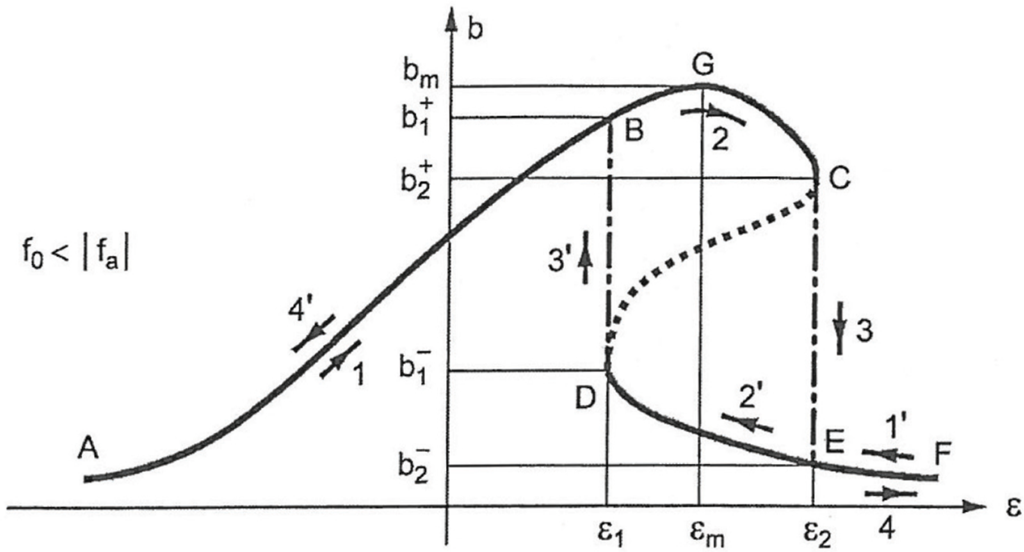
Mathematics and Physics for Science and Technology

Non-Linear Differential Equations and Dynamical Systems



L.M.B.C. Campos

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Non-Linear Differential Equations and Dynamical Systems

Mathematics and Physics for Science and Technology
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Director of the Center for Aeronautical
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Mathematics and Physics for Science and Technology

Volume IV

Ordinary Differential Equations with Applications to Trajectories and Vibrations

Book 5

Non-Linear Differential Equations and Dynamical Systems

By

L.M.B.C. Campos

Director of the Center for Aeronautical and Space Science and Technology

Lisbon University

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Preface

Volume IV ("*Ordinary Differential Equations with Applications to Trajectories and Oscillations*") is organized like the preceding three volumes of the series "*Mathematics and Physics Applied to Science and Technology*": (volume III) "*Generalized Calculus with Applications to Matter and Forces*"; (volume II) "*Elementary Transcendentals with Applications to Solids and Fluids*"; (volume I) "*Complex Analysis with Applications to Flows and Fields*". These three volumes on complex, transcendental, and generalized functions complete Topic A, "Theory of Functions and Potential Fields". Topic B, "Boundary and Initial-Value Problems" starts with volume IV on "*Ordinary Differential Equations with Applications to Trajectories and Oscillations*".

Volume IV consists of ten chapters: (i) the odd-numbered chapters present mathematical developments; (ii) the even-numbered chapters contain physical and engineering applications; (iii) the last chapter is a set of 20 detailed examples of (i) and (ii). The first book "*Linear Differential Equations and Oscillators*" of volume IV corresponds to the fourth book of the series and consists of chapters 1 and 2 of volume IV. The present second book, "*Non-linear Differential Equations and Dynamical Systems*", corresponds to the fifth book of the series and consists of chapters 3 and 4 of volume IV.

Chapter 1 described linear differential equations of any order with constant or power coefficients, and chapter 3 focuses on non-linear differential equations of the first-order, including variable coefficients, with extensions to differentials of order higher than the first and in more than two variables, and applications to the representation of vector fields by potentials. Chapter 2 discussed linear oscillators with damping/amplification and forcing with constant coefficients, including ordinary resonance. Chapter 4 considers linear oscillators with variable coefficients leading to parametric resonance, and non-linear oscillators leading to non-linear resonance amplitude jumps and hysteresis. Together with electromechanical dynamos and non-linear damping, these are examples of dynamical systems that may have bifurcations leading to chaotic motions.

Organization of the Book

The chapters are divided into sections and subsections, for example chapter 3, section 3.1, and subsection 3.1.1. The formulas are numbered by chapters in curved brackets; for example, (3.2) is equation 2 of chapter 3. When referring to volume I the symbol I is inserted at the beginning, for example: (i) chapter I.36, section I.36.1, subsection I.36.1.2; (ii) equation (I.36.33a). The final part of each chapter includes: (i) a conclusion referring to the figures as a kind of visual summary; (ii) the note(s), list(s), table(s), diagram(s), and classification(s) as additional support. The latter (ii) apply at the end of each chapter, and are numbered within the chapter (for example note N3.1, table T4.1); if there is more than one they are numbered sequentially (for example, notes N4.1 to N4.13). The chapter starts with an introductory preview, and related topics may be mentioned in the notes at the end. The "Series Preface" and "Mathematical Symbols" in the first book of volume IV are not repeated, and the "Physical Quantities", "References", and "Index" focus on the contents of the present second book of volume IV.

Acknowledgments

The fourth volume of the series justifies renewing some of the acknowledgments also made in the first three volumes, to those who contributed more directly to the final form of the book, namely, Ms. Ana Moura, L. Sousa, and S. Pernadas for help with the manuscripts; Mr. J. Coelho for all the drawings, and at last, but not least, to my wife as my companion in preparing this work.

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About the Author

L.M.B.C. Campos was born on March 28, 1950, in Lisbon, Portugal. He graduated in 1972 as a mechanical engineer from the Instituto Superior Tecnico (IST) of Lisbon Technical University. The tutorials as a student (1970) were followed by a career at the same institution (IST) through all levels: assistant (1972), assistant with tenure (1974), assistant professor (1978), associate professor (1982), chair of Applied Mathematics and Mechanics (1985). He has served as the coordinator of undergraduate and postgraduate degrees in Aerospace Engineering since the creation of the programs in 1991. He is the coordinator of the Scientific Area of Applied and Aerospace Mechanics in the Department of Mechanical Engineering. He is also the director and founder of the Center for Aeronautical and Space Science and Technology.

In 1977, Campos received his doctorate on "waves in fluids" from the Engineering Department of Cambridge University, England. Afterwards, he received a Senior Rouse Ball Scholarship to study at Trinity College, while on leave from IST. In 1984, his first sabbatical was as a Senior Visitor at the Department of Applied Mathematics and Theoretical Physics of Cambridge University, England. In 1991, he spent a second sabbatical as an Alexander von Humboldt scholar at the Max-Planck Institut fur Aeronomic in Katlenburg-Lindau, Germany. Further sabbaticals abroad were excluded by major commitments at the home institution. The latter were always compatible with extensive professional travel related to participation in scientific meetings, individual or national representation in international institutions, and collaborative research projects.

Campos received the von Karman medal from the Advisory Group for Aerospace Research and Development (AGARD) and Research and Technology Organization (RTO). Participation in AGARD/RTO included serving as a vice-chairman of the System Concepts and Integration Panel, and chairman of the Flight Mechanics Panel and of the Flight Vehicle Integration Panel. He was also a member of the Flight Test Techniques Working Group. Here he was involved in the creation of an independent flight test capability, active in Portugal during the last 30 years, which has been used in national and international projects, including Eurocontrol and the European Space Agency. The participation in the European Space Agency (ESA) has afforded Campos the opportunity to serve on various program boards at the levels of national representative and Council of Ministers.

His participation in activities sponsored by the European Union (EU) has included: (i) 27 research projects with industry, research, and academic institutions; (ii) membership of various Committees, including Vice-Chairman of the Aeronautical Science and Technology Advisory Committee; (iii) participation on the Space Advisory Panel on the future role of EU in space. Campos has been a member of the Space Science Committee of the European Science Foundation, which works with the Space Science Board of the National Science Foundation of the United States. He has been a member of the Committee for Peaceful Uses of Outer Space (COPUOS) of the United Nations. He has served as a consultant and advisor on behalf of these organizations and other institutions. His participation in professional societies includes member and vice-chairman of the Portuguese Academy of Engineering, fellow of the Royal Aeronautical Society, Astronomical Society and Cambridge Philosophical Society, associate fellow of the American Institute of Aeronautics and Astronautics, and founding and life member of the European Astronomical Society.

Campos has published and worked on numerous books and articles. His publications include 10 books as a single author, one as an editor, and one as a co-editor. He has published 152 papers (82 as the single author, including 12 reviews) in 60 journals, and 254 communications to symposia. He has served as reviewer for 40 different journals, in addition to 23 reviews published in *Mathematics Reviews*. He is or has been member of the editorial boards of several journals, including *Progress in Aerospace Sciences*, *International Journal of Aeroacoustics*, *International Journal of Sound and Vibration*, and *Air & Space Europe*.

Campos's areas of research focus on four topics: acoustics, magnetohydrodynamics, special functions, and flight dynamics. His work on acoustics has concerned the generation, propagation, and refraction of sound in flows with mostly aeronautical applications. His work on magnetohydrodynamics has concerned magneto-acoustic-gravity-inertial waves in solar-terrestrial and stellar physics. His developments on special functions have used differintegration operators, generalizing the ordinary derivative and primitive to complex order; they have led to the introduction of new special functions. His work on flight dynamics has concerned aircraft and rockets, including trajectory optimization, performance, stability, control, and atmospheric disturbances.

The range of topics from mathematics to physics and engineering fits with the aims and contents of the present series. Campos's experience in university teaching and scientific and industrial research has enhanced his ability to make the series valuable to students from undergraduate level to research level.

Campos's professional activities on the technical side are balanced by other cultural and humanistic interests. Complementary non-technical interests include classical music (mostly orchestral and choral), plastic arts (painting, sculpture, architecture), social sciences (psychology and biography), history (classical, renaissance and overseas expansion) and technology (automotive, photo, audio). Campos is listed in various biographical publications, including *Who's Who in the World* since 1986, *Who's Who in Science and Technology* since 1994, and *Who's Who in America* since 2011.

Physical Quantities

The location of first appearance is indicated, for example "2.7" means 'section 2.7' "6.8.4" means "subsection 6.8.4", "N8.8" means "note 8.8", and "E10.13.1" means "example 10.13.1".

1 Small Arabic Letters

a — moment arm: 4.7.3

a_n — coefficients of a series: 4.4.10

h — amplitude of excitation of parametric resonance: 4.3.2

q — non-linearity parameter for pendular motion: 4.7.13

2 Capital Arabic Letters

A — anharmonic factor: 4.4.10

$\vec{A}$ — vector potential: 3.9.9

D — drag force: 4.8.13

F — force: 3.9.11

I — moment of inertia: 4.7.3

J — electric current: 4.7.1

L — induction of a coil or self: 4.4.7.1

— lift force: 4.8.13

Q — heat: 3.9.11

R — electrical resistance: 4.7.1

S — entropy: 3.9.11

— dynamo parameter: 4.7.3

T — temperature: 3.9.11

— thrust: 4.8.13

U — internal energy: 3.9.11

W — work: 3-9.11
— weight: 4.8.13

3 Small Greek Letters

α — angle of cylindrical helix: 3.9.5
— coefficient of the cubic term of the quartic potential: 4.5.1
β — coefficient of the quartic term in the biquadratic potential: 4.4.4
γ — dimensionless non-linearity parameter: 4.4.8
ω_0 — natural frequency: 4.3.2
ω_ϵ — excitation frequency of parametric resonance: 4.3.2
ψ — non-linearity parameter for a quartic potential: 4.6.1

4 Capital Greek Letters

Φ — scalar potential: 3.9.9
— mechanical potential energy: 4.4.2
Θ — Euler potential: 3.9.9
Ω — angular velocity of rotation: 4.7.1
Ψ — Euler potential 3.9.9
Ξ — Clebsch potential: 3.9.9

3

Differentials and First-Order Equations

If a differential equation involves derivatives with regard to one (several) variables, it is an ordinary (partial) differential equation; ordinary differential equations will be considered in the present volume IV, and partial equations in the next volume V. In the dynamical example of the second-order linear system with constant (variable) coefficients (chapter 2(4)), the independent variable is time only, leading to an ordinary differential equation. If the particle or system has one (several) degrees of freedom, for example its position is specified by one (several) coordinates, then the motion is specified by an equation (system of equations); thus the solution of an ordinary differential equation (system of N simultaneous ordinary differential equations), is a function (N functions) of one variable. The derivative of highest order appearing in a differential equation specifies the order of that equation; for example in the dynamics of a particle with one degree of freedom the ordinary differential equation is of order two, because the highest order derivative of position with regard to time is the second, specifying the acceleration. Not all differential equations are readily solvable; this is why it is important to classify them into (i) ordinary or partial, (ii) equations or systems, (iii) order one or higher, and (iv) particular standards or sub-types. The aim is to identify classes of differential equations that have certain properties, making possible specific methods of solution. The starting point was the ordinary linear differential equations of any order N with constant or homogeneous coefficients that have a characteristic polynomial of degree N (chapter 1); by analogy other equations with one characteristic polynomial. Considering (i) linear differential equations with variable coefficients or (ii) non-linear differential equations, it is simpler to start with those of first order (chapter 3).

After the discussion of some general properties of first-order differential equations (section 3.1), methods of solution are presented for eight classes or standards: (i) the separable equation (section 3.2), for which the derivative is the ratio of two separate functions of the independent and dependent variable, which can be solved by quadratures, that is the solution reduces to one (or two) integration(s); (ii) the linear unforced equation, a particular case of a separable equation and hence always solvable (section 3.2); (iii) the solution of the linear forced equation is obtained from its unforced part by the method of variation of parameters (section 3.3), used before (notes 1.2–1.5 and section 2.9); (iv) the Bernoulli equation, which is non-linear but can be transformed to the linear type, and hence is also always solvable (section 3.4); (v) the Riccati equation, which, while there is no known general solution, is a non-linear generalization of the Bernoulli equation (section 3.6), and it can be shown (section 3.5) that if one/two/three particular integrals are known, then the general integral can be obtained via two/one/zero quadratures.

A sixth method is (vi) a change of variable that may render a differential equation solvable; for example the homogeneous first-order differential equation, in which the dependent and independent variable appear only through their ratio, can be transformed into a separable type, and hence integrated in all cases (section 3.7). Proceeding from the first six to the next two classes, any first-order differential equation is equivalent to a first-order differential in two variables (section 3.8) leading to two cases: (vii) if it satisfies a condition of integrability, it is an exact differential that is the differential of a function, that equated to an arbitrary constant supplies the general integral; (viii) if the integrability condition is not satisfied, the first-order differential equation is equivalent to an inexact differential, and it is not the differential of a function, though it becomes so when multiplied by an integrating factor that always exists. This is no longer the case for a first-order differential in three variables (section 3.9), when there are three possibilities: (i/ii) exact (inexact) differential satisfying (not satisfying) an integrability condition and not needing (needing) an integration factor, as for a first-order differential in two variables; (ii/iii) the existence (non-existence) of an integrating factor depends on the satisfaction (non-satisfaction) of a more general integrability condition. The first-order differential may be extended to: (i) more than three variables (notes 3.1–3.15); (ii) homogeneous differentials (notes 3.16–3.20); (iii) higher-order differentials (notes 3.21–3.24).

3.1 General Properties of First-Order Equations

The general properties include a classification of solutions (subsection 3.1.1) and the determination of the arbitrary constant in the general integral (subsection 3.1.2). The differential equation is considered to be solved when it is reduced to an integration (subsection 3.1.3) or quadrature (subsection 3.1.4) that may be elementary or not. The general integral of a first-order differential equation is one family (or several families) of integral curves [subsection 3.1.1 (3.1.5)], and each value of the arbitrary constant corresponds to one curve of the (of each) family.

3.1.1 General Integral and Integral Curves

An ordinary differential equation of the first order is a relation between a function $y(x)$, its variable x, and its first derivative y':

$$y' \equiv \frac{dy}{dx}: \qquad F\left(x, y; y'\right) = 0. \qquad \text{(3.1a, b)}$$

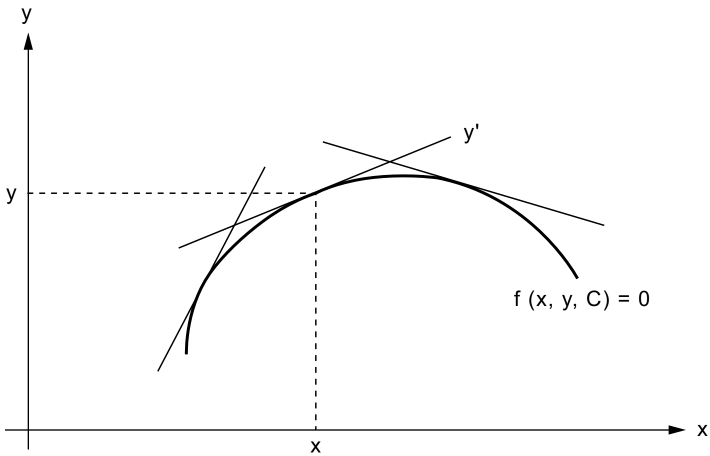

FIGURE 3.1
An explicit first-order differential equation specifies one slope at each point of the plane and its solution is an integral curve with the corresponding tangents.

The function $y(x)$ is the **dependent variable**, and the variable x is the **independent variable**. A solution of the differential equation is a function $y(x)$ that satisfies it (3.1b) when substituted together with its first derivative (3.1a). The solution of the differential equation (3.1a, b), may be given a geometric interpretation (Figure 3.1): the first-order ordinary differential equation (3.1b) specifies at each point of the (x, y)-plane one (or more) slopes y'. A solution must be a curve, whose tangent at the point (x, y), has a slope y' and is called an **integral curve**. Since it specifies one (or more) slopes or tangents at each point, the most general solution is a **family of curves** (Figure 3.2):

$$f(x, y, C) = 0, \tag{3.2}$$

involving a parameter C, and this is the **general integral** of the first-order ordinary differential equation (3.1b). Giving to the parameter C a particular value $C = C_1$, specifies a particular curve, or a **particular integral** of the differential equation. Thus *an ordinary differential equation of first-order (3.1b) relates a family of curves (3.2) to the slopes (3.1a) of its tangents at each point (x, y).* The cases in which the tangent is not unique may lead to more than one family of integral curves or to integral curves with multiple points, the simplest being as double points (Figure 3.3). If the family of integral curves has an envelope this a singular or special integral (section 1.1); the locus of double points or other loci may also lead to special integrals (sections 5.1–5.4).

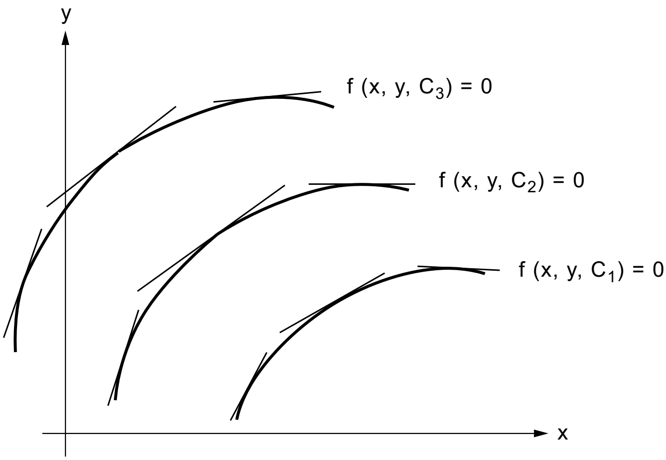

FIGURE 3.2
Each integral curve is a particular integral of the first-order differential equation for a particular value of the constant of integration; the general integral is the family of curves with the value of the arbitrary constant identifying each curve.

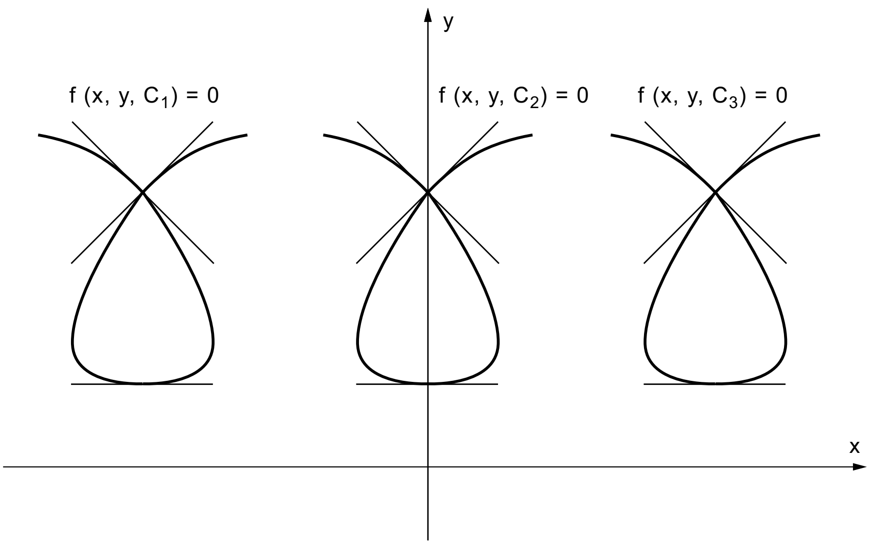

FIGURE 3.3
If there are multiple slopes at some points these are multiple points, for example double points through which the curve crosses itself in two different directions.

3.1.2 Constant of Integration and Boundary Condition

The preceding geometric property may be re-stated in analytical terms: *the general integral (3.2) of the first-order ordinary differential equation (3.1a, b), involves one arbitrary constant of integration* C, *and includes, for particular values of* C, *all particular integrals*. A rigorous proof will be given in section 9.1, and here is provided a heuristic explanation; there may also be singular integrals not included in the general integral (chapter 5). If the general integral (3.2) is differentiated with regard to x it yields:

$$\frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} = 0, \tag{3.3}$$

where the partial derivatives may depend on the constant of integration C. Taking C as a parameter, and eliminating it between (3.2) and (3.3), leads back to the differential equation of first order (3.1b). If the general integral (3.2) involved more than one constant of integration, say two C, C_o, then after elimination of C with (3.3) the remaining constant C_o would have appeared in (3.1b). If the general integral (3.2) involved no constant of integration, then elimination with (3.3) for (3.1b) would generally be impossible, unless the two equations were redundant, that is the case of singular or special (sections 5.1–5.4) integrals. Excepting the case of special integrals, the general integral (3.2) of a first-order (3.1a) differential equation (3.1b) involves one arbitrary constant of integration that can be determined from a boundary condition:

$$f(x_0, y_0, C) = 0, \tag{3.4}$$

selecting the integral curve passing through a given point, assuming there is one and only one.

3.1.3 Algebra, Analysis, and Differential Equations

A differential equation may be considered a **third-level problem** in the sense that it is considered solved when it reduces to a problem of analysis, like an integration. For example, the simplest ordinary differential equation of first order (3.5b):

$$f \in \mathcal{E}(|R), \qquad y' = f(x), \qquad y = \int^x f(\xi)\, d\xi + C, \tag{3.5a–c}$$

where f(x) is an integrable function (3.5a), has general integral (3.5c) obtained by a single integration or **quadrature**, involving one arbitrary constant of integration C. Performing the integration, regardless of whether it is elementary or not, is a **second-level problem of analysis**. The solution of the problem of analysis may lead to a general integral in implicit form (3.2) or in explicit form (3.6) involving the roots of an algebraic equation, such as the characteristic polynomial of a linear differential equations with constant (homogeneous) coefficients [sections (1.3–1.5 (1.6–1.8)]. Finding the roots or obtaining one or more explicit solution(s) in the form:

$$y = y(x; C), \tag{3.6}$$

is a **first-level problem of algebra.** Thus there is a four-level **hierarchy of problems:** *(i) algebraic at level one; (ii) analysis at level two involving integrations or quadratures; (iii) differential equations at level three, involving relations between functions and derivatives; and (iv) variational problems at level four whose solution lead to differential equations.*

3.1.4 Solution by Quadratures and Indefinite Integrals

A differential equation is solved by **quadratures** when its solution is expressed by an integral, for example (3.5c) is the solution by quadratures of the differential equation (3.5b) where the function f is integrable (3.5a). The general integral (3.5b) specifies a family of integral curves, differing from each other by a translation along the y-axis (Figure 3.3); hence only one curve passes through each point (x_0, y_0):

$$y - y_0 = \int^{x} f(\xi)\,d\xi - \int^{x_0} f(\xi)\,d\xi = \int_{x_0}^{x} f(\xi)\,d\xi, \tag{3.7a}$$

thus the indefinite integral in the general solution (3.5c) is replaced by a definite integral (3.7a) in the particular solution corresponding to the constant of integration:

$$C \equiv y_0 - \int^{x_0} f(\xi)\,d\xi. \tag{3.7b}$$

An example of first-order ordinary differential is the case of linear slope (3.8a):

$$y' = 2x, \qquad y = x^2 + C, \tag{3.8a, b}$$

whose general integral (3.8b) is a family of parabolas (Figure 3.4) cutting the y-axis $x = 0$ at $y = C$; for example the parabola through the origin is the particular integral $C = 0$. The solution of a first-order differential equation may consist of N families of functions if it is of degree N (subsection 5.4.4); next is given an example of degree $N = 2$ (subsection 3.1.5).

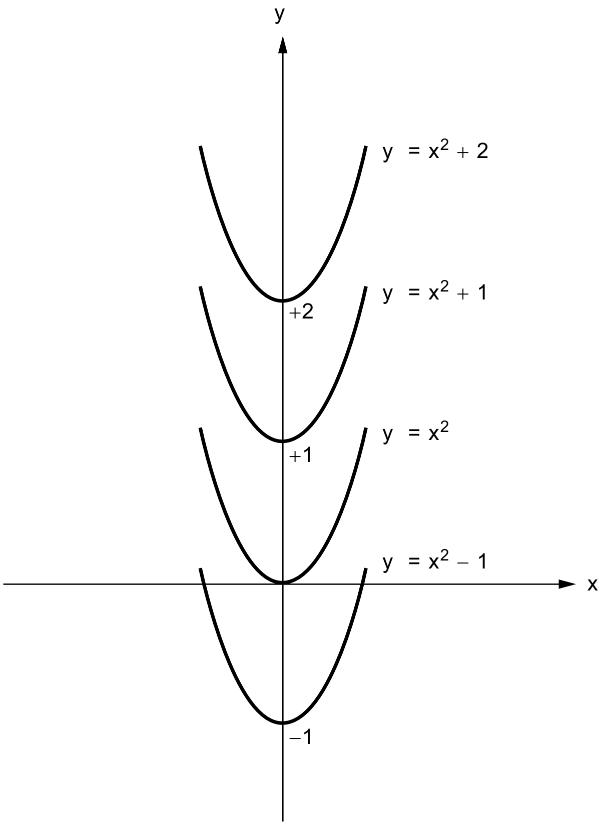

FIGURE 3.4
Through the regular points of a differential equation passes only one integral curve, for example the family of parabolas symmetric relative to the y-axis with unit curvature at the apex has particular integrals identified by the intersection with the y-axis.

3.1.5 Double Points of Quadratic Differential Equation

A first-order differential equation is **explicit** if it can be solved for the slope (3.9a), and is of **degree M** if is a polynomial of the derivatives (3.9b):

$$y' = f(x,y), \qquad \prod_{m=1}^{M}\left[y' - f_m(x,y)\right] = 0. \tag{3.9a, b}$$

A simple case of quadratic differential equation is (3.10a):

$$0 = y'^2 - \left[f_+(x) + f_-(x)\right]y' + f_+(x)f_-(x) = \left[y' - f_+(x)\right]\left[y' - f_-(x)\right], \tag{3.10a}$$

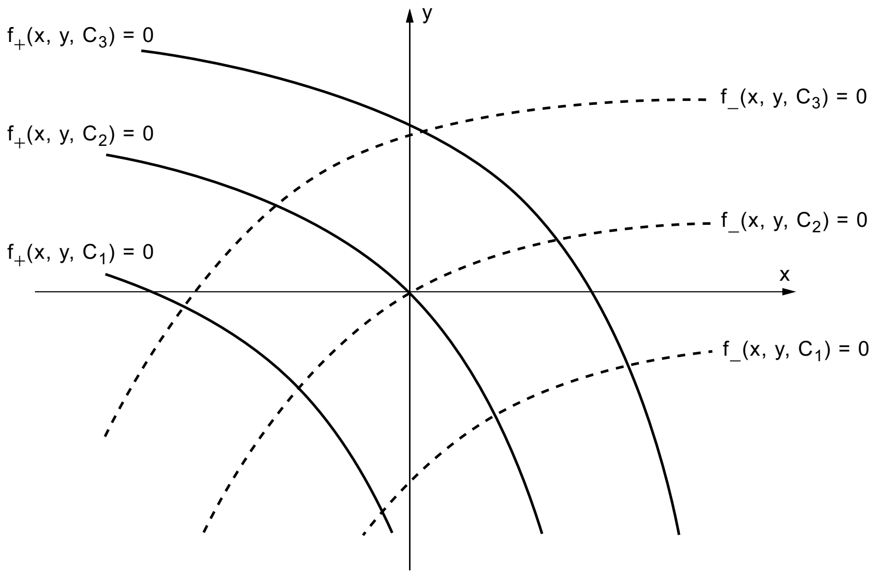

FIGURE 3.5
If there are multiple slopes at all points, then the general integral consists of several families of curves, for example two for a first-order differential equation quadratic on the slope.

whose general integral is two families of curves (3.10b, c):

$$y_\pm(x) = C + \int^x f_\pm(\xi)\, d\xi. \tag{3.10b, c}$$

For each value of the constant C there are two curves, and thus the regular points in the plane (Figure 3.5) are double points; unlike those in Figure 3.3 they do not arise from the curve crossing itself. The general integral is (3.10d):

$$0 = \left(y_+(x) - \int^x f_+(\xi)\, d\xi - C \right)\left(y_-(x) - \int^x f_-(\xi)\, d\xi - C \right), \tag{3.10d}$$

because it is satisfied by either of (3.10b, c); there is only one constant of integration in (3.10b, c) ≡ (3.10d) because the ordinary differential equation (3.10a) is of the first-order.

3.2 Integration by Quadratures of a Separable Equation

Integration by quadratures is possible for a separable equation (subsection 3.2.1) for which several examples can be given, including the particular case of the linear unforced equation (subsection 3.2.2).