 6–17.

In order to have a better understanding of the material presented on graph coloring, it is important to be familiar with other fundamental areas of graph theory. To achieve the goal of having the book self-contained, Chapters 1-5 have been written to contain many of the fundamentals, concepts, and results of graph theory that lie outside of graph colorings.

- ⋆ Chapter 1 describes much of the basic terminology of graph theory, introducing the notation of the concepts that commonly appear in graph theory.

- ⋆ Much of graph theory deals with graphs that are connected. This is the primary topic of Chapter 2. The simplest type of connected graphs are the trees, also discussed in this chapter. Among the measures that describe how connected a graph may be, the two best known are connectivity and edge-connectivity, both of which are discussed in Chapter 2.

- ⋆ There are various ways that one may proceed about a connected graph. The two best known of these, each with an interesting origin and history, result in graphs referred to as Eulerian graphs and Hamiltonian graphs, named for the two famous mathematicians Leonhard Euler and Sir Willian Rowan Hamilton, each of whom played a role in the history of graph theory. These are the primary topics of Chapter 3.

- ⋆ Much of the interest and research in graph theory deals with whether a given graph contains a particular subgraph or possesses properties under which a graph may contain such a subgraph. One of these subgraphs deals with what is called a matching. A related topic concerns the problem of decomposing a graph into these types of subgraphs or other subgraphs of interest. These are the topics of Chapter 4.

- ⋆ Since the subject of graph coloring came from the Four Color Problem, which deals with coloring maps drawn on a sphere or in a plane, there has been an interest for many years on those graphs that can be drawn in the plane without any of its edges crossing. A discussion of these planar graphs is the primary topic of Chapter 5. If a graph is not planar, then there are more complex surfaces on which the graphs can be drawn or embedded. This is discussed in Chapter 5 as well.

- ⋆ The main topic of this book, namely graph colorings, is introduced in Chapter 6, where the subject of vertex coloring is described. As was the goal of solving the Four Color Problem, the major interest in a vertex coloring is one in which as few colors are used as possible so that no two neighboring vertices are colored the same. The basic terminology of vertex coloring is described here as well as a number of possibly unexpected applications of vertex colorings. If a subgraph of a given graph requires a certain number of colors, then at least that many colors are needed to color the vertices of the graph itself. This observation gave rise to the concept of perfect graphs, a topic also discussed in Chapter 6.

- ⋆ It is often extraordinarily difficult to determine the exact minimum number of colors needed to color the vertices of a graph so that no two neighboring vertices are colored the same. Because of this, there has been interest in knowing a number of colors, at most of which can be used to color the vertices of a graph. A number of such upper bounds are discussed in Chapter 7, the primary topic of this chapter.

- ⋆ One way of stating the Four Color Problem is that of determining whether 4 is the minimum number of colors needed to color the vertices of all planar graphs so that no two neighboring vertices are colored the same. During the 124-year period in which this problem was known but a solution unknown, other results, problems, and concepts came about in hopes of understanding graph coloring better. While the Four Color Problem was difficult, the Five Color Theorem was not. The number of ways to color the vertices of a graph with a specified number of colors was introduced and turned out to be a polynomial in the number of colors. These are the topics of Chapter 8.

- ⋆ The minimum number of colors needed to color the vertices of a graph G so that no two neighboring vertices are colored the same is the chromatic number of G. If the chromatic number of a graph G is k, then it's possible to partition the vertex set into k subsets, each subset containing vertices no two of which are neighboring. There are occasions when there is only one such partition. This is a topic discussed in Chapter 9. When there is a vertex coloring of a graph G, with k colors say, then each vertex of G can be assigned any one of these k colors with the only condition being that at the conclusion no two neighboring vertices are colored the same. There has been interest in vertex colorings where each vertex can be colored with a color in a prescribed proper subset or list of the k colors. Such list colorings are also discussed in this chapter. There are also situations where the vertices of a subgraph of a graph G is colored with k colors, bringing up the question as to whether this coloring can be extended to a vertex coloring of G itself – another topic discussed in Chapter 9.

- ⋆ Of the many attempts to solve the Four Color Problem and show that the vertices of every planar graph could be colored with four colors so that no two neighboring vertices are colored the same, one such attempt involved coloring the edges of certain planar graphs so that no two edges incident with a common vertex are colored the same. The number of edges incident with a vertex is its degree. While the minimum number of colors needed to color the edges of a graph is always at least the maximum degree, an important theorem states that it never exceeds this maximum degree by more than 1. Consequently, the minimum number of colors needed to color the edges of a graph is always one of two numbers. This is the primary topic of Chapter 10.

- ⋆ There is a theorem, called Ramsey's theorem, that implies that for every two graphs, say F and H, there are complete graphs (every two vertices are joined by an edge) such that if the edges of these complete graphs are colored red or blue in any manner whatsoever, then there results either a subgraph F all of whose edges are colored red or a subgraph H all of whose edges are colored blue. The minimum number of vertices in such a complete graph is the Ramsey number of F and H. This and related Ramsey numbers where complete graphs are replaced by other graphs are the primary topics of Chapter 11.

- ⋆ There have been numerous variations of Ramsey numbers introduced, many of which deal with properties required of the colors of given subgraphs. A number of these types of Ramsey numbers are described in Chapter 12.

- ⋆ The September 11, 2001 terrorist attacks have given rise to many concepts related to security of communication networks. Connected edge-colored graphs can be used to model and study the transfer of information. This has created an interest in edge colorings of connected graphs so that every two vertices are connected by a path whose edges satisfy some prescribed property and in the minimum number of colors that will accomplish this. One of these requires that the colors of the edges in one such path be distinct for each pair of vertices. This results in the concept of rainbow-connected graphs. Another concept deals with coloring the edges of a graph with as few colors as possible so that for each pair of vertices there always exists a set of edges no two of which are colored the same and whose removal results in a graph where the vertices are not connected. These are the major topics of Chapter 13.

- ⋆ Chapters 14 and 15 return to vertex colorings, each chapter discussing conditions to be satisfied by a coloring.

  Chapter 14 deals with distance and lengths of paths connecting two vertices. Many of these concepts have come from graphs that model the Channel Assignment Problem in a certain way. Here the goal is to assign channels to transmitters located in some region so that clear reception of the transmitted signals results. In these cases, the channels are the colors and requirements on the channels result in conditions on the colors.

⋆ A topic in graph theory that has gained increasing interest in recent decades is domination. Typically, a vertex in a graph dominates itself and each neighboring vertex and a dominating set is a set of vertices having the property that each vertex in the graph is dominated by at least one vertex in the set. The primary interest here has been determining the minimum number of vertices in such a dominating set. There are other sets of vertices satisfying other types of domination. In Chapter 15, it is shown that many of the best-known types of domination can be looked in terms of coloring the vertices of a graph with one of two colors.

⋆ Chapter 16 deals with both edge colorings and vertex colorings of graphs. In each case, every edge of a graph is assigned a color which induces a color for each vertex in some manner. In the first instance, each edge is assigned a positive integer for its color and the vertex coloring is a set, namely the set of colors of the edges incident with the vertex. The main problem here is to minimize the number of colors needed for the edges so that the resulting vertex coloring has some prescribed property. This has been generalized to where each edge is assigned a nonempty subset of some set of colors so that each vertex is assigned either the union or the intersection of the colors of its incident edges and, again, the resulting vertex coloring satisfies some prescribed property. In the final edge coloring, each edge is assigned a positive integer for its color, resulting in a positive integer color for each vertex, where again, the vertex coloring satisfies some prescribed property.

⋆ In the final Chapter 17, we close where we began, namely with the Four Color Problem. In this case, we describe a vertex labeling of planar graphs due to Cooroo Egan. Here, the problem is whether the vertices of a planar graph embedded in the plane can be labeled with the nonzero elements of the additive group $\mathbb{Z}_3$ in such a way that the sum of the labels of the vertices on the boundary of each zone (region) is a constant, namely 0. It is seen that there is a connection with this type of labeling, called a zonal labeling, with the Four Color Problem.

TO THE STUDENT

In the 1950s and 1960s, it was unusual for colleges and universities to offer courses in graph theory or even in discrete mathematics. However, as time went by, the importance and applicability of discrete mathematics has become increasingly clear. One of the most important areas within discrete mathematics is graph theory, a subject that is considered to have begun in 1736 when the famous Swiss mathematician Leonhard Euler solved the equally famous Königsberg Bridge Problem (which is discussed in Chapter 3). What Euler accomplished did not cause graph theory to become an area of mathematics soon afterward, however. Indeed, for the next 150-200 years, graphs primarily occurred indirectly in puzzles and other recreational mathematics. However, during the second half of the 19th century and especially beginning in 1936, when the first book on graph theory by the Hungarian mathematician Dénes König was published, graph theory began developing into a "theory".

As with much of mathematics, graph theory only grew significantly after World War II. In the case of graph theory, though, the early interest in this subject had much to do with a problem that occurred in 1852: The Four Color Problem. It was attempts to solve this problem during the next 124 years that was the impetus for graph coloring becoming a major area of graph theory. Indeed, this problem was the beginning of graph coloring becoming perhaps the most popular area within graph theory. In this second edition, Chapter 0 gives some historical background on the subject of graph coloring. The next five chapters provide some of the fundamental concepts and topics within graph theory that do not involve coloring. All of the remaining twelve chapters deal with graph coloring, including material that has become standard, followed by more recent material that mathematicians are investigating even now. Many of the topics discussed in these later chapters are still developing. Studying these topics gives one ideas on how some areas of research in graph theory came about and may suggest ideas for new contributions to graph theory.

TO THE INSTRUCTOR

If this book is being used as a textbook for a course or a reading course, then the way the material is to be covered depends on whether this is a first course in graph theory for the students. Regardless of the primary purpose, since this book is written with a graph coloring theme, it would be good for students to read Chapter 0 on their own to give them background on this subject.

Let's first consider the case where this is a first course in graph theory for the students. Chapters 1–5 cover some fundamental material on graph theory not dealing with graph colorings, but which is important in the study of various aspects of graph colorings. Chapter 6 covers vertex colorings. Consequently, it would be good to design a course to cover various parts of Chapters 1–6. Topics could then be chosen from the remaining chapters that are of interest to the instructor.

If the students have already had a beginning course on graph theory, then it would be good to quickly go through the material in Chapter 1 so that everyone is familiar with the terminology and notation being used throughout the book. Since the students already had a course in graph theory, it is likely that they have already encountered much of the material from Chapters 1-6. Nevertheless, it is good for students to review this material. The instructor can then design a course from the remaining chapters that not only interest the instructor but is likely to interest the students as well. Depending on what the purpose of the course is, topics can be chosen so that the students can be creative and ask questions of their own.

ACKNOWLEDGMENT

We thank Bob Ross (Senior Editor at CRC Press/Chapman & Hall, Taylor & Francis Group) for his constant encouragement and interest and for suggesting this writing project to us.

G.C. & P.Z.

List of Symbols

Symbol	**Meaning**	**Page**
V, $V(G)$	vertex set of G	27
E, $E(G)$	edge set of G	27
$N(v)$	(open) neighborhood of v	28
$N[v]$	closed neighborhood of v	28
$[A, B]$	set of edges between A and B	28
$\deg v$	degree of a vertex v	28
$\deg_G v$	degree of a vertex v in G	28
$\Delta(G)$	maximum degree of G	28
$\delta(G)$	minimum degree of G	28
$G[S]$	vertex-induced subgraph of G	29
$G[X]$	edge-induced subgraph of G	29
$G - v$, $G - U$	deleting vertices from G	30
$G - e$, $G - X$	deleting edges from G	30
$G + e$	adding edge e to G	30
$k(G)$	number of components of G	33
$\sigma_2(G)$	minimum degree sum nonadjacent vertices	33
$d(u, v)$	distance between u and v	34
$e(v)$	eccentricity of a vertex v	35
$\mathrm{diam}(G)$	diameter of G	35
$\mathrm{rad}(G)$	radius of G	35
$\mathrm{Cen}(G)$	center of G	35
$\mathrm{Per}(G)$	periphery of G	35
$D(u, v)$	detour distance between u and v	37
$G \cong H$	G is isomorphic to H	37
C_n	cycle of order n	39
P_n	path of order n	39
K_n	complete graph of order n	39
$\mathrm{cir}(G)$	circumference of G	40
$K_{s,t}$	complete bipartite graph	41
$K_{n_1,n_2,\cdots,n_k}$	complete k-partite graph	42
$\overline{G}$	complement of G	42
$G + H$	union of G and H	42
$G \vee H$	join of G and H	42
W_n	$C_n \vee K_1$, wheel of order $n + 1$	42
$G_1 + G_2 + \cdots + G_k$	union of G_i $(1 \le i \le k)$	43
$G_1 \vee G_2 \vee \cdots \vee G_k$	join of G_i $(1 \le i \le k)$	43
$G \,\square\, H$	Cartesian product of G and H	43
$P_s \,\square\, P_t$	grid	44
Q_n	n-cube	44
$L(G)$	line graph of G	44
$\mathrm{od}\, v$	outdegree of v	46
$\mathrm{id}\, v$	indegree of v	46

Chapter 0

The Origin of Graph Colorings

If the countries in a map of South America (see Figure 1) were to be colored in such a way that every two countries with a common boundary are colored differently, then this map could be colored using only four colors. Is this true of every map?

While it is not difficult to color a map of South America with four colors, it is not possible to color this map with less than four colors. In fact, every two of Brazil, Argentina, Bolivia, and Paraguay are neighboring countries and so four colors are required to color only these four countries.

It is probably clear why we might want two countries colored differently if they have a common boundary – so they can easily be distinguished as different countries in the map. It may not be clear, however, why we would think that four colors would be enough to color the countries of every map. After all, we can probably envision a complicated map having a large number of countries with some countries having several neighboring countries, so constructed that a great many colors might possibly be needed to color the entire map. Here we understand neighboring countries to mean two countries with a boundary *line* in common, not simply a single point in common.

While this problem may seem nothing more than a curiosity, it is precisely this problem that would prove to intrigue so many for so long and whose attempted solutions would contribute so significantly to the development of the area of mathematics known as Graph Theory and especially to the subject of graph colorings: Chromatic Graph Theory. This map coloring problem would eventually acquire a name that would become known throughout the mathematical world.

The Four Color Problem *Can the countries of every map be colored with four or fewer colors so that every two countries with a common boundary are colored differently?*

Figure 1: Map of South America

Many of the concepts, theorems, and problems of Graph Theory lie in the shadows of the Four Color Problem. Indeed ...

Graph Theory is an area of mathematics whose past is always present.

Since the maps we consider can be real or imagined, we can think of maps being divided into more general regions, rather than countries, states, provinces, or some other geographic entities.

So just how did the Four Color Problem come about? It turns out that this question has a rather well-documented answer. On 23 October 1852, a student, namely Frederick Guthrie (1833–1886), at University College London visited his mathematics professor, the famous Augustus De Morgan (1806–1871), to describe an apparent mathematical discovery of his older brother Francis. While coloring the counties of a map of England, Francis Guthrie (1831–1899) observed that he could color them with four colors, which led him to conjecture that no more than four colors would be needed to color the regions of any map.

The Four Color Conjecture *The regions of every map can be colored with four or fewer colors in such a way that every two regions sharing a common boundary are colored differently.*

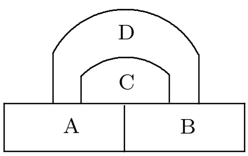
Two years earlier, in 1850, Francis had earned a Bachelor of Arts degree from University College London and then a Bachelor of Laws degree in 1852. He would later become a mathematics professor himself at the University of Cape Town in South Africa. Francis developed a lifelong interest in botany and his extensive collection of flora from the Cape Peninsula would later be placed in the Guthrie Herbarium in the University of Cape Town Botany Department. Several rare species of flora are named for him.

Francis Guthrie attempted to prove the Four Color Conjecture and although he thought he may have been successful, he was not completely satisfied with his proof. Francis discussed his discovery with Frederick. With Francis's approval, Frederick mentioned the statement of this apparent theorem to Professor De Morgan, who expressed pleasure with it and believed it to be a new result. Evidently Frederick asked Professor De Morgan if he was aware of an argument that would establish the truth of the theorem.

This led De Morgan to write a letter to his friend, the famous Irish mathematician Sir William Rowan Hamilton (1805–1865), on 23 October 1852. These two mathematical giants had corresponded for years, although apparently had met only once. De Morgan wrote (in part):

> *My dear Hamilton:*
>
> *A student of mine asked me to day to give him a reason for a fact which I did not know was a fact – and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary lines are differently coloured – four colours may be wanted but not more – the following is his case in which four are wanted.*

A B C D are names of colours



> *Query cannot a necessity for five or more be invented . . .*
>
> *My pupil says he guessed it colouring a map of England . . . . The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did . . .*

In De Morgan's letter to Hamilton, he refers to the "Sphynx" (or Sphinx). While the Sphinx is a male statue of a lion with the head of a human in ancient Egypt which guards the entrance to a temple, the Greek Sphinx is a female creature of bad luck who sat atop a rock posing the following riddle to all those who pass by:

> *What animal is that which in the morning goes on four feet, at noon on two, and in the evening upon three?*

Those who did not solve the riddle were killed. Only Oedipus (the title character in *Oedipus Rex* by Sophocles, a play about how people do not control their own destiny) answered the riddle correctly as "Man", who in childhood (the morning of life) creeps on hands and knees, in manhood (the noon of life) walks upright, and in old age (the evening of life) walks with the aid of a cane. Upon learning that her riddle had been solved, the Sphinx cast herself from the rock and perished, a fate De Morgan had envisioned for himself if his riddle (the Four Color Problem) had an easy and immediate solution.

In De Morgan's letter to Hamilton, De Morgan attempted to explain why the problem appeared to be difficult. He followed this explanation by writing:

> *But it is tricky work and I am not sure of all convolutions – What do you say? And has it, if true been noticed?*

Among Hamilton's numerous mathematical accomplishments was his remarkable work with quaternions. Hamilton's quaternions are a 4-dimensional system of numbers of the form $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = k^2 = -1$. When $c = d = 0$, these numbers are the 2-dimensional system of complex numbers; while when $b = c = d = 0$, these numbers are simply real numbers. Although it is commonplace for binary operations in algebraic structures to be commutative, such is not the case for products of quaternions. For example, $i \cdot j = k$ but $j \cdot i = -k$. Since De Morgan had shown an interest in Hamilton's research on quaternions as well as other subjects Hamilton had studied, it is likely that De Morgan expected an enthusiastic reply to his letter to Hamilton. Such was not the case, however. Indeed, three days later, on 26 October 1852, Hamilton gave an unexpected response:

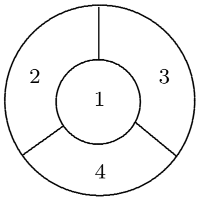
> *I am not likely to attempt your "quaternion" of colours very soon.*

Hamilton's response did nothing however to diminish De Morgan's interest in the Four Color Problem.

Since De Morgan's letter to Hamilton did not mention Frederick Guthrie by name, there may be reason to question whether Frederick was in fact the student to whom De Morgan was referring and that it was Frederick's older brother Francis who was the originator of the Four Color Problem.

In 1852 Frederick Guthrie was a teenager. He would go on to become a distinguished physics professor and founder of the Physical Society in London. An area that he studied was the science of thermionic emission – first reported by Frederick Guthrie in 1873. He discovered that a red-hot iron sphere with a positive charge would lose its charge. This effect was rediscovered by the famous American inventor Thomas Edison early in 1880. It was during 1880 (only six years before Frederick died) that Frederick wrote:

> *Some thirty years ago, when I was attending Professor De Morgan's class, my brother, Francis Guthrie, who had recently ceased to attend them* (*and who is now professor of mathematics at the South African University, Cape Town*)*, showed me the fact that the greatest necessary number of colours to be used in colouring a map so as to avoid identity colour in lineally contiguous districts is four. I should not be justified, after this lapse of time, in trying to give his proof, but the critical diagram was as in the margin.*



> *With my brother's permission I submitted the theorem to Professor De Morgan, who expressed himself very pleased with it; accepted it as new; and, as I am informed by those who subsequently attended his classes, was in the habit of acknowledging where he had got his information.*
>
> *If I remember rightly, the proof which my brother gave did not seem altogether satisfactory to himself; but I must refer to him those interested in the subject. . . .*

The first statement in print of the Four Color Problem evidently occurred in an anonymous review written in the 14 April 1860 issue of the literary journal *Athenaeum*. Although the author of the review was not identified, De Morgan was quite clearly the writer. This review led to the Four Color Problem becoming known in the United States.

The Four Color Problem came to the attention of the American mathematician Charles Sanders Peirce (1839–1914), who found an example of a map drawn on a torus (a donut-shaped surface) that required six colors. (As we will see in Chapter 5, there is an example of a map drawn on a torus that requires seven colors.) Peirce expressed great interest in the Four Color Problem. In fact, he visited De Morgan in 1870, who by that time was experiencing poor health. Indeed, De Morgan died the following year. Not only had De Morgan made little progress towards a solution of the Four Color Problem at the time of his death, overall interest in this problem had faded. While Peirce continued to attempt to solve the problem, De Morgan's British acquaintances appeared to pay little attention to the problem – with at least one notable exception.

Arthur Cayley (1821–1895) graduated from Trinity College, Cambridge in 1842 and then received a fellowship from Cambridge, where he taught for four years. Afterwards, because of the limitations on his fellowship, he was required to choose a profession. He chose law, but only as a means to make money while he could continue to do mathematics. During 1849–1863, Cayley was a successful lawyer but published some 250 research papers during this period, including many for which he is well known. One of these was his pioneering paper on matrix algebra. Cayley was famous for his work on algebra, much of which was done with the British mathematician James Joseph Sylvester (1814–1897), a former student of De Morgan.

In 1863 Cayley was appointed a professor of mathematics at Cambridge. Two years later, the London Mathematical Society was founded at University College London and would serve as a model for the American Mathematical Society, founded in 1888. De Morgan became the first president of the London Mathematical Society, followed by Sylvester and then Cayley. During a meeting of the Society on 13 June 1878, Cayley raised a question about the Four Color Problem that brought renewed attention to the problem:

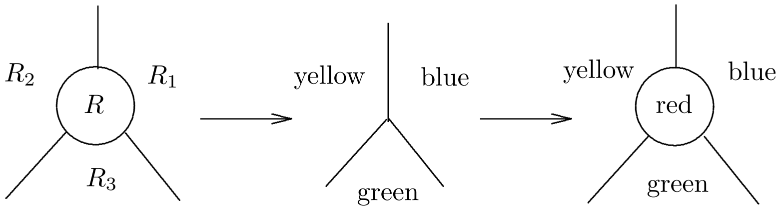
> *Has a solution been given of the statement that in colouring a map of a country, divided into counties, only four distinct colours are required, so that no two adjacent counties should be painted in the same colour?*

This question appeared in the Proceedings of the Society's meeting. In the April 1879 issue of the *Proceedings of the Royal Geographical Society*, Cayley reported:

> *I have not succeeded in obtaining a general proof; and it is worth while to explain wherein the difficulty consists.*

Cayley observed that if a map with a certain number of regions has been colored with four colors and a new map is obtained by adding a new region, then there is no guarantee that the new map can be colored with four colors – without first recoloring the original map. This showed that any attempted proof of the Four Color Conjecture using a proof by mathematical induction would not be straightforward. Another possible proof technique to try would be proof by contradiction. Applying this technique, we would assume that the Four Color Conjecture is false. This would mean that there are some maps that cannot be colored with four colors. Among the maps that require five or more colors are those with a smallest number of regions. Any one of these maps constitutes a minimum counterexample. If it could be shown that no minimum counterexample could exist, then this would establish the truth of the Four Color Conjecture.

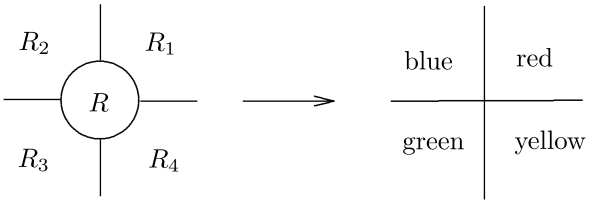
For example, no minimum counterexample M could possibly contain a region R surrounded by three regions R_1, R_2, and R_3 as shown in Figure 2(a). In this case, we could shrink the region R to a point, producing a new map M' with one less region. The map M' can then be colored with four colors, only three of which are used to color R_1, R_2, and R_3 as in Figure 2(b). Returning to the original map M, we see that there is now an available color for R as shown in Figure 2(c), implying that M could be colored with four colors after all, thereby producing a contradiction. Certainly, if the map M contains a region surrounded by fewer than three regions, a contradiction can be obtained in the same manner.



(a) in M (b) in M' (c) in M

Figure 2: A region surrounded by three regions in a map

Suppose, however, that the map M contained no region surrounded by three or fewer regions but did contain a region R surrounded by four regions, say R_1, R_2, R_3, R_4, as shown in Figure 3(a). If, once again, we shrink the region R to a point, producing a map M' with one less region, then we know that M' can be colored with four colors. If two or three colors are used to color R_1, R_2, R_3, R_4, then we can return to M and there is a color available for R. However, this technique does not work if the regions R_1, R_2, R_3, R_4 are colored with four distinct colors, as shown in Figure 3(b).



(a) in M (b) in M'

Figure 3: A region surrounded by four regions in a map

What we can do in this case, however, is to determine whether the map M' has a chain of regions, beginning at R_1 and ending at R_3, all of which are colored red or green. If no such chain exists, then the two colors of every red-green chain of regions beginning at R_1 can be interchanged. We can then return to the map M, where the color red is now available for R. That is, the map M can be colored with four colors, producing a contradiction. But what if a red-green chain of regions beginning at R_1 and ending at R_3 exists? (See Figure 4, where r, b, g, y denote the colors red, blue, green, yellow.) Then interchanging the colors red and green offers no benefit to us. However, in this case, there can be no blue-yellow chain of regions, beginning at R_2 and ending at R_4. Then the colors of every blue-yellow chain of regions beginning at R_2 can be interchanged. Returning to M, we see that the color blue is now available for R, which once again says that M can be colored with four colors and produces a contradiction.

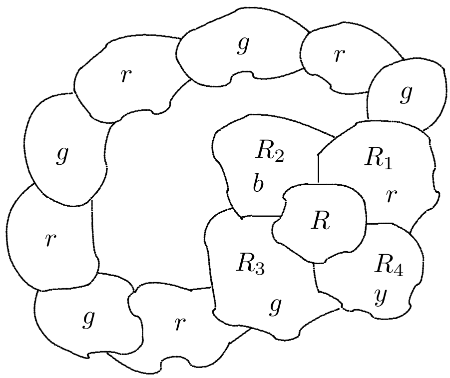


Figure 4: A red-green chain of regions from R_1 to R_3

It is possible to show (as we will see in Chapter 5) that a map may contain no region that is surrounded by four or fewer neighboring regions. Should this occur however, such a map must contain a region surrounded by exactly five neighboring regions.

We mentioned that James Joseph Sylvester worked with Arthur Cayley and served as the second president of the London Mathematical Society. Sylvester, a superb mathematician himself, was invited to join the mathematics faculty of the newly founded Johns Hopkins University in Baltimore, Maryland in 1875. Included among his attempts to inspire more research at the university was his founding in 1878 of the *American Journal of Mathematics*, of which he held the position of editor-in-chief. While the goal of the journal was to serve American mathematicians, foreign submissions were encouraged as well, including articles from Sylvester's friend Cayley.

Among those who studied under Arthur Cayley was Alfred Bray Kempe (1849–1922). Despite his great enthusiasm for mathematics, Kempe took up a career in the legal profession. Kempe was present at the meeting of the London Mathematical Society in which Cayley had inquired about the status of the Four Color Problem. Kempe worked on the problem and obtained a solution in 1879. Indeed, on 17 July 1879 a statement of Kempe's accomplishment appeared in the British journal *Nature*, with the complete proof published in Volume 2 of Sylvester's *American Journal of Mathematics*.

Kempe's approach for solving the Four Color Problem essentially followed the technique described earlier. His technique involved locating a region R in a map M such that R is surrounded by five or fewer neighboring regions and showing that for every coloring of M (minus the region R) with four colors, there is a coloring of the entire map M with four colors. Such an argument would show that M could not be a minimum counterexample. We saw how such a proof would proceed if R were surrounded by four or fewer neighboring regions. This included looking for chains of regions whose colors alternate between two colors and then interchanging these colors, if appropriate, to arrive at a coloring of the regions of M (minus R) with four colors so that the neighboring regions of R used at most three of these colors and thereby leaving a color available for R. In fact, these chains of regions became known as *Kempe chains*, for it was Kempe who originated this idea.

There was one case, however, that still needed to be resolved, namely the case where no region in the map was surrounded by four or fewer neighboring regions. As we noted, the map must then contain some region R surrounded by exactly five neighboring regions. At least three of the four colors must be used to color the five neighboring regions of R. If only three colors are used to color these five regions, then a color is available for R. Hence, we are left with the single situation in which all four colors are used to color the five neighboring regions surrounding R (see Figure 5), where once again r, b, g, y indicate the colors red, blue, green, yellow.

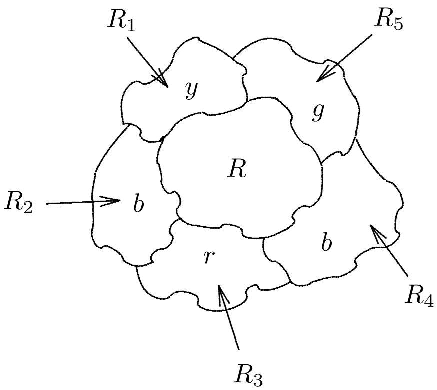


Figure 5: The final case in Kempe's solution of the Four Color Problem

Let's see how Kempe handled this final case. Among the regions adjacent to R, only the region R_1 is colored yellow. Consider all the regions of the map M that are colored either yellow or red and that, beginning at R_1, can be reached by an alternating sequence of neighboring yellow and red regions, that is, by a yellow-red Kempe chain. If the region R_3 (which is the neighboring region of R colored red) cannot be reached by a yellow-red Kempe chain, then the colors yellow and red can be interchanged for all regions in M that can be reached by a yellow-red Kempe chain beginning at R_1. This results in a coloring of all regions in M (except R) in which neighboring regions are colored differently and such that each neighboring region of R is colored red, blue, or green. We can then color R yellow to arrive at a 4-coloring of the entire map M. From this, we may assume that the region R_3 can be reached by a yellow-red Kempe chain beginning at R_1. (See Figure 6.)

Let's now look at the region R_5, which is colored green. We consider all regions of M colored green or red and that, beginning at R_5, can be reached by a green-red Kempe chain. If the region R_3 cannot be reached by a green-red Kempe chain that begins at R_5, then the colors green and red can be interchanged for all regions in M

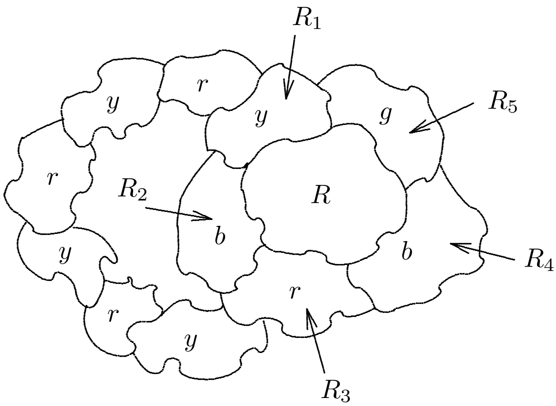


Figure 6: A yellow-red Kempe chain in the map M

that can be reached by a green-red Kempe chain beginning at R_5. Upon doing this, a 4-coloring of all regions in M (except R) is obtained, in which each neighboring region of R is colored red, blue, or yellow. We can then color R green to produce a 4-coloring of the entire map M. We may therefore assume that R_3 can be reached by a green-red Kempe chain that begins at R_5. (See Figure 7.)

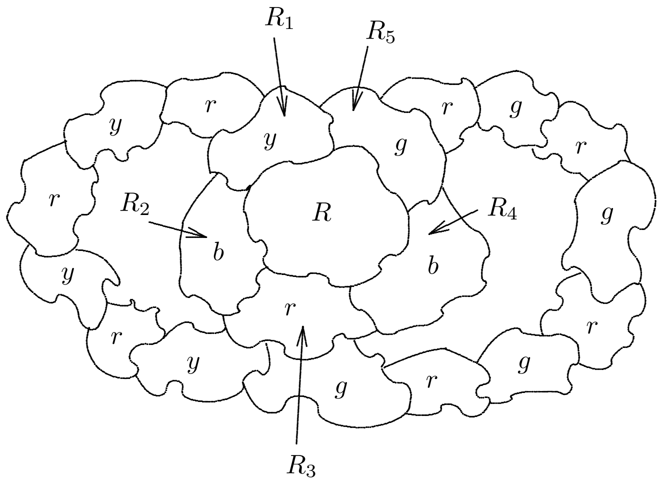


Figure 7: Yellow-red and green-red Kempe chains in the map M

Because there is a ring of regions consisting of R and a green-red Kempe chain, there cannot be a blue-yellow Kempe chain in M beginning at R_4 and ending at R_1. In addition, because there is a ring of regions consisting of R and a yellow-red Kempe chain, there is no blue-green Kempe chain in M beginning at R_2 and ending at R_5. Hence, we interchange the colors blue and yellow for all regions in M that can be reached by a blue-yellow Kempe chain beginning at R_4 and interchange the colors blue and green for all regions in M that can be reached by a blue-green Kempe chain beginning at R_2. Once these two color interchanges have been performed, each of the five neighboring regions of R is colored red, yellow, or green. Then R can be colored blue and a 4-coloring of the map M has been obtained, completing the proof.

As it turned out, the proof given by Kempe contained a fatal flaw, but one that would go unnoticed for a decade. Despite the fact that Kempe's attempted proof of the Four Color Problem was erroneous, he made a number of interesting observations in his article. He noticed that if a piece of tracing paper was placed over a map and a point was marked on the tracing paper over each region of the map and two points were joined by a line segment whenever the corresponding regions had a common boundary, then a diagram of a "linkage" was produced. Furthermore, the problem of determining whether the regions of the map can be colored with four colors so that neighboring regions are colored differently is the same problem as determining whether the points in the linkage can be colored with four colors so that every two points joined by a line segment are colored differently. (See Figure 8.)

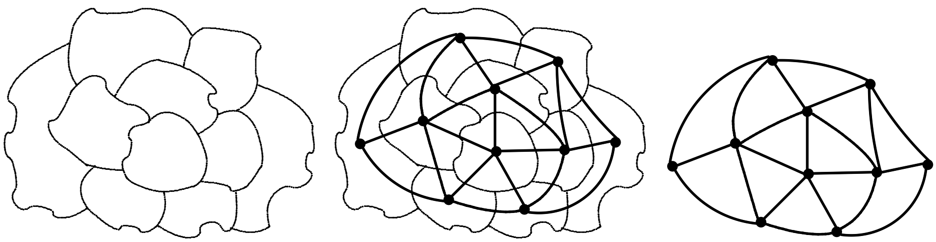


Figure 8: A map and corresponding planar graph

In 1878 Sylvester referred to a linkage as a graph and it is this terminology that became accepted. Later it became commonplace to refer to the points and lines of a linkage as the vertices and edges of the graph (with "vertex" being the singular of "vertices"). Since the graphs constructed from maps in this manner (referred to as the dual graph of the map) can themselves be drawn in the plane without two edges (line segments) intersecting, these graphs were called *planar graphs.* A planar graph that is actually drawn in the plane without any of its edges intersecting is called a *plane graph.* In terms of graphs, the Four Color Conjecture could then be restated.

The Four Color Conjecture *The vertices of every planar graph can be colored with four or fewer colors in such a way that every two vertices joined by an edge are colored differently.*

Indeed, the vast majority of this book will be devoted to coloring graphs (not coloring maps) and, in fact, to coloring graphs in general, not only planar graphs.

> *The colouring of abstract graphs is a generalization of the colouring of maps, and the study of the colouring of abstract graphs ... opens a new chapter in the combinatorial part of mathematics.*
>
> Gabriel Andrew Dirac (1951)

For the present, however, we continue our discussion in terms of coloring the regions of maps.

Kempe's proof of the theorem, which had become known as the Four Color Theorem, was accepted both within the United States and England. Arthur Cayley had accepted Kempe's argument as a valid proof. This led to Kempe being elected as a Fellow of the Royal Society in 1881.

The Four Color Theorem *The regions of every map can be colored with four or fewer colors so that every two adjacent regions are colored differently.*

Among the many individuals who had become interested in the Four Color Problem was Charles Lutwidge Dodgson (1832–1898), an Englishman with a keen interest in mathematics and puzzles. Dodgson was better known, however, under his pen-name Lewis Carroll and for his well-known books *Alice's Adventures in Wonderland* and *Through the Looking-Glass and What Alice Found There.*

Another well-known individual with mathematical interests, but whose primary occupation was not that of a mathematician, was Frederick Temple (1821–1902), Bishop of London and who would later become the Archbishop of Canterbury. Like Dodgson and others, Temple had a fondness for puzzles. Temple showed that it was impossible to have five mutually neighboring regions in any map and from this concluded that no map required five colors. Although Temple was correct about the non-existence of five mutually neighboring regions in a map, his conclusion that this provided a proof of the Four Color Conjecture was incorrect.

There was historical precedence about the non-existence of five mutually adjacent regions in any map. In 1840 the famous German mathematician August Möbius (1790–1868) reportedly stated the following problem, which was proposed to him by the philologist Benjamin Weiske (1748–1809).

> **Problem of Five Princes**
>
> *There was once a king with five sons. In his will, he stated that after his death his kingdom should be divided into five regions in such a way that each region should have a common boundary with the other four. Can the terms of the will be satisfied?*

As we noted, the conditions of the king's will cannot be met. This problem illustrates Möbius's interest in topology, a subject of which Möbius was one of the early pioneers. In a memoir written by Möbius and only discovered after his death, he discussed properties of one-sided surfaces, which became known as Möbius strips (even though it was determined that Johann Listing (1808–1882) had discovered these earlier).

In 1885 the German geometer Richard Baltzer (1818–1887) also lectured on the non-existence of five mutually adjacent regions. In the published version of his lecture, it was incorrectly stated that the Four Color Theorem followed from this. This error was repeated by other writers until the famous geometer Harold Scott MacDonald Coxeter (1907–2003) corrected the matter in 1959.

Mistakes concerning the Four Color Problem were not limited to mathematical errors however. Prior to establishing Francis Guthrie as the true and sole originator of the Four Color Problem, it was often stated in print that cartographers were aware that the regions of every map could be colored with four or less colors so that adjacent regions are colored differently. The well-known mathematical historian Kenneth O. May (1915–1977) investigated this claim and found no justification to it. He conducted a study of atlases in the Library of Congress and found no evidence of attempts to minimize the number of colors used in maps. Most maps used more than four colors and even when four colors were used, often less colors could have been used. There was never a mention of a "four color theorem".

Another mathematician of note around 1880 was Peter Guthrie Tait (1831–1901). In addition to being a scholar, he was a golf enthusiast. His son Frederick Guthrie Tait was a champion golfer and considered a national hero in Scotland. The first golf biography ever written was about Frederick Tait. Indeed, the Freddie Tait Golf Week is held every year in Kimberley, South Africa to commemorate his life as a golfer and soldier. He was killed during the Anglo-Boer War of 1899–1902.

Peter Guthrie Tait had heard of the Four Color Conjecture through Arthur Cayley and was aware of Kempe's solution. He felt that Kempe's solution of the Four Color Problem was overly long and gave several shorter solutions of the problem, all of which turned out to be incorrect. Despite this, one of his attempted proofs contained an interesting and useful idea. A type of map that is often encountered is a *cubic map*, in which there are exactly three boundary lines at each meeting point. In fact, every map M that has no region completely surrounded by another region can be converted into a cubic map M' by drawing a circle about each meeting point in M' and creating new meeting points and one new region (see Figure 9). If the map M' can be colored with four colors, then so can M.

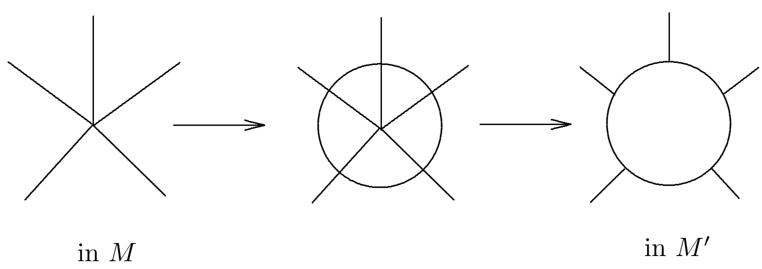


Figure 9: Converting a map into a cubic map

Tait's idea was to consider coloring the boundary lines of cubic maps. In fact, he stated as a lemma that:

The boundary lines of every cubic map can always be colored with three colors so that the three lines at each meeting point are colored differently.

Tait also mentioned that this lemma could be easily proved and showed how the lemma could be used to prove the Four Color Theorem. Although Tait was correct that this lemma could be used to prove the Four Color Theorem, he was incorrect when he said that the lemma could be easily proved. Indeed, as it turned out, this lemma is equivalent to the Four Color Theorem and, of course, is equally difficult to prove. (We will discuss Tait's coloring of the boundary lines of cubic maps in Chapter 10.)

The next important figure in the history of the Four Color Problem was Percy John Heawood (1861–1955), who spent the period 1887–1939 as a lecturer, professor, and vice-chancellor at Durham College in England. When Heawood was a student at Oxford University in 1880, one of his teachers was Professor Henry Smith who spoke often of the Four Color Problem. Heawood read Kempe's paper and it was he who discovered the serious error in the proof. In 1889 Heawood wrote a paper of his own, published in 1890, in which he presented the map shown in Figure 10.

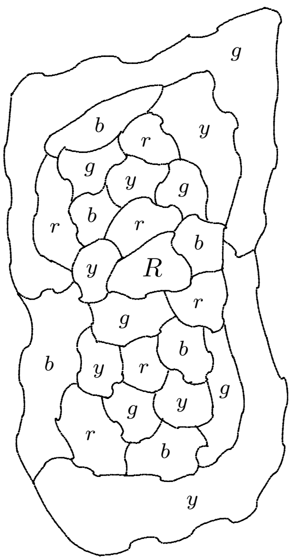


Figure 10: Heawood's counterexample to Kempe's proof

In the Heawood map, two of the five neighboring regions surrounding the uncolored region R are colored red; while for each of the colors blue, yellow, and green, there is exactly one neighboring region of R with that color. According to Kempe's argument, since blue is the color of the region that shares a boundary with R as well as with the two neighboring regions of R colored red, we are concerned with whether this map contains a blue-yellow Kempe chain between two neighboring regions of R as well as a blue-green Kempe chain between two neighboring regions of

R. It does. These Kempe chains are shown in Figures 11(a) and 11(b), respectively.

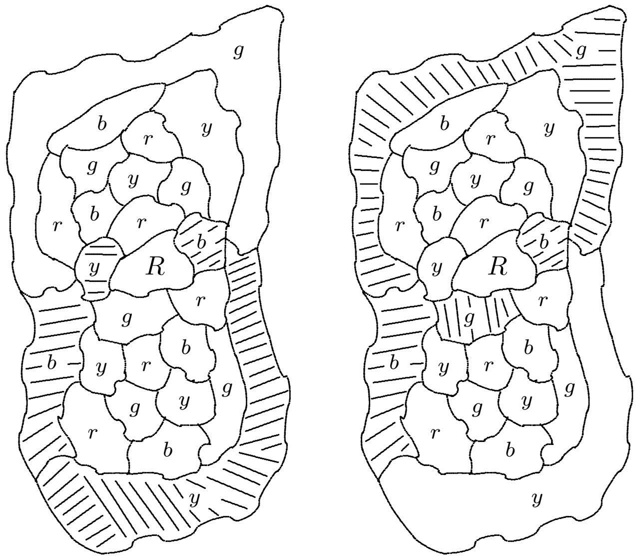
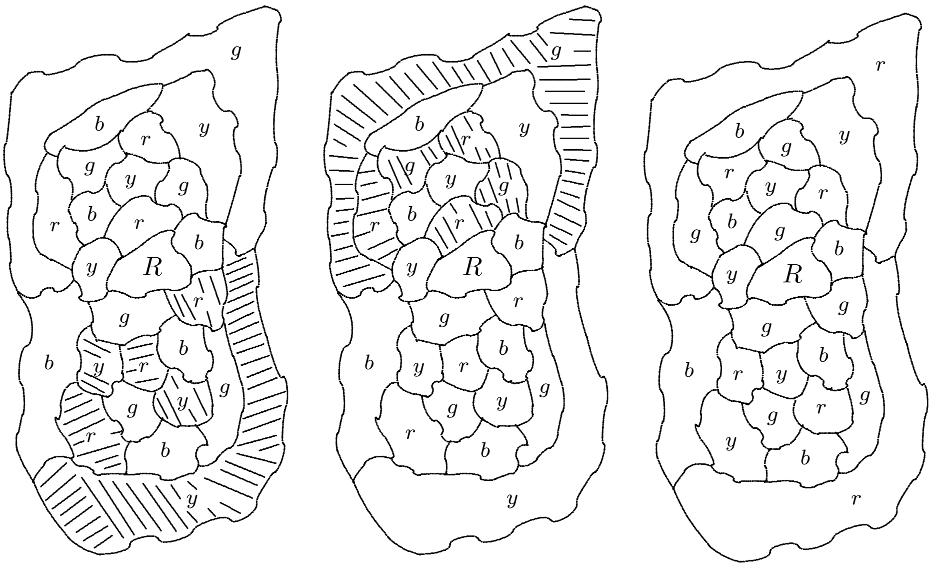


Figure 11: Blue-yellow and blue-green Kempe chains in the Heawood map

Because the Heawood map contains these two Kempe chains, it follows by Kempe's proof that this map does not contain a red-yellow Kempe chain between the two neighboring regions of R that are colored red and yellow *and* does not contain a red-green Kempe chain between the two neighboring regions of R that are colored red and green. This is, in fact, the case. Figure 12(a) indicates all regions that can be reached by a red-yellow Kempe chain beginning at the red region that borders R and that is not adjacent to the yellow region bordering R. Furthermore, Figure 12(b) indicates all regions that can be reached by a red-green Kempe chain beginning at the red region that borders R and that is not adjacent to the green region bordering R.

In the final step of Kempe's proof, the two colors within each Kempe chain are interchanged resulting in a coloring of the Heawood map with four colors. This double interchange of colors is shown in Figure 12(c). However, as Figure 12(c) shows, this results in neighboring regions with the same color. Consequently, Kempe's proof is unsuccessful when applied to the Heawood map, as colored in Figure 10. What Heawood had shown was that Kempe's method of proof was incorrect. That is, Heawood had discovered a counterexample to Kempe's technique, not to the Four Color Conjecture itself. Indeed, it is not particularly difficult to give a 4-coloring of the regions of the Heawood map so that every two neighboring regions are colored differently.

Other counterexamples to Kempe's proof were found after the publication of Heawood's 1890 paper, including a rather simple example (see Figure 13) given in



(a) (b) (c)

Figure 12: Steps in illustrating Kempe's technique

1921 by Alfred Errera (1886–1960), a student of Edmund Landau, well known for his work in analytic number theory and the distribution of primes.

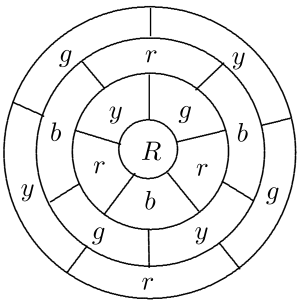


Figure 13: The Errera example

In addition to the counterexample to Kempe's proof, Heawood's paper contained several interesting results, observations, and comments. For example, although Kempe's attempted proof of the Four Color Theorem was incorrect, Heawood was able to use this approach to show that the regions of every map could be colored with five or fewer colors so that neighboring regions were colored differently (see Chapter 8).

Heawood also considered the problem of coloring maps that can be drawn on other surfaces. Maps that can be drawn in the plane are precisely those maps that can be drawn on the surface of a sphere. There are considerably more complex surfaces on which maps can be drawn, however. In particular, Heawood proved that the regions of every map drawn on the surface of a torus can be colored with seven or fewer colors and that there is, in fact, a map on the torus that requires seven colors (see Chapter 8). More generally, Heawood showed that the regions of every map drawn on a pretzel-shaped surface consisting of a sphere with k holes ($k > 0$) can be colored with $\left\lfloor \frac{7+\sqrt{1+48k}}{2} \right\rfloor$ colors. In addition, he stated that such maps requiring this number of colors exist. He never proved this latter statement, however. In fact, it would take another 78 years to verify this statement (see Chapter 8).

Thus, the origin of a curious problem by the young scholar Francis Guthrie was followed over a quarter of a century later by what was thought to be a solution to the problem by Alfred Bray Kempe. However, we were to learn from Percy John Heawood a decade later that the solution was erroneous, which returned the problem to its prior status. Well not quite – as these events proved to be stepping stones along the path to chromatic graph theory.

Is it five? Is it four?
Heawood rephrased the query.
Sending us back to before,
But moving forward a theory.

At the beginning of the 20th century, the Four Color Problem was still unsolved. Although possibly seen initially as a rather frivolous problem, not worthy of a serious mathematician's attention, it would become clear that the Four Color Problem was a very challenging mathematics problem. Many mathematicians, using a variety of approaches, would attack this problem during the 1900s. As noted, it was known that if the Four Color Conjecture could be verified for cubic maps, then the Four Color Conjecture would be true for all maps. Furthermore, every cubic map must contain a region surrounded by two, three, four, or five neighboring regions. These four kinds of configurations (arrangements of regions) were called *unavoidable* because every cubic map had to contain at least one of them. Thus, the arrangements of regions shown in Figure 14 make up an unavoidable set of configurations.

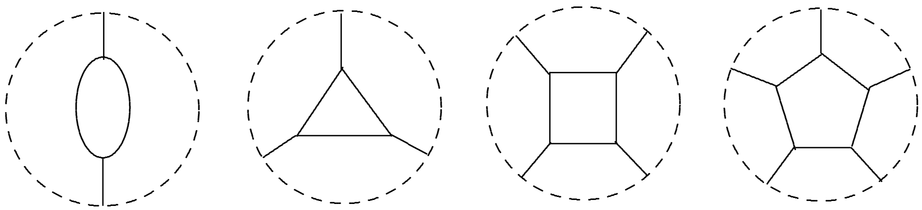


Figure 14: An unavoidable set of configurations in a cubic map

A region surrounded by k neighboring regions is called a *k-gon*. It is possible to show that any map that contains no k-gon where $k < 5$ must contain at least twelve pentagons (5-gons). In fact, there is a map containing exactly twelve regions, each of which is a pentagon. Such a map is shown in Figure 15, where one of the regions is the "exterior region". Since this map can be colored with four colors, any counterexample to the Four Color Conjecture must contain at least thirteen regions. Alfred Errera proved that no counterexample could consist only of pentagons and hexagons (6-gons).

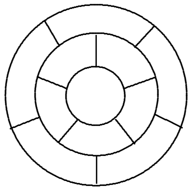


Figure 15: A cubic map with twelve pentagons

A *reducible configuration* is any configuration of regions that cannot occur in a minimum counterexample of the Four Color Conjecture. Many mathematicians who attempted to solve the Four Color Problem attempted to do so by trying to find an unavoidable set S of reducible configurations. Since S is unavoidable, this means that every cubic map must contain at least one configuration in S. Because each configuration in S is reducible, this means that it cannot occur in a minimum counterexample. Essentially then, a proof of the Four Color Conjecture by this approach would be a proof by minimum counterexample resulting in a number of cases (one case for each configuration in the unavoidable set S) where each case leads to a contradiction (that is, each configuration is shown to be reducible).

Since the only configuration in the unavoidable set shown in Figure 14 that could not be shown to be reducible was the pentagon, this suggested searching for more complex configurations that must also be part of an unavoidable set with the hope that these more complicated configurations could somehow be shown to be reducible. For example, in 1903 Paul Wernicke proved that every cubic map containing no k-gon where $k < 5$ must either contain two adjacent pentagons or two adjacent regions, one of which is a pentagon and the other a hexagon (see Chapter 5). That is, the troublesome case of a cubic map containing a pentagon could be eliminated and replaced by two different cases.

Finding new, large unavoidable sets of configurations was not a problem. Finding reducible configurations was. In 1913 the distinguished mathematician George David Birkhoff (1884–1944) published a paper called *The reducibility of maps* in which he considered rings of regions for which there were regions interior to as well as exterior to the ring. Since the map was a minimum counterexample, the ring together with the interior regions and the ring together with the exterior regions could both be colored with four colors. If two 4-colorings could be chosen so that they match along the ring, then there is a 4-coloring of the entire map. Since this can always be done if the ring consists of three regions, rings of three regions can never appear in a minimum counterexample. Birkhoff proved that rings of four regions also cannot appear in a minimum counterexample. In addition, he was successful in proving that rings of five regions cannot appear in a minimum counterexample either – unless the interior of the region consisted of a single region. This generalized Kempe's approach. While Kempe's approach to solving the Four Color Problem involved the removal of a single region from a map, Birkhoff's method allowed the removal of regions inside or outside some ring of regions. For example, a configuration that Birkhoff was able to prove was reducible consisted of a ring of six pentagons enclosing four pentagons. This became known as the *Birkhoff diamond* (see Figure 16).

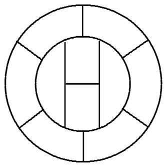


Figure 16: The Birkhoff diamond (a reducible configuration)

Philip Franklin (1898–1965) wrote his doctoral dissertation in 1921 titled *The Four Color Problem* under the direction of Oswald Veblen (1880–1960). Veblen was the first professor at the Institute for Advanced Study at Princeton University. He was well known for his work in geometry and topology (called analysis situs at the time) as well as for his lucid writing. In his thesis, Franklin showed that if a cubic map does not contain a k-gon, where $k < 5$, then it must contain a pentagon adjacent to two other regions, each of which is a pentagon or a hexagon (see Chapter 5). This resulted in a larger unavoidable set of configurations.

In 1922 Franklin showed that every map with 25 or fewer regions could be colored with four or fewer colors. This number gradually worked its way up to 96 in a result established in 1975 by Jean Mayer, curiously a professor of French literature.

Favorable impressions of new areas of mathematics clearly did not occur quickly. Geometry of course had been a prominent area of study in mathematics for centuries. The origins of topology may only go back to 19th century however. In his 1927 survey paper about the Four Color Problem, Alfred Errera reported that some mathematicians referred to topology as the "geometry of drunkards". Graph theory belongs to the more general area of combinatorics. While combinatorial arguments can be found in all areas of mathematics, there was little recognition of combinatorics as a major area of mathematics until later in the 20th century, at which time topology was gaining in prominence. Indeed, John Henry Constantine Whitehead (1904–1960), one of the founders of homotopy theory in topology, reportedly said that "Combinatorics is the slums of topology." However, by the latter part of the 20th century, combinatorics had come into its own. The famous mathematician Israil Moiseevich Gelfand (1913–2009) stated (in 1990):

> *The older I get, the more I believe that at the bottom of most deep mathematical problems there is a combinatorial problem.*

Heinrich Heesch (1906–1995) was a German mathematician who was an assistant to Hermann Weyl, a gifted mathematician who was a colleague of Albert Einstein (1879–1955) and a student of the famous mathematician David Hilbert (1862–1943), whom he replaced as mathematics chair at the University of Göttingen. In 1900, Hilbert gave a lecture before the International Congress of Mathematicians in Paris in which he presented 23 extremely challenging problems. In 1935, Heesch solved one of these problems (Problem 18) dealing with tilings of the plane. One of Heesch's friends at Göttingen was Ernst Witt (1911–1991), who thought he had solved an even more famous problem: the Four Color Problem. Witt was anxious to show his proof to the famous German mathematician Richard Courant (1888–1972), who later moved to the United States and founded the Courant Institute of Mathematical Sciences. Since Courant was in the process of leaving Göttingen for Berlin, Heesch joined Witt to travel with Courant by train in order to describe the proof. However, Courant was not convinced and the disappointed young mathematicians returned to Göttingen. On their return trip, however, Heesch discovered an error in Witt's proof. Heesch too had become captivated by the Four Color Problem.

As Heesch studied this famous problem, he had become increasingly convinced that the problem could be solved by finding an unavoidable set of reducible configurations, even though such a set may very well be extremely large. He began lecturing on his ideas in the 1940s at the Universities of Hamburg and Kiel. A 1948 lecture at the University of Kiel was attended by the student Wolfgang Haken (born in 1928), who recalls Heesch saying that an unavoidable set of reducible configurations may contain as many as ten thousand members. Heesch discovered a method for creating many unavoidable sets of configurations. Since the method had an electrical flavor to it, electrical terms were chosen for the resulting terminology.

What Heesch did was to consider the dual planar graphs constructed from cubic maps. Thus, the configurations of regions in a cubic map became configurations of vertices in the resulting dual planar graph. These planar graphs themselves had regions, each necessarily a triangle (a 3-gon). Since the only cubic maps whose coloring was still in question were those in which every region was surrounded by five or more neighboring regions, five or more edges of the resulting planar graph met at each vertex of the graph. If k edges meet at a vertex, then the vertex is said to have *degree* k. Thus every vertex in each planar graph of interest had degree 5 or more. Heesch then assigned each vertex in the graph a "charge" of $6 - k$ if the degree of the vertex was k (see Chapter 5). The only vertices receiving a positive charge were therefore those of degree 5, which were given a charge of $+1$. The vertices of degree 6 had a charge of 0, those of degree 7 a charge of -1, and so on. It can be proved (see Chapter 5) that the sum of the charges of the vertices in such a planar graph is always positive (in fact exactly 12).

Heesch's plan consisted of establishing rules, called *discharging rules*, for moving a positive charge from one vertex to others in a manner that did not change the sum of the charges. The goal was to use these rules to create an unavoidable set of configurations by showing that if a minimum counterexample to the Four Color Conjecture contained none of these configurations, then the sum of the charges of its vertices was not 12.

Since Heesch's discharging method was successful in finding unavoidable sets, much of the early work in the 20th century on the Four Color Problem was focused on showing that certain configurations were reducible. Often showing that even one configuration was reducible became a monumental task. In the 1960s Heesch had streamlined Birkhoff's approach of establishing the reducibility of certain configurations. One of these techniques, called D-reduction, was sufficiently algorithmic in nature to allow this technique to be executed on a computer and, in fact, a computer program for implementing D-reducibility was written on the CDC 1604A computer by Karl Dürre, a graduate of Hanover.

Because of the large number of ways that the vertices on the ring of a configuration could be colored, the amount of computer time needed to analyze complex configurations became a major barrier to their work. Heesch was then able to develop a new method, called C-reducibility, where only some of the colorings of the ring vertices needed to be considered. Of course, one possible way to deal with the obstacles that Heesch and Dürre were facing was to find a more powerful computer on which to run Dürre's program.

While Haken had attended Heesch's talk at the University of Kiel on the Four Color Problem, the lectures that seemed to interest Haken the most were those on topology given by Karl Heinrich Weise in which he described three long-standing unsolved problems. One of these was the Poincaré Conjecture posed by the great mathematician and physicist Henri Poincaré in 1904 and which concerned the relationship of shapes, spaces, and surfaces. Another was the Four Color Problem and the third was a problem in knot theory. Haken decided to attempt to solve all three problems. Although his attempts to prove the Poincaré Conjecture failed, he was successful with the knot theory problem. A proof of the Poincaré Conjecture by the Russian mathematician Grigori Perelman was confirmed and reported in Trieste, Italy on 17 June 2006. For this accomplishment, he was awarded a Fields Medal (the mathematical equivalent of the Nobel Prize) on 22 August 2006. However, Perelman declined to attend the ceremony and did not accept the prize. As for the Four Color Problem, the story continues.

Haken's solution of the problem in knot theory led to his being invited to the University of Illinois as a visiting professor. After leaving the University of Illinois to spend some time at the Institute for Advanced Study in Princeton, Haken then returned to the University of Illinois to take a permanent position.

Heesch inquired, through Haken, about the possibility of using the new supercomputer at the University of Illinois (the ILLAC IV) but much time was still needed to complete its construction. The Head of the Department of Computer Science there suggested that Heesch contact Yoshio Shimamoto, Head of the Applied Mathematics Department at the Brookhaven Laboratory at the United States Atomic Energy Commission, which had access to the Stephen Cray-designed Control Data 6600, which was the fastest computer at that time.

Shimamoto himself had an interest in the Four Color Problem and had even thought of writing his own computer program to investigate the reducibility of configurations. Shimamoto arranged for Heesch and Dürre to visit Brookhaven in the late 1960s. Dürre was able to test many more configurations for reducibility. The configurations that were now known to be D-reducible still did not constitute an unavoidable set, however, and Heesch and Dürre returned to Germany. In August of 1970 Heesch visited Brookhaven again – this time with Haken visiting the following month. At the end of September, Shimamoto was able to show that if a certain configuration that he constructed (known as the *horseshoe configuration*) was D-reducible, then the Four Color Conjecture is true. Figure 17 shows the dual planar graph constructed from the horseshoe configuration. This was an amazing development. To make matters even more interesting, Heesch recognized the horseshoe configuration as one that had earlier been shown to be D-reducible. Because of the importance of knowing, with complete certainty, that this configuration was D-reducible, Shimamoto took the cautious approach of having a totally new computer program written to verify the D-reducibility of the horseshoe configuration.

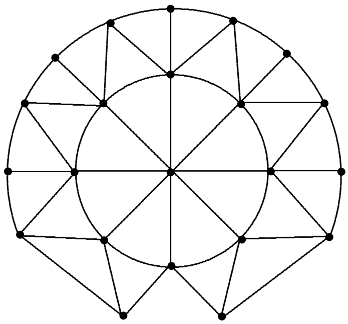


Figure 17: The Shimamoto horseshoe

Dürre was brought back from Germany because of the concern that the original verification of the horseshoe configuration being D-reducible might be incorrect. Also, the printout of the computer run of this was nowhere to be found. Finally, the new computer program was run and, after 26 hours, the program concluded that this configuration was *not* D-reducible. It was not only that this development was so very disappointing to Shimamoto but, despite the care he took, rumors had begun to circulate in October of 1971 that the Four Color Problem had been solved – using a computer!

Haken had carefully checked Shimamoto's mathematical reasoning and found it to be totally correct. Consequently, for a certain period, the only obstacle standing in the way of a proof of the Four Color Conjecture had been a computer. William T. Tutte (1917–2002) and Hassler Whitney (1907–1989), two of the great graph theorists at that time, had also studied Shimamoto's method of proof and found no flaw in his reasoning. Because this would have resulted in a far simpler proof of the Four Color Conjecture than could reasonably be expected, Tutte and Whitney concluded that the original computer result must be wrong. However, the involvement of Tutte and Whitney in the Four Color Problem resulted in a clarification of D-reducibility. Also because of their stature in the world of graph theory, there was even more interest in the problem.

It would not be hard to present the history of graph theory as an account of the struggle to prove the four color conjecture, or at least to find out why the problem is difficult.

William T. Tutte (1967)

In the April 1, 1975 issue of the magazine *Scientific American* the popular mathematics writer Martin Gardner (1914–2010) stunned the mathematical community (at least momentarily) when he wrote an article titled "Six Sensational Discoveries that Somehow Have Escaped Public Attention" that contained a map (see Figure 18) advertised as one that could *not* be colored with four colors. However, several individuals found that this map could in fact be colored with four colors, only to learn that Gardner had intended this article as an April Fool's joke.

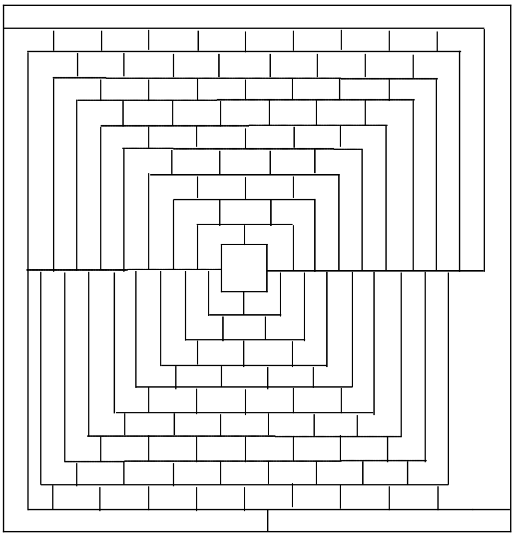


Figure 18: Martin Gardner's April Fool's Map

In the meantime, Haken had been losing faith in a computer-aided solution of the Four Color Problem despite the fact that he had a doctoral student at the University of Illinois whose research was related to the problem. One of the members of this student's thesis committee was Kenneth Appel (1932–2013). After completing his undergraduate degree at Queens College with a special interest in actuarial mathematics, Appel worked at an insurance company and shortly afterwards was drafted and began a period of military service. He then went to the University of Michigan for his graduate studies in mathematics. During the spring of 1956, the University of Michigan acquired an IBM 650 and the very first programming course offered at the university was taught by John W. Carr, III, one of the pioneers of computer education in the United States. Curiously, Carr's doctoral advisor at the Massachusetts Institute of Technology was Phillip Franklin, who, as we mentioned, wrote his dissertation on the Four Color Problem. Appel audited this programming course. Since the university did not offer summer financial support to Appel and Douglas Aircraft was recruiting computer programmers, he spent the summer of 1956 writing computer programs concerning the DC-8 jetliner, which was being designed at the time. Appel had become hooked on computers.

Kenneth Appel's area of research was mathematical logic. In fact, Appel asked Haken to give a talk at the logic seminar in the Department of Mathematics so he could better understand the thesis. In his talk, Haken included a discussion of the computer difficulties that had been encountered in his approach to solve the Four Color Problem and explained that he was finished with the problem for the present. Appel, however, with his knowledge of computer programming, convinced Haken that the two of them should "take a shot at it".

Together, Appel and Haken took a somewhat different approach. They devised an algorithm that tested for "reduction obstacles". The work of Appel and Haken was greatly aided by Appel's doctoral student John Koch who wrote a very efficient program that tested certain kinds of configurations for reducibility. Much of Appel and Haken's work involved refining Heesch's method for finding an unavoidable set of reducible configurations.

The partnership in the developing proof concerned the active involvement of a team of three, namely Appel, Haken, and a computer. As their work progressed, Appel and Haken needed ever-increasing amounts of time on a computer. Because of Appel's political skills, he was able to get time on the IBM 370-168 located in the university's administration building. Eventually, everything paid off. In June of 1976, Appel and Haken had constructed an unavoidable set of 1936 reducible configurations, which was later reduced to 1482. The proof was finally announced at the 1976 Summer Meeting of the American Mathematical Society and the Mathematical Association of America at the University of Toronto. Shortly afterwards, the University of Illinois employed the postmark

FOUR COLORS SUFFICE

on its outgoing mail.

In 1977 Frank Harary (1921–2005), editor-in-chief of the newly founded *Journal of Graph Theory*, asked William Tutte if he would contribute something for the first volume of the journal in connection with this announcement. Tutte responded with a short but pointed poem (employing his often-used pen-name Blanche Descartes) with the understated title *Some Recent Progress in Combinatorics*:

Wolfgang Haken
Smote the Kraken
One! Two! Three! Four!
Quoth he: "The monster is no more".

In the poem, Tutte likened the Four Color Problem to the legendary sea monster known as a kraken and proclaimed that Haken (along with Appel, of course) had slain this monster.

With so many mistaken beliefs that the Four Color Theorem had been proved during the preceding century, it was probably not surprising that the announced proof by Appel and Haken was met with skepticism by many. While the proof was received with enthusiasm by some, the reception was cool by others, even to the point of not being accepted by some that such an argument was a proof at all. It certainly didn't help matters that copying, typographical, and technical errors were found – even though corrected later. In 1977, the year following the announcement of the proof of the Four Color Theorem, Wolfgang Haken's son Armin, then a graduate student at the University of California at Berkeley, was asked to give a talk about the proof. He explained that

> *the proof consisted of a rather short theoretical section, four hundred pages of detailed checklists showing that all relevant cases had been covered, and about 1800 computer runs totaling over a thousand hours of computer time.*

He went on to say that the audience seemed split into two groups, largely by age and roughly at age 40. The older members of the audience questioned a proof that made such extensive use of computers, while the younger members questioned a proof that depended on hand-checking 400 pages of detail.

The proof of the Four Color Theorem initiated a great number of philosophical discussions as to whether such an argument was a proof and, in fact, what a proof is. Some believed that it was a requirement of a proof that it must be possible for a person to be able to read through the entire proof, even though it might be extraordinarily lengthy. Others argued that the nature of proof had changed over the years. Centuries ago a mathematician might have given a proof in a conversational style. As time went on, proofs had become more structured and were presented in a very logical manner. While some were concerned with the distinct possibility of computer error in a computer-aided proof, others countered this by saying that the literature is filled with incorrect proofs and misstatements since human error is always a possibility, perhaps even more likely. Furthermore, many proofs written by modern mathematicians, even though not computer-aided, were so long that it is likely that few, if any, had read through these proofs with care. Also, those who shorten proofs by omitting arguments of some claims within a proof may in fact be leaving out key elements of the proof, improving the opportunity for human error. Then there are those who stated that knowing the Four Color Theorem is true is not what is important. What is crucial is to know *why* only four colors are needed to color all maps. A computer-aided proof does not supply this information.

A second proof of the Four Color Theorem, using the same overall approach but a different discharging procedure, a different unavoidable set of reducible configurations, and more powerful proofs of reducibility was announced and described by Frank Allaire of Lakehead University, Canada in 1977, although the complete details were never published.

As Robin Thomas of the Georgia Institute of Technology reported, there appeared to be two major reasons for the lack of acceptance by some of the Appel-Haken proof: (1) part of the proof uses a computer and cannot be verified by hand; (2) the part that is supposed to be checked by hand is so complicated that no one may have independently checked it at all. For these reasons, in 1996, Neil Robertson, Daniel P. Sanders, Paul Seymour, and Thomas constructed their own (computer-aided) proof of the Four Color Theorem. While Appel and Haken's unavoidable set of configurations consisted of 1482 graphs, this new proof had an unavoidable set of 633 graphs. In addition, while Appel and Haken used 487 discharging rules to construct their set of configurations, Robertson, Sanders, Seymour, and Thomas used only 32 discharging rules to construct their set of configurations. Thomas wrote:

> *Appel and Haken's use of a computer 'may be a necessary evil', but the complication of the hand proof was more disturbing, particularly since the 4CT has a history of incorrect "proofs". So in 1993, mainly for our own peace of mind, we resolved to convince ourselves that the 4CT really was true.*

The proof of the Four Color Theorem given by Robertson, Sanders, Seymour, and Thomas rested on the same idea as the Appel-Haken proof, however. These authors proved that none of the 633 configurations can be contained in a minimum counterexample to the Four Color Theorem and so each of these configurations is reducible.

As we noted, the Four Color Theorem could have been proved if any of the following could be shown to be true.

(1) The regions of every map can be colored with four or fewer colors so that neighboring regions are colored differently.

(2) The vertices of every planar graph can be colored with four or fewer colors so that every two vertices joined by an edge are colored differently.

(3) The edges of every cubic map can be colored with exactly three colors so that every three edges meeting at a vertex are colored differently.

Coloring the regions, vertices, and edges of maps and planar graphs, inspired by the desire to solve the Four Color Problem, has progressed far beyond this – to coloring more general graphs and even to reinterpreting what is meant by coloring.

For many decades, a coloring of (the vertices of) a graph G was always meant as an assignment of colors (elements of some set) to the vertices of G, one color to each vertex, so that adjacent vertices are colored differently. Of course, this is quite understandable as this came from the Four Color Problem where neighboring regions were required to be colored differently. In more recent decades, this interpretation of a coloring has changed – often dramatically. As we will see in the later chapters, there are occasions when we might not want any two vertices to be colored the same. And this is the case when coloring edges as well. Indeed, and opposite to this, we may want all vertices (or edges) to be colored the same. Of course, all this depends on what the goal of a coloring is. There are also occasions when a coloring (or the related concept of a labeling) of the vertices or edges of a graph itself gives rise to another coloring. It is the study of these topics into which we are about to venture.

Chapter 1

Introduction to Graphs

In the preceding chapter we were introduced to the famous map coloring problem known as the Four Color Problem. We saw that this problem can also be stated as a problem dealing with coloring the vertices of a certain class of graphs called *planar graphs* or as a problem dealing with coloring the edges of a certain subclass of planar graphs. This gives rise to coloring the vertices or coloring the edges of graphs in general. In order to provide the background needed to discuss this subject, we will describe, over the next five chapters, some of the fundamental concepts and theorems we will encounter in our investigation of graph colorings as well as some common terminology and notation in graph theory.

1.1 Fundamental Terminology

A **graph** G is a finite nonempty set V of objects called **vertices** (the singular is **vertex**) together with a set E of 2-element subsets of V called **edges**. Vertices are sometimes called **points** or **nodes**, while edges are sometimes referred to as **lines** or **links**. Each edge $\{u, v\}$ of G is commonly denoted by uv or vu. If $e = uv$, then the edge e is said to **join** u and v. The number of vertices in a graph G is the **order** of G and the number of edges is the **size** of G. We often use n for the order of a graph and m for its size. To indicate that a graph G has **vertex set** V and **edge set** E, we sometimes write $G = (V, E)$. To emphasize that V is the vertex set of a graph G, we often write V as $V(G)$. For the same reason, we also write E as $E(G)$. A graph of order 1 is called a **trivial graph** and so a **nontrivial graph** has two or more vertices. A graph of size 0 is an **empty graph** and so a **nonempty graph** has one or more edges.

Graphs are typically represented by diagrams in which each vertex is represented by a point or small circle (open or solid) and each edge is represented by a line segment or curve joining the corresponding small circles. A diagram that represents a graph G is referred to as the graph G itself and the small circles and lines representing the vertices and edges of G are themselves referred to as the vertices and edges of G.

Figure 1.1 shows a graph G with vertex set $V = \{t, u, v, w, x, y, z\}$ and edge set $E = \{tu,\ ty,\ uv,\ uw,\ vw,\ vy,\ wx,\ wz,\ yz\}$. Thus, the order of this graph G is 7 and its size is 9. In this drawing of G, the edges tu and vw intersect. This has no significance. In particular, the point of intersection of these two edges is not a vertex of G.

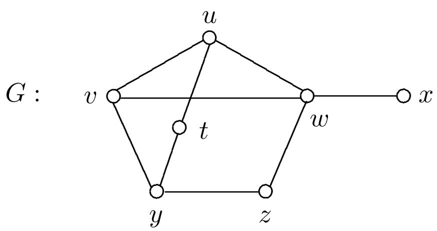


Figure 1.1: A graph

If uv is an edge of G, then u and v are **adjacent vertices**. Two adjacent vertices are referred to as **neighbors** of each other. The set of neighbors of a vertex v is called the **open neighborhood** of v (or simply the **neighborhood** of v) and is denoted by $N(v)$. The set $N[v] = N(v) \cup \{v\}$ is called the **closed neighborhood** of v. If uv and vw are distinct edges in G, then uv and vw are **adjacent edges**. The vertex u and the edge uv are said to be **incident** with each other. Similarly, v and uv are incident.

For the graph G of Figure 1.1, the vertices u and w are therefore adjacent in G, while the vertices u and x are not adjacent. The edges uv and uw are adjacent in G, while the edges vy and wz are not adjacent. The vertex v is incident with the edge vw but is not incident with the edge wz.

For nonempty disjoint sets A and B of vertices of G, we denote by $[A, B]$ the set of edges of G joining a vertex of A and a vertex of B. For the sets $A = \{u, v, y\}$ and $B = \{w, z\}$ in the graph G of Figure 1.1, $[A, B] = \{uw, vw, yz\}$.

The **degree of a vertex** v in a graph G is the number of vertices in G that are adjacent to v. Thus, the degree of a vertex v is the number of the vertices in its neighborhood $N(v)$. Equivalently, the degree of v is the number of edges of G incident with v. The degree of a vertex v is denoted by $\deg_G v$ or, more simply, by $\deg v$ if the graph G under discussion is clear. A vertex of degree 0 is referred to as an **isolated vertex** and a vertex of degree 1 is an **end-vertex** or a **leaf**. An edge incident with an end-vertex is called a **pendant edge**. The largest degree among the vertices of G is called the **maximum degree** of G is denoted by $\Delta(G)$. The **minimum degree** of G is denoted by $\delta(G)$. Thus, if v is a vertex of a graph G of order n, then

$$0 \leq \delta(G) \leq \deg v \leq \Delta(G) \leq n - 1.$$

For the graph G of Figure 1.1,

$$\deg x = 1,\ \deg t = \deg z = 2,\ \deg u = \deg v = \deg y = 3, \text{ and } \deg w = 4.$$

Thus, $\delta(G) = 1$ and $\Delta(G) = 4$.

A well-known theorem in graph theory deals with the sum of the degrees of the vertices of a graph. This theorem was indirectly observed by the great Swiss mathematician Leonhard Euler in a 1736 paper [80] that is now considered the first paper ever written on graph theory – even though graphs were never mentioned in the paper. It is often referred to as the First Theorem of Graph Theory. (Some have called this theorem the **Handshaking Lemma**, although Euler never used this name.)

Theorem 1.1 (**The First Theorem of Graph Theory**) *If G is a graph of size m, then*

$$\sum_{v \in V(G)} \deg v = 2m.$$

Proof. When summing the degrees of the vertices of G, each edge of G is counted twice, once for each of its two incident vertices. ■

The sum of the degrees of the vertices of the graph G of Figure 1.1 is 18, which is twice the size 9 of G, as is guaranteed by Theorem 1.1.

A vertex v in a graph G is **even** or **odd**, according to whether its degree in G is even or odd. Thus, the graph G of Figure 1.1 has three even vertices and four odd vertices. While a graph can have either an even or odd number of even vertices, this is not the case for odd vertices.

Corollary 1.2 *Every graph has an even number of odd vertices.*

Proof. Suppose that G is a graph of size m. By Theorem 1.1,

$$\sum_{v \in V(G)} \deg v = 2m,$$

which is, of course, an even number. Since the sum of the degrees of the even vertices of G is even, the sum of the degrees of the odd vertices of G must be even as well, implying that G has an even number of odd vertices. ■

A graph H is said to be a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If $V(H) = V(G)$, then H is a **spanning subgraph** of G. If H is a subgraph of a graph G and either $V(H)$ is a proper subset of $V(G)$ or $E(H)$ is a proper subset of $E(G)$, then H is a **proper subgraph** of G. For a nonempty subset S of $V(G)$, the **subgraph** $G[S]$ **of** G **induced by** S has S as its vertex set and two vertices u and v in S are adjacent in $G[S]$ if and only if u and v are adjacent in G. (The subgraph of G induced by S is also denoted by $\langle S \rangle_G$ or simply by $\langle S \rangle$ when the graph G is understood.) A subgraph H of a graph G is called an **induced subgraph** if there is a nonempty subset S of $V(G)$ such that $H = G[S]$. Thus, $G[V(G)] = G$. For a nonempty set X of edges of a graph G, the **subgraph** $G[X]$ **induced by** X has X as its edge set and a vertex v belongs to $G[X]$ if v is incident with at least one edge in X. A subgraph H of G is **edge-induced** if there is a nonempty subset X of $E(G)$ such that $H = G[X]$. Thus, $G[E(G)] = G$ if and only if G has no isolated vertices.

Figure 1.2 shows six graphs, namely G and the graphs H_i for $i = 1, 2, \ldots, 5$. All six of these graphs are proper subgraphs of G, except G itself and H_1. Since G is a subgraph of itself, it is not a proper subgraph of G. The graph H_1 contains the edge uz, which G does not and so H_1 is not even a subgraph of G. The graph H_3 is a spanning subgraph of G since $V(H_3) = V(G)$. Since $xy \in E(G)$ but $xy \notin E(H_4)$, the subgraph H_4 is not an induced subgraph of G. On the other hand, the subgraphs H_2 and H_5 are both induced subgraphs of G. Indeed, for $S_1 = \{v, x, y, z\}$ and $S_2 = \{u, v, y, z\}$, $H_2 = G[S_1]$ and $H_5 = G[S_2]$. The subgraph H_4 of G is edge-induced; in fact, $H_4 = G[X]$, where $X = \{uw, wx, wy, xz, yz\}$.

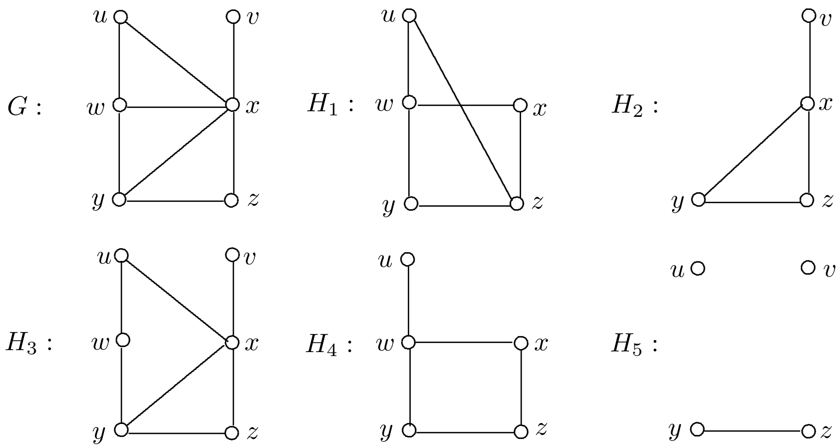


Figure 1.2: Graphs and subgraphs

For a vertex v and an edge e in a nonempty graph $G = (V, E)$, the subgraph $G - v$, obtained by deleting v from G, is the induced subgraph $G[V - \{v\}]$ of G and the subgraph $G - e$, obtained by deleting e from G, is the spanning subgraph of G with edge set $E - \{e\}$. More generally, for a proper subset U of V, the graph $G - U$ is the induced subgraph $G[V - U]$ of G. For a subset X of E, the graph $G - X$ is the spanning subgraph of G with edge set $E - X$. If u and v are distinct nonadjacent vertices of G, then $G + uv$ is the graph with $V(G + uv) = V(G)$ and $E(G + uv) = E(G) \cup \{uv\}$. Thus, G is a spanning subgraph of $G + uv$. For the graph G of Figure 1.3, the set $U = \{t, x\}$ of vertices, and the set $X = \{tw, ux, vx\}$ of edges, the subgraphs $G - u$, $G - wx$, $G - U$, and $G - X$ of G are also shown in that figure, as is the graph $G + uv$.

1.2 Connected Graphs

There are several types of sequences of vertices in a graph as well as subgraphs of a graph that can be used to describe ways in which one can move about within the graph. For two (not necessarily distinct) vertices u and v in a graph G, a $u - v$ **walk** W in G is a sequence of vertices in G, beginning at u and ending at v such that