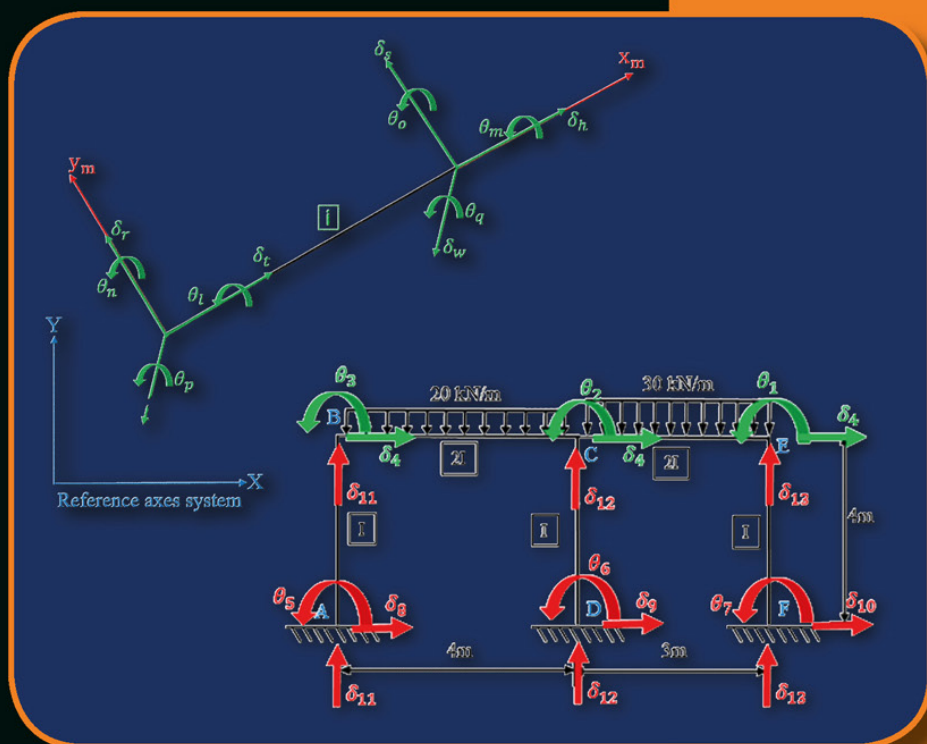


Advanced Structural Analysis with MATLAB®

Srinivasan Chandrasekaran



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By
Srinivasan Chandrasekaran



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Foreword

There has been tremendous growth in the field of civil engineering in recent times. The structures have evolved from being simple and straightforward in configuration to more complex ones in terms of shape, geometry and design requirements. Modern-day architecture demands very intricate structural design in making a built environment efficient and sustainable. In the era of fast-paced growth, there are numerous finite element packages available to reduce the computational effort and time. However, one should have a strong foothold in the basic concepts of structural engineering as most of the numerical work is handled by computer.

This book addresses the basic structural forms such as orthogonal and non-orthogonal planar frames, space frames and trusses that go into the making of complicated structures. The approach adopted is based on the intuition that an engineer's ability to perceive a concept is through simple models. Numerous examples have been included to illustrate the fundamental concepts more clearly. Hence, in addition to conceptual understanding, an effort is made to include the basics of computation with detailed examples. The book consists of five chapters.

The analysis steps have been explained in a classroom style of teaching and the computer programs for MATLAB® platform have been introduced in the form of examples. These computer programs cover matrix operation for a variety of structural forms and responses. The illustrative examples in the book enhance the understanding of the structural concepts stimulating interest in learning, creative thinking and design. In conclusion, the book stems from a void in conceptual understanding of the structural behavior, based on problem solving experience with students exposed to engineering mechanics and mechanics of materials.

The author of this book, Professor Srinivasan Chandrasekaran, is a renowned teacher and researcher with diverse industrial experience in structural engineering. He has already authored many peer-reviewed journal articles, conference papers, textbooks and reports on international projects. His rich expertise and experience in the teaching of fundamentals of structural analysis at IIT Madras has been brought out now in the form of this new book. I strongly believe this textbook to be an ideal resource for students and teachers, and a comprehensive reference for practitioners. I congratulate Professor Chandrasekaran for his total commitment to the advancement of technical education. I hope that many will learn from this book and apply its principles in their profession.

Katta Venkataramana

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Preface

Analysis of civil engineering structures is becoming more complex essentially due to different structural forms that are conceived by architects and engineers to accommodate various functional requirements. Conventional analysis tools guide engineering graduates and practicing professionals in addressing such issues, but accuracy and compactness, in terms of varied solutions, are difficult. Matrix methods in general, and the stiffness method in particular, are very powerful tools to model complicated structural forms and to perform the required analysis. However, the application of matrix algorithms in a more generic form to solve all types of problems, namely beams, trusses, planar orthogonal frames, planar non-orthogonal frames, three-dimensional trusses and space frames, needs to be addressed in a step-by-step manner to resolve all possible doubts that may arise during solution procedures. While acknowledging the ingenious efforts made by authors from all over the world on this front, this book is a humble attempt to revisit these concepts with more elaborate explanations and very strong hand-supportive computer coding. One of the main objectives of this book is to help solve problems using matrix methods along with a familiarization of computer coding to solve such problems. MATLAB® is a well-established and proven tool to handle such complex problems in a very simple and highly supportive manner.

This book starts with an analysis of beams and planar orthogonal frames, and it also addresses problems of truss elements, special elements, planar non-orthogonal frames, three-dimensional trusses and space frames. One of the most attractive features of this book is how it explains the problem solution which is highly compatible with computer coding using MATLAB. Each problem is carefully examined and degrees-of-freedom (both restrained and unrestrained) are marked in a more generic manner with a uniform sign convention throughout the text of this book. Matrix formulation of the problem is clearly presented step by step; this is also followed while writing the computer code for solving the problem. Example problems given in each chapter are solved using MATLAB coding, while input data to use the coding is explained in detail. The output obtained from the coding is plotted as a screenshot for better inference of results. Numerous exercise problems are given along with solutions that enable the readers to use the same computer code with a minor modification to suit the inputs for the problems.

One of the salient features of this book is that similar computer code as that for two-dimensional is used for three-dimensional analysis, except transformations that are required from local to global axes systems. The book also supports many practice papers to ensure a high level of confidence while solving such problems. A Solutions Manual and additional instructor resources are available as downloadable e-resources on the book's CRC Press webpage at <https://www.crcpress.com/Advanced-Structural-Analysis-with-MATLAB/Chandrasekaran/p/book/9780367026455>. MATLAB files are also available in downloadable format on same webpage. The author sincerely thanks the Centre for Continuing Education, Indian Institute of Technology (IIT) Madras, for extending administrative support in preparing the

manuscript of this book. The author also thanks MATLAB for permitting usage of MATLAB codes throughout the text of this book.

The computer programs used in the book are written using MATLAB, following well-established programming concepts. Program coding is written following the same steps as those for conventional analysis using the stiffness method. The program codes were written by Nagavinothini.R and verified by a team of research scholars, Department of Ocean Engineering, IIT Madras. Nagavinothini.R is senior research scholar in the Department of Ocean Engineering at IIT Madras, Chennai, India. She is currently working on dynamic analysis of offshore new generation compliant platforms in ultra-deep waters under environmental and accidental loads. She is a University Rank Holder and Gold Medalist, who has published many research papers in refereed journals. Her research interests include dynamic analysis of structures, computer-aided analysis of structures, design and optimization of structures.

Utmost care has been taken to check solutions and to correct errors, but the author does not claim or guarantee the correctness of outputs using the provided computer codes. Readers are asked to verify based on their independent capacity and then use the codes for practical applications.

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1 Planar Orthogonal Structures

1.1 INTRODUCTION TO STRUCTURAL ANALYSIS

The first and foremost step in structural analysis is problem formulation using an appropriate mathematical model. There are two models widely used in classical structural analysis, namely: (1) the statically determinate model and (2) the statically indeterminate model. Statically determinate models are relatively easier as they use only the basic equations of static equilibrium to solve the problem. Hence, one should look for ways to solve problems related to statically indeterminate models through computer methods. In order to solve the problems, it is important to formulate a standard procedure, which should be generic in nature and not problem-specific. Thus, it is important to note that the models should be restrained from any action to enable solution of the problem by using a standard equation of statics. This can be done by grouping the formation. Grouping is done through two methods, namely: (1) the flexibility method; and (2) the stiffness method.

Both of the previously mentioned methods are frequently used to simplify the model to be solved by using only the standard equation of statics. Both methods are equally powerful and useful; there is no supremacy of one method over the other. Any method can be used for grouping based on the user's convenience. But, a method that is easily programmable is preferred, as the main objective of this book is to make the problems solvable through computer methods. There is a significant difference in identifying the unknowns for formulating the problem. In the flexibility method, the unknowns are actions such as shear force, axial force and bending moment. In the stiffness method, however, the unknowns are displacements such as translational and rotational displacements. Basic assumptions applicable to both methods are as follows:

1. A linear relationship exists between an applied load and the resulting displacement of the structure. This makes the principle of superposition valid through the formulation.
2. The material of the structure must obey Hooke's Law, which says that the material must not be stressed beyond its elastic limit.
3. The equations of static equilibrium shall be developed using the geometry of the un-deflected model. The change in geometry caused by the imposed loads is negligible when compared to original geometry.

1.2 INDETERMINACY

Both flexibility and stiffness methods circumscribe the problem formulations around the term indeterminacy. It is important to understand indeterminacy in terms of

problem formulation by either the flexibility or the stiffness method. There are two types of indeterminacy, namely:

- 1. Static indeterminacy
- 2. Kinematic indeterminacy

1.2.1 STATIC INDETERMINACY

Static indeterminacy is the term related to the flexibility approach. It is defined as the number of actions (e.g. shear force, axial force, bending moment) that can be either external or internal, that must be released in order to transform the structural system into a stable statically determinate system. Thus, the objective is to convert the known structural system into a statically determinate and stable system, for which the number of actions has to be identified. The *degree of static indeterminacy* is defined as the number of released actions, which specify the number of special independent equations that must be developed in terms of the released actions to analyze the system. So, the approach used in the formulation and solution is the *flexibility approach*.

1.2.2 KINEMATIC INDETERMINACY

Kinematic indeterminacy is the term related to the stiffness approach. It refers to the number of independent components of joint displacements (both translational and rotational) with respect to a specified coordinate axis that is required to describe the response of the system under any arbitrary load. It can be seen that the kinematic indeterminacy or stiffness method is trying to reach a generic solution. This problem formulation needs to identify the number of independent displacement components, which will be invoked under the external forces acting on the system of any nature. It is important to note that the structure must be restrained to convert or transform the system into a kinematically determinate structure. A structure with all joint displacements restrained is the formulation. The *degree of kinematic indeterminacy* is defined as the number of unrestrained components of the joint displacements (both rotational and translational). It is important to know that the degree of kinematic indeterminacy specifies the number of independent equations that must be written in terms of unrestrained displacements, if the system is to be analyzed using the stiffness approach.

The differences between flexibility and stiffness methods are summarized as follows:

Flexibility Approach	Stiffness Approach
This deals with static indeterminacy	This deals with kinematic indeterminacy
The unknowns are actions such as shear force, axial force, bending moments, etc.	The unknowns are joint displacements such as rotational and translational displacements
The problem formulation converts the structural system into a statically determinate structure	The problem formulation converts the structural system into a kinematically determinate structure.

Thus, the static and kinematic indeterminacies of any structural system are indicators of the amount or extent of computational effort required to analyze the structural system either using the flexibility approach or the stiffness approach. It is clear that the structural analysis can be carried out by any of the two methods, which are equally useful and powerful numerically. The unknowns are released to convert the structural system into a statically or kinematically determinate system, so that the standard equations can be used to solve the system under applied loads.

If the problem formulation reduces the number of unknowns, then it is the best formulation attempted by a mathematician or an engineer. The number of unknowns in the system of equations purely depends on the choice of the method demanded. For a computer method of structural analysis, one should keep in mind that the method recommended should be more or less generic and not problem-specific. The degree of static and kinematic indeterminacies of standard problems are given subsequently.

1.2.2.1 Continuous Beam

Consider a continuous beam of three spans with one hinged joint and three roller joints with reactions R_1 , R_2 , R_3 and R_4 , as shown in Figure 1.1. The displacement unknowns, neglecting axial deformations are θ_1 , θ_2 , θ_3 and θ_4 . We all know that there are three systems of standard equations available to solve the problem. The degree of static indeterminacy and the degree of kinematic indeterminacy are as follows:

$$\begin{aligned}\text{Degree of static indeterminacy} &= \text{Number of unknown reactions} \\ &\quad - \text{system of standard equations} \\ &= 5 - 3 = 2.\end{aligned}$$

$$\begin{aligned}\text{Degree of kinematic indeterminacy} &= \text{Number of displacements} \\ &= 4\end{aligned}$$

1.2.2.2 Fixed Beam

Let us now consider a fixed beam with reactions R_1 , R_2 and R_3 at support A. Similarly, the reactions at support B are R_4 , R_5 and R_6 , as shown in Figure 1.2. The rotational displacement which is free to move is zero. Thus,

$$\text{Degree of static indeterminacy} = 6 - 3 = 3$$

$$\text{Degree of kinematic indeterminacy} = 0$$

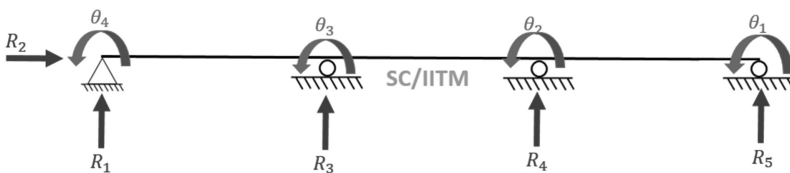


FIGURE 1.1 Continuous beam.

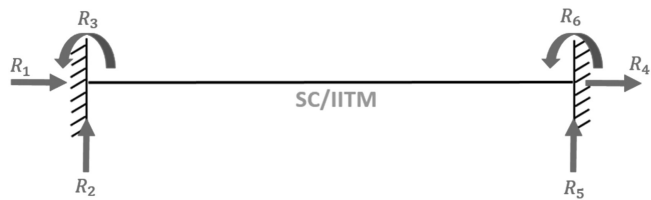


FIGURE 1.2 Fixed beam.

1.2.2.3 Simply Supported Beam

Let us consider a simply supported beam with reaction components R_1 , R_2 and R_3 . The displacements are θ_1 and θ_2 as shown in Figure 1.3. Thus,
Degree of static indeterminacy = $3 - 3 = 0$
Degree of kinematic indeterminacy = 2

1.2.2.4 Frame

Let us consider a single story single bay frame with one end fixed and the other end on a roller support as shown in Figure 1.4. The unknown reactions are R_1 , R_2 , R_3 ,



FIGURE 1.3 Simply supported beam.

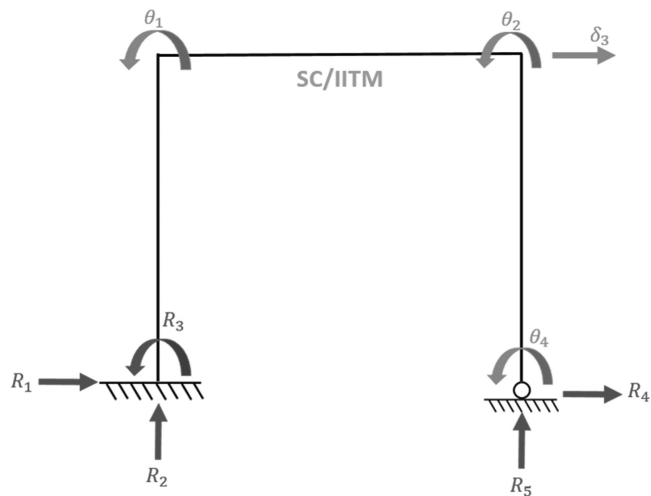


FIGURE 1.4 Single story single bay frame.

R4 and R5. The independent displacements are θ_1 , θ_2 , δ_3 and δ_4 by neglecting axial deformation.

Degree of static indeterminacy = $5 - 3 = 2$

Degree of kinematic indeterminacy = 4

From the previous examples, it can be seen that the number of unknowns is different depending on the choice of method. In order to summarize for the choice of method of analysis which governs the system of equations, the following is noteworthy:

1. There are essentially two methods to solve the statically indeterminate structures, such as the flexibility method and stiffness method. So, the choice of the method depends on the computational convenience.
2. For the flexibility method, there are several alternatives for the redundant or unknowns. Thus, the choice of the redundant has a significant effect on the computational effort.
3. On the other hand, there is no choice of unknown quantities in the stiffness method because there is only one possible restrained structure. Therefore, this has a set of standard procedures.
4. Based on computer methods of structural analysis, one can say that the choice of the method should not be geometry specific. It should be more generic and repetitive in nature.

Thus, the stiffness method is a better choice fulfilling all the previously mentioned requirements. In this book, the stiffness method is used for elaborating the application procedures. After identifying the variables in a given system, such as rotational or translational displacements for every joint, it will result in the formation of a set of linear equations.

1.3 LINEAR EQUATIONS

A system of ' m ' linear equations with ' n ' unknowns is expressed as follows:

$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\
 &\dots\dots\dots \\
 a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m
 \end{aligned}
 \tag{1.1}$$

The previous set of equations can be written in matrix form as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ a_{m1} & a_{m2} & a_{mn} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_n \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ b_m \end{Bmatrix}$$

$$[A]\{x\} = \{B\} \quad (1.2)$$

Pre-multiply the previous equation with A^{-1} on both sides,

$$\begin{aligned} [A]^{-1}[A]\{x\} &= [A]^{-1}\{B\} \\ [I]\{x\} &= [A]^{-1}\{B\} \\ \{x\} &= [A]^{-1}\{B\} \end{aligned} \quad (1.3)$$

Thus, the previous equation gives the unknown 'x' by multiplying the inverse of the matrix A and B vector. Now, the problem is to compute the inverse of a matrix.

1.3.1 INVERSE OF A MATRIX

Inverse of matrix A is given by,

$$[A]^{-1} = \frac{adj A}{|A|} \quad (1.4)$$

In a given square matrix, replace each element a_{ij} of the matrix $[A]$ by its cofactor α_{ij} . Transform the cofactor matrix to obtain adjoint matrix. The following simple example will give the procedure to find the inverse of matrix.

Consider a matrix, $A = \begin{bmatrix} 1 & 5 & 2 \\ 0 & 4 & 1 \\ 0 & 2 & 1 \end{bmatrix}$. Find $[A]^{-1}$ by adjoint method.

$$|A| = 1\{(4 \times 1) - (2 \times 1)\} = 2$$

Cofactors are given by,

$$\alpha_{11} = (-1)^{1+1} \begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix} = 2$$

$$\alpha_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix} = 0$$

$$\alpha_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 4 \\ 0 & 2 \end{vmatrix} = 0$$

$$\alpha_{21} = (-1)^{1+3} \begin{vmatrix} 5 & 2 \\ 2 & 1 \end{vmatrix} = -1$$

$$\alpha_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1$$

$$\alpha_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix} = -2$$

$$\alpha_{31} = (-1)^{3+1} \begin{vmatrix} 5 & 2 \\ 4 & 1 \end{vmatrix} = -3$$

$$\alpha_{32} = (-1)^{3+2} \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = -1$$

$$\alpha_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 5 \\ 0 & 4 \end{vmatrix} = 4$$

The cofactor matrix is written as:

$$\alpha_{ij} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & -2 \\ -3 & -1 & 4 \end{bmatrix}$$

$$\text{Adj } A = [\alpha_{ij}]^T = \begin{bmatrix} 2 & -4 & -3 \\ 0 & 1 & -1 \\ 0 & -2 & 4 \end{bmatrix}$$

Thus,

$$[A]^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{2} \begin{bmatrix} 2 & -4 & -3 \\ 0 & 1 & -1 \\ 0 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1/2 & -1/2 \\ 0 & -1 & 2 \end{bmatrix}$$

To check:

$$[A]^{-1}[A] = \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1/2 & -1/2 \\ 0 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 5 & 2 \\ 0 & 4 & 1 \\ 0 & 2 & 1 \end{bmatrix} = [I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.3.2 SOLUTION OF LINEAR EQUATIONS

Let us express the matrix A as a system of equations,

$$x_1 + 5x_2 + 2x_3 = 2$$

$$4x_2 + x_3 = 5$$

$$2x_2 + x_3 = 4$$

Now, these set of equations have to be solved to get the variables (x_1, x_2, x_3) . In matrix form, the equations can be written as follows:

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 4 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 5 \\ 4 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} 2 \\ 5 \\ 4 \end{Bmatrix} = \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1/2 & -1/2 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} 2 \\ 5 \\ 4 \end{Bmatrix} = \begin{Bmatrix} -6.5 \\ 0.5 \\ 3 \end{Bmatrix}$$

If one can generate a system of equations with unknowns as variables, then this set of equations can be solved using matrix inversion. This is an easy method to solve for the variable as given by equation 1.3.

$$\{x\} = [A]^{-1} \{B\}$$

This is true only when $[A]^{-1}$ exists. It should also be noted that $\{x\}$ purely depends on $\{B\}$ and $[A]^{-1}$ does not change to get the value of $\{x\}$. Assume matrix A as a stiffness matrix of a given system, B as a load vector and x as a displacement vector. Through this comparison, it can be seen that the value of the displacement vector for a changed load vector can be found without changing the inverse of the stiffness matrix. In the case that $\{B\}$ is zero and when $[A]^{-1}$ also exists, then the possible solution is said to be a trivial solution, that is, $x=0$. In this case, $[A]^{-1}$ does not exist, then the previous set of equations will lead to non-trivial solution.

1.4 MATRIX OPERATIONS

1.4.1 SUBMATRIX

Let A be the given matrix, then the submatrix is defined as a matrix formed by deleting specified rows and columns of the matrix A . Instead of deleting the rows and columns, partitioning can also be done. This is a useful technique when the matrix size is very large.

Let us assume a set of algebraic equations as follows:

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1q}x_q + a_{1,q+1}x_{q+1} + \dots + a_{1n}x_n \\ y_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2q}x_q + a_{2,q+1}x_{q+1} + \dots + a_{2n}x_n \\ &\dots \\ &\dots \\ &\dots \\ y_n &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nq}x_q + a_{n,q+1}x_{q+1} + \dots + a_{nn}x_n \end{aligned} \tag{1.5}$$

Then, the previous set of equations can also be grouped.

$$\begin{aligned}
 y_1 &= (a_{11}x_1 + a_{12}x_2 + \dots + a_{1q}x_q) + (a_{1,q+1}x_{q+1} + \dots + a_{1n}x_n) \\
 y_2 &= (a_{21}x_1 + a_{22}x_2 + \dots + a_{2q}x_q) + (a_{2,q+1}x_{q+1} + \dots + a_{2n}x_n) \\
 &\quad \dots \\
 y_n &= (a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nq}x_q) + (a_{n,q+1}x_{q+1} + \dots + a_{nn}x_n)
 \end{aligned} \tag{1.6}$$

Now let us express both sets of equations in matrix form:

$$\begin{Bmatrix} y_1 \\ y_2 \\ y_n \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{1q} \\ a_{21} & a_{22} & a_{2q} \\ a_{n1} & a_{n2} & a_{nq} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_q \end{Bmatrix} + \begin{bmatrix} a_{1,q+1} & \dots & a_{1n} \\ a_{2,q+2} & \dots & a_{2n} \\ a_{n,q+1} & \dots & a_{nn} \end{bmatrix} \begin{Bmatrix} x_{q+1} \\ x_{q+2} \\ x_n \end{Bmatrix} \tag{1.7}$$

Now the vector y can be written as,

$$\{y\} = [A_1]\{x_1\} + [A_2]\{x_2\} \tag{1.8}$$

where,

$$[A_1] = \begin{bmatrix} a_{11} & a_{12} & a_{1q} \\ a_{21} & a_{22} & a_{2q} \\ a_{n1} & a_{n2} & a_{nq} \end{bmatrix}$$

$$[A_2] = \begin{bmatrix} a_{1,q+1} & \dots & a_{1n} \\ a_{2,q+2} & \dots & a_{2n} \\ a_{n,q+1} & \dots & a_{nn} \end{bmatrix}$$

$$\{x_1\} = \begin{Bmatrix} x_1 \\ x_2 \\ x_q \end{Bmatrix}$$

$$\{x_2\} = \begin{Bmatrix} x_{q+1} \\ x_{q+2} \\ x_n \end{Bmatrix}$$

There should be a perfect compatibility among the multiplying matrices, shown as follows:

$$\{y\}_{n \times 1} = [A_1]_{n \times q} \{x_1\}_{q \times 1} + [A_2]_{n \times (n-q)} \{x_2\}_{(n-q) \times 1} \tag{1.9}$$

The number of columns and the number of rows of the adjacent multipliers should be same. The compatibility is required to ensure grouping. Now, it can be said

that $[A_1]$ is a submatrix of $[A]$ of size $n \times q$ and $[A_2]$ is a submatrix of $[A]$ of size $n \times (n - q)$.

1.4.2 PARTITIONING OF MATRIX

Let,

$$\{y\} = [A]\{x\} \quad (1.10)$$

After partitioning,

$$\{y\} = \left[[A_1] \quad | \quad [A_2] \right] \begin{Bmatrix} \{x_1\} \\ - \\ \{x_2\} \end{Bmatrix} \quad (1.11)$$

Thus,

$$\{y\} = [A_1]\{x_1\} + [A_2]\{x_2\} \quad (1.12)$$

The previous equation is called partitioned matrix. Matrix $[A]$ is vertically partitioned and vector $\{x\}$ is horizontally partitioned. To make the valid partition of $[A]$ and $\{x\}$, it is important to establish compatibility; that is, the number of columns of $[A_1]$ must correspond to the number of rows of $\{x_1\}$, to make $[A_1]\{x_1\}$ valid.

1.4.3 CROSS-PARTITIONING OF MATRIX

$$\{y\} = [A]\{x\}$$

Let $[A]$ be partitioned both horizontally and vertically into submatrices,

$$[A] = \begin{bmatrix} [A_{11}]_{p \times q} & [A_{12}]_{p \times (n-q)} \\ [A_{21}]_{(m-p) \times q} & [A_{22}]_{(m-p) \times (n-q)} \end{bmatrix}_{m \times n} \quad (1.13)$$

Let $\{x\}$ also be partitioned horizontally,

$$\{x\}_{n \times 1} = \begin{Bmatrix} \{x_1\}_{q \times 1} \\ \{x_2\}_{(n-q) \times 1} \end{Bmatrix}_{n \times 1} \quad (1.14)$$

Therefore, the resulting matrix $\{y\}$ will also be a horizontally partitioned matrix.

$$\{y\}_{m \times 1} = \begin{Bmatrix} \{y_1\}_{p \times 1} \\ \{y_2\}_{(m-p) \times 1} \end{Bmatrix}_{m \times 1} \quad (1.15)$$

Therefore,

$$\begin{Bmatrix} \{y_1\} \\ - \\ \{y_2\} \end{Bmatrix} = \begin{bmatrix} [A_{11}] & | & [A_{12}] \\ [A_{21}] & | & [A_{22}] \end{bmatrix} \begin{Bmatrix} \{x_1\} \\ - \\ \{x_2\} \end{Bmatrix} \quad (1.16)$$

This means that the matrix $[A]$, which has both a horizontal and vertical partitioning, is called a cross-partitioned matrix.

Once partitioning is done, the following equations are valid.

$$\begin{aligned} \{y_1\} &= [A_{11}]\{x_1\} + [A_{12}]\{x_2\} \\ \{y_2\} &= [A_{21}]\{x_1\} + [A_{22}]\{x_2\} \end{aligned} \quad (1.17)$$

The inverse is also valid for a partitioned matrix, which is very advantageous.

Let $[A]$ be the following matrix with horizontal and vertical partitioning,

$$A = \begin{bmatrix} 1 & 4 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 3 & -1 \\ 0 & 0 & -5 & 2 \end{bmatrix}$$

Now, $[A]$ can be written as,

$$A = \begin{bmatrix} [A_{11}] & [A_{12}] \\ [A_{21}] & [A_{22}] \end{bmatrix}$$

It can also be written as,

$$A = \begin{bmatrix} [A_{11}] & 0 \\ 0 & [A_{22}] \end{bmatrix}$$

Now, A^{-1} can also be expressed as a set of submatrices as follows:

$$[A]^{-1} = \begin{bmatrix} [B_{11}] & [B_{12}] \\ [B_{21}] & [B_{22}] \end{bmatrix}$$

where,

$$[B_{12}] = \text{Inverse of } [A_{12}] = 0$$

$$[B_{21}] = \text{Inverse of } [A_{21}] = 0$$

$$[B_{11}] = \text{Inverse of } [A_{11}] = \frac{1}{-6} \begin{bmatrix} 2 & -4 \\ -2 & 1 \end{bmatrix}$$

$$[B_{22}] = \text{Inverse of } [A_{22}] = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$$

Now the advantage is that $[A]^{-1}$ can be easily written as:

$$[A]^{-1} = \begin{bmatrix} -1/3 & 2/3 & 0 & 0 \\ 1/3 & -1/6 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 5 & 3 \end{bmatrix}$$

Thus, determining the inverse of the 4×4 matrix is made easier with cross-partitioning of the matrix. Partitioning benefits inverting a 2×2 matrix, instead of a 4×4 . This can lead to substantial savings in time and computational efforts.

1.4.4 BANDED MATRIX

Matrices in structural analysis show certain special properties. The matrices are real, symmetric, positive definite and banded. They can be utilized for solving a large system of equations. Given matrix $[A]$ is said to be positive definite only when the following condition is satisfied.

$$X^T A X > 0, \quad \text{for all non-zero column matrix of } \{x\}$$

For example, consider the following matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} & 0 \\ 0 & 0 & 0 & a_{54} & a_{55} & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & a_{66} \end{bmatrix}$$

$[A]$ is said to be a banded matrix with width $(2m+1)$ if all elements of a_{ij} for which $|i-j| > m$ are zero. For $m=1$, the band width of the previously mentioned matrix is 3.

1.5 STANDARD BEAM ELEMENT

A beam element is one of the basic elements to be used in the structural analysis of planar orthogonal frames. There are some sign conventions that need to be followed before deriving the stiffness matrix of the beam element. The anticlockwise end moment, joint rotation and joint moments are taken as positive; the upward force or displacement of the joint is positive; the force toward the right or axial displacement toward the right of the joint is positive; the upward end shear at the ends of the beam is positive; the right direction force at the ends of the beam is positive. Let us limit

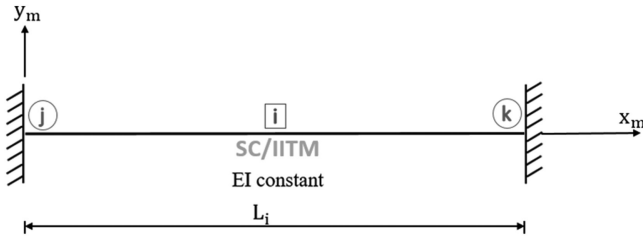


FIGURE 1.5 Standard beam element.

the discussion to planar orthogonal structures, which have either horizontal or vertical members.

A fixed beam is considered as a basic model. Consider a fixed beam undergoing deformation due to bending, neglecting the axial deformation. The standard fixed beam is shown in Figure 1.5. The two joints of the beam element are 'j' and 'k', and the length of the member is L_i . In terms of mathematical conditions, the considered beam element is fixed at nodes 'j' and 'k'; it has constant EI over its entire length. The left end of the beam is designated as the j th node and the right end of the beam is designated as the k th node. The member is designated as the i th member. (x_m, y_m) are local axes of the member. It is very important to note the axis system. The axis system is such that it has an origin at the j th end; x_m is directed toward the k th end. y_m is counterclockwise 90 degrees to the x_m axis. Therefore, the (x_m, y_m) plane defines the plane of bending the beam element. This is the conventional way of explaining the standard beam element. Let us neglect the axial deformation. For the stiffness method, one should identify possible displacements, both translational and rotational, at each end of the beam. So, the possible rotational and translational moments are shown in Figure 1.6. Suitable subscripts are used to denote the rotational and translational moments. Note the order by which the moments are marked, which allow the readers to understand the computer programs easily. The displacements at the j th end and the k th end are (θ_p, δ_r) and (θ_q, δ_s) respectively. All these displacements happen in the x_m, y_m plane and there is no out-of-plane bending.

Now, let us derive the stiffness coefficient, k_{ij} . According to classical definition, k_{ij} is the force in the i th degree-of-freedom by imposing unit displacement, which can

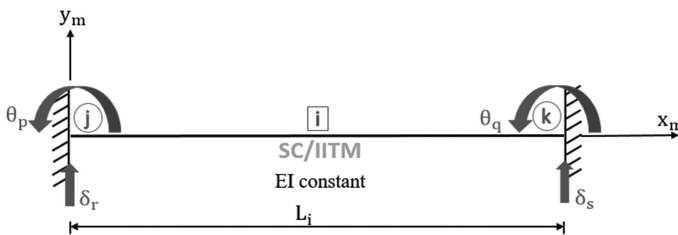


FIGURE 1.6 Rotational and translational moments in a beam element.