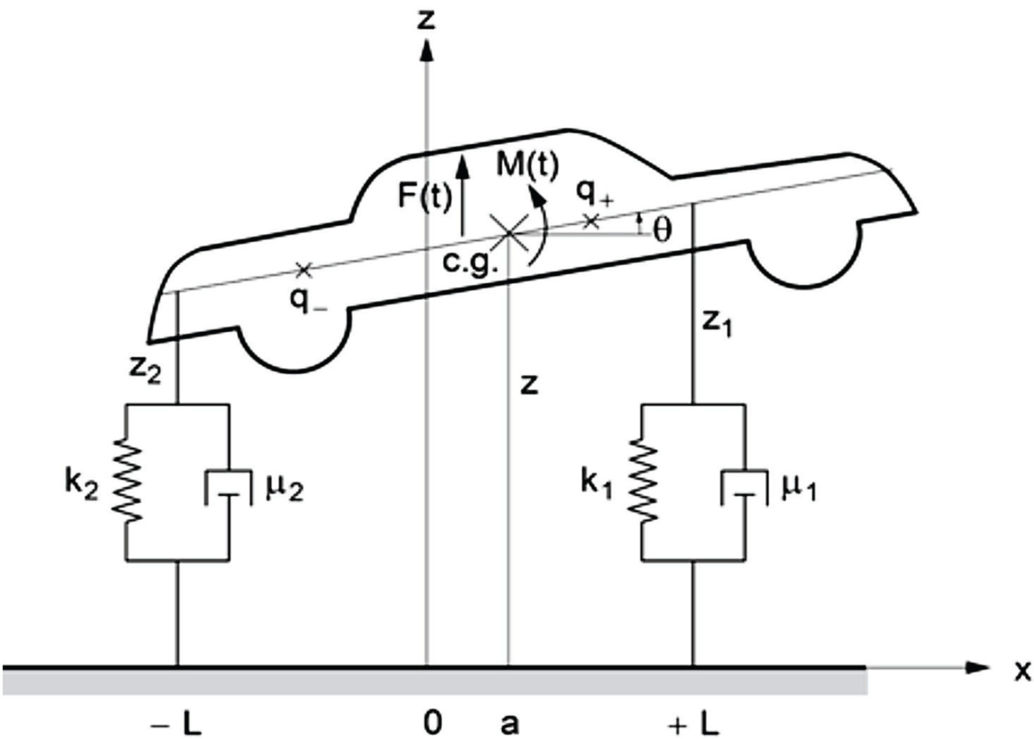
Mathematics and Physics for Science and Technology

Simultaneous Systems of Differential Equations and Multi-Dimensional Vibrations



L.M.B.C. Campos

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Simultaneous Systems of Differential Equations and Multi-Dimensional Vibrations

Mathematics and Physics for Science and Technology

Series Editor: L.M.B.C. Campos

Director of the Center for Aeronautical
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Mathematics and Physics for Science and Technology

Volume IV

Ordinary Differential Equations with Applications to Trajectories and Vibrations

Book 7

Simultaneous Systems of Differential Equations and Multi-Dimensional Vibrations

By

L.M.B.C. Campos

Director of the Center for Aeronautical and Space Science and Technology

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to Leonor Campos

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Preface

Volume IV (*Ordinary Differential Equations with Applications to Trajectories and Oscillations*) is organized like the preceding three volumes of the series *Mathematics and Physics Applied to Science and Technology*: (volume III) *Generalized Calculus with Applications to Matter and Forces*; (volume II) *Transcendental Representations with Applications to Solids and Fluids*; and (volume I) *Complex Analysis with Applications to Flows and Fields*. The first book, *Linear Differential Equations and Oscillators*; the second book, *Non-Linear Differential Equations and Dynamical Systems*; and the third book, *Higher-Order Differential Equations and Elasticity* of volume IV, correspond, respectively, to books four to six of the series, and consist of chapters 1 to 6 of volume IV. The present book, *Simultaneous Differential Equations and Multidimensional Vibrations*, is the fourth book of volume IV and the seventh book of the series; it consists of chapters 7 and 8 of volume IV.

Chapters 1, 3, and 5 focus on single differential equations, starting with (i) linear differential equations of any order with constant or homogeneous coefficients; and continuing with (ii) non-linear first-order differential equations, including variable coefficients and (iii) non-linear differential second-order and higher-order equations. Chapter 7 discusses simultaneous systems of ordinary differential equations, and focuses mostly on the cases that have a matrix of characteristic polynomials, namely linear systems with constant or homogeneous power coefficients. The method of the matrix of characteristic polynomials also applies to simultaneous systems of linear finite difference equations with constant coefficients.

Chapters 2 and 4 focus on, respectively, linear and non-linear oscillators described by second-order differential equations, like the elastic bodies without stiffness in the chapter 6; the elastic bodies with stiffness in chapter 6 lead to fourth-order differential equations equivalent to coupled second-order systems. Chapter 8 considers linear multi-dimensional oscillators with any number of degrees of freedom, including damping, forcing, and multiple resonance. The discrete oscillators may be extended from a finite number of degrees-of-freedom to infinite chains. The continuous oscillators correspond to waves in homogeneous or inhomogeneous media, including elastic, acoustic, electromagnetic, and water surface waves. The combination of propagation and dissipation leads to the equations of mathematical physics.

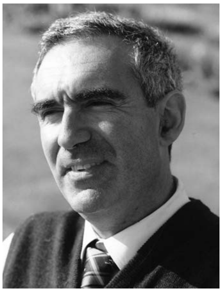
Organization of the Contents

Volume IV consists of ten chapters: (i) the odd-numbered chapters present mathematical developments; (ii) the even-numbered chapters contain physical applications; (iii) the last chapter is a set of 20 detailed examples of (i) and (ii). The chapters are divided into sections and subsections, for example, chapter 7, section 7.1, and subsection 7.1.1. The formulas are numbered by chapters in curved brackets; for example, (8.2) is equation 2 of chapter 8. When referring to volume I the symbol I is inserted at the beginning, for example: (i) chapter I.36, section I.36.1, subsection I.36.1.2; (ii) equation (I.36.33a). The final part of each chapter includes: (i) a conclusion referring to the figures as a kind of visual summary; (ii) the notes, lists, tables, diagrams, and classifications as additional support. The latter (ii) appear at the end of each chapter, and are numbered within the chapter (for example, diagram—D7.2, note—N8.10, table—T7.4); if there is more than one diagram, note, or table, they are numbered sequentially (for example, notes—N7.1 to N7.55). The chapter starts with an introductory preview, and related topics may be mentioned in the notes at the end. The sections "Series Preface," and "Mathematical Symbols" from the first book of volume IV are not repeated. The sections "Physical Quantities," "References," and "Index" focus on the contents of the present fourth book of volume IV.

Acknowledgments

The fourth volume of the series justifies renewing some of the acknowledgments made in the first three volumes. Thank you to those who contributed more directly to the final form of this volume: Ms. Ana Moura, L. Sousa, and S. Pernadas for help with the manuscripts; Mr. J. Coelho for all the drawings; and at last, but not least, to my wife, my companion in preparing this work.

About the Author



L.M.B.C. Campos was born on March 28, 1950, in Lisbon, Portugal. He graduated in 1972 as a mechanical engineer from the Instituto Superior Tecnico (IST) of Lisbon Technical University. The tutorials as a student (1970) were followed by a career at the same institution (IST) through all levels: assistant (1972), assistant with tenure (1974), assistant professor (1978), associate professor (1982), chair of Applied Mathematics and Mechanics (1985). He has served as the coordinator of undergraduate and postgraduate degrees in Aerospace Engineering since the creation of the programs in 1991. He is the coordinator of the Scientific Area of Applied and Aerospace Mechanics in the Department of Mechanical Engineering. He is also the director and founder of the Center for Aeronautical and Space Science and Technology.

In 1977, Campos received his doctorate on "waves in fluids" from the Engineering Department of Cambridge University, England. Afterwards, he received a Senior Rouse Ball Scholarship to study at Trinity College, while on leave from IST. In 1984, his first sabbatical was as a Senior Visitor at the Department of Applied Mathematics and Theoretical Physics of Cambridge University, England. In 1991, he spent a second sabbatical as an Alexander von Humboldt scholar at the Max-Planck Institut fur Aeronomic in Katlenburg-Lindau, Germany. Further sabbaticals abroad were excluded by major commitments at the home institution. The latter were always compatible with extensive professional travel related to participation in scientific meetings, individual or national representation in international institutions, and collaborative research projects.

Campos received the von Karman medal from the Advisory Group for Aerospace Research and Development (AGARD) and Research and Technology Organization (RTO). Participation in AGARD/RTO included serving as a vice-chairman of the System Concepts and Integration Panel, and chairman of the Flight Mechanics Panel and of the Flight Vehicle Integration Panel. He was also a member of the Flight Test Techniques Working Group. Here he was involved in the creation of an independent flight test capability, active in Portugal during the last 30 years, which has been used in national and international projects, including Eurocontrol and the European Space Agency. The participation in the European Space Agency (ESA) has afforded Campos the opportunity to serve on various program boards at the levels of national representative and Council of Ministers.

His participation in activities sponsored by the European Union (EU) has included: (i) 27 research projects with industry, research, and academic institutions; (ii) membership of various Committees, including Vice-Chairman of the Aeronautical Science and Technology Advisory Committee; (iii) participation on the Space Advisory Panel on the future role of EU in space. Campos has been a member of the Space Science Committee of the European Science Foundation, which works with the Space Science Board of the National Science Foundation of the United States. He has been a member of the Committee for Peaceful Uses of Outer Space (COPUOS) of the United Nations. He has served as a consultant and advisor on behalf of these organizations and other institutions. His participation in professional societies includes member and vice-chairman of the Portuguese Academy of Engineering, fellow of the Royal Aeronautical Society, Astronomical Society and Cambridge Philosophical Society, associate fellow of the American Institute of Aeronautics and Astronautics, and founding and life member of the European Astronomical Society.

Campos has published and worked on numerous books and articles. His publications include 10 books as a single author, one as an editor, and one as a co-editor. He has published 152 papers (82 as the single author, including 12 reviews) in 60 journals, and 254 communications to symposia. He has served as reviewer for 40 different journals, in addition to 23 reviews published in *Mathematics Reviews*. He is or has been member of the editorial boards of several journals, including *Progress in Aerospace Sciences*, *International Journal of Aeroacoustics*, *International Journal of Sound and Vibration*, and *Air & Space Europe.*

Campos's areas of research focus on four topics: acoustics, magnetohydrodynamics, special functions, and flight dynamics. His work on acoustics has concerned the generation, propagation, and refraction of sound in flows with mostly aeronautical applications. His work on magnetohydrodynamics has concerned magneto-acoustic-gravity-inertial waves in solar-terrestrial and stellar physics. His developments on special functions have used differintegration operators, generalizing the ordinary derivative and primitive to complex order; they have led to the introduction of new special functions. His work on flight dynamics has concerned aircraft and rockets, including trajectory optimization, performance, stability, control, and atmospheric disturbances.

The range of topics from mathematics to physics and engineering fits with the aims and contents of the present series. Campos's experience in university teaching and scientific and industrial research has enhanced his ability to make the series valuable to students from undergraduate level to research level.

Campos's professional activities on the technical side are balanced by other cultural and humanistic interests. Complementary non-technical interests include classical music (mostly orchestral and choral), plastic arts (painting, sculpture, architecture), social sciences (psychology and biography), history (classical, renaissance and overseas expansion) and technology (automotive, photo, audio). Campos is listed in various biographical publications, including *Who's Who in the World* since 1986, *Who's Who in Science and Technology* since 1994, and *Who's Who in America* since 2011.

Physical Quantities

The location of first appearance is indicated, for example "2.7" means "section 2.7," "6.8.4" means "subsection 6.8.4," "N8.8" means "note 8.8," and "E10.13.1" means "example 10.13.1."

1 Small Arabic Letters

b — width of a water channel: N7.12

c — phase speed of waves: N8.1

c_e — speed of transversal waves in an elastic string: N7.9

c_{em} — speed of electromagnetic waves: N7.15

c_ℓ — speed of longitudinal waves in an elastic rod: N7.11

c_s — adiabatic sound speed: N7.16

c_t — speed of torsional waves along an elastic rod: N7.10

c_w — speed of water waves along a channel: N7.13

$\vec{e}_i$ — non-unit base vector: N8.8

f_r — reduced external force: 8.2.1

g — determinant of the covariant metric tensor: N8.8

g_ℓ — reduced modal force: 8.1.1

g_{ij} — covariant metric tensor: N8.8

g^{ij} — contravariant metric tensor: N8.8

h — depth of water channel: N7.12

h_i — friction force vector: 8.1.1

—scale factors: N8.8

j_i — restoring force vector: 8.1.1

k — wavenumber: N7.25, N8.2

k_{rs} — resilience matrix: 8.1.2

ℓ — lengthscale for horns: N7.28

m_{rs} — mass matrix: 8.2.1

q_ℓ — modal coordinates: 8.2.4

z — specific impedance: N7.47

2 Capital Arabic Letters

A — cross-sectional area of a horn: N7.16
— admittance: 8.9.1
A_n — modal amplitudes: 8.8.5
B_{mn} — terms in the modal matrix: 8.8.5
E_v — kinetic energy: 6.8.5
F_ℓ — reduced external force: 8.2.1
G_ℓ — modal forces: 8.2.9
$J^\pm$ — invariants of acoustic horns: N7.28
K — reduced wavenumber: N7.29
$\vec{K}$ — enhanced wavenumber: N7.33
L_A — lengthscale of variation of the cross-sectional area: N7.17
L_b — lengthscale of variation of the width of a water channel: N7.12
L_c — lengthscale of variation of the torsional stiffness: N7.10
L_E — lengthscale of change of the Young modulus: N7.11
L_T — lengthscale of change of tension: N7.9
L_ε — lengthscale of variation of the dielectric permittivity: N7.15
L_μ — lengthscale of variation of the magnetic permeability: N7.15
M_n— mass of n-th element of a radioactive disintegration chain: 8.7.1
M_{rs} — modal matrix: 8.2.13
N — number of particles: 5.5.17
N_{rs} — undamped modal matrix: 8.2.14
P — reduced pressure perturbation spectrum: N7.23
P_{2N} — dispersion polynomial: 8.2.2
P_{ij} — dispersion matrix: 8.2.2
$Q_{r\ell}$ — transformation matrix: 8.2.7
R — electrical resistance: 8.9.2
— reflection coefficient: N7.47
S — surface adsorption coefficient: N7.48
S_{ij} — scattering matrix: N7.53
T — transmission coefficient: N7.51
V — reduced velocity perturbation spectrum: N7.23
Y — inductance: 8.9.1

Z — impedance: 8.9.1
$\tilde{Z}$ — overall impedance: N7.52
Z_0 — impedance of a plane sound wave: N7.47

3 Small Greek Letters

α — diffusivity: N8.1
β — amplification/attenuation factor: N7.50
—potential: N8.1
δ_{ij} — identity matrix: 8.1.4
λ_ℓ — modal dampings: 8.2.5
λ_{rs} — damping matrix: 8.2.1
μ_{rs} — kinematic friction matrix: 8.1.3
ν_n — disintegration rate of the mass of the *n*-th element in a chain: 8.7.1
σ — mass density per unit length: N7.8
ω_ℓ —modal frequencies: 8.2.4
$\bar{\omega}_\ell$ — oscillation frequency of modes: 8.2.6
ω^2_{rs} — oscillation matrix: 8.2.1

4 Capital Greek Letters

Φ — primal wave variable: N7.19
Φ_m — mechanical potential energy: 8.1.2
Ψ — dual wave variable: N7.19
Ψ_m — dissipation by mechanical friction: 8.1.2

7

Simultaneous Differential Equations

An ordinary differential equation of order N relates one independent and one dependent variable, with a set of derivatives of the latter with regard to the former up to and including the order N. In the case of a system of simultaneous differential equations, there are M dependent variables and only one dependent variable (section 7.1). The derivatives of the former with regard to the latter appear in a set of simultaneous equations, which cannot be separated for each dependent variable (at least without some manipulation).

The simplest case of a system of M simultaneous ordinary differential equations is an autonomous system (section 7.1) in which the first-order derivative of each of the M dependent variables is an explicit function of all the dependent variables, and does not involve any derivatives. The autonomous system of ordinary differential equations is related to the problem of finding the family of curves tangent to a given continuous vector field (section 7.2), which always has a solution. In contrast, the problem of finding a family of hypersurfaces orthogonal to a given continuous vector field leads to a differential of first order in M variables, which has (does not have) a solution (sections 3.8–3.9 and notes 3.1–3.15) if the differential is exact or has an integrating factor (is inexact and has no integrating factor). Any system of M simultaneous ordinary differential equations can be transformed to an autonomous system (section 7.1); this leads to a method to eliminate a system of M simultaneous ordinary differential equations in M dependent variables into a single ordinary differential equation of order N for one of the dependent variables (section 7.3). This specifies the order N of the system of M simultaneous ordinary differential equations that equals (is less than) the sum of the higher-order derivatives in all M equations if the system is independent (redundant).

The most important class of single (simultaneous) ordinary differential equation(s) is the linear case with constant coefficients [sections 1.3–1.5 (7.4–7.5)] to which can be reduced the case of power coefficients [sections 1.6–1.8 (7.6–7.7)]. In all of the cases, the solution is determined by the roots of a single characteristic polynomial. For a single linear ordinary differential equation with constant coefficients, the characteristic polynomial is the differential operator acting on the dependent variable (section 1.3). In the case of a simultaneous system of M equations with M dependent variables, there is (section 7.4) an $M \times M$ matrix of linear operators with constant coefficients, and its determinant specifies the characteristic polynomial of the system of simultaneous differential equations, whose order N is the degree of the characteristic polynomial. The solutions corresponding to single or multiple, real or complex roots, are similar for a single (set of simultaneous) differential equation(s), and each dependent variable is a linear combination of them, with coefficients determined by the initial conditions; the number of independent and compatible initial conditions needed to specify a unique solution is equal the order of the single (set of simultaneous) differential equation(s). This implies that in the solution of a linear simultaneous system of M ordinary differential equations with constant (homogeneous) coefficients without forcing [section 7.4 (7.5)]: (i) each of the M dependent variables is a linear combination of N linearly independent particular integrals specified by the roots of the characteristic polynomial; (ii) there are N arbitrary constants of integration, for example, those in the first dependent variable; (iii) the coefficients in all other dependent variables involve the same N arbitrary constants of integration, in a way that is compatible with substitution back into the system of simultaneous ordinary differential equations.

The case of a single (set of simultaneous) linear ordinary differential equation(s) with constant coefficients and a forcing term, can be considered using [sections 1.4–1.5 (7.5)] the characteristic polynomial directly or as an inverse operator. A characteristic polynomial also exists for a single (set of simultaneous) linear ordinary differential equation(s) with power coefficients, leading to similar methods of solution [sections 1.6–1.8 (7.6–7.7)]. The characteristic polynomial also exists for a single (set of simultaneous) linear finite difference equation(s), again leading to similar methods of solution [section(s) 1.9 (7.8–7.9)].

7.1 Reduction of General to Autonomous Systems

A general system of M simultaneous ordinary differential equations (subsection 7.1.2) can be reduced to an autonomous system of differential equations (subsection 7.1.1).

7.1.1 Autonomous System of Differential Equations

A **generalized autonomous system of order** M of ordinary differential equations (standard CXXI) has one independent variable x, and M dependent variables (7.1a) whose first-order derivatives (7.1b) depend explicitly only on all the dependent variables and the dependent variable:

$$m = 1,\dots,M: \qquad y'_m(x) \equiv \frac{dy_m}{dx} = Y_m\left(x; y_1,\dots.,y_M\right); \qquad (7.1\text{a, b})$$

This excludes the appearance of any derivatives of any order on the right-hand side (r.h.s.) of (7.1b). *An ordinary differential equation (1.1a, b) of order N with independent (dependent) variable x(y), which is explicit in the highest-order derivative (7.2):*

$$y^{(N)}(x) = G\left(x; y, y', \ldots, y^{(N-1)}\right), \tag{7.2}$$

can be transformed into (standard CXXI) an autonomous system of order N:

$$r = 1, \ldots, M-1: \qquad y_r(x) \equiv y^{(r)}(x), \qquad y'_{N-1}(x) = G\left(x; y_1, \ldots., y_{N-1}\right), \tag{7.3a–c}$$

by: (i) defining $N-1$ new dependent variables (7.3a, b) as the derivatives of the dependent variable up to the order $N-1$; (ii) rewriting the original differential equation (7.2) in autonomous form (7.3c). For example, the third-order differential equation (7.4a) explicit for the third-order derivative:

$$y''' = F\left(x; y, y', y''\right): \qquad y' \equiv y_1, \quad y'_1 \equiv y_2 = y'', \quad y'_2 = y''' = F\left(x; y, y_1, y_2\right), \tag{7.4a–d}$$

can be reduced to the autonomous system (7.4b–d) also of order 3. The preceding method of reduction to an autonomous system of differential equations applies both [subsection 7.1.1 (7.1.2)] to a single (set of simultaneous) differential equation(s) of any order(s).

7.1.2 General System of Simultaneous Differential Equations

A set of M differential equations with one independent variable x and M dependent variables is **decoupled** if like (7.2):

$$m = 1, \ldots, M: \qquad F_m\left(x; y_m, y'_m, y'_m, \ldots., y_m^{(N_m)}\right) = 0, \tag{7.5a, b}$$

each dependent variable (7.5a) satisfies an ordinary differential equation (7.5b) of order N_m involving only the same dependent variable and its derivatives of order up to N_m; in this case each of the M differential equations (7.5b) can be solved separately from the others. This is not the case if each differential equation involves more than one dependent variable and/or its derivatives:

$$m, s = 1, \ldots, M; \quad N_{m,s} \in |N: \qquad 0 = F_s\left(x; y_m, y'_m, y''_m, \ldots., y_m^{(N_{m,s})}\right). \tag{7.6a–c}$$

*The **general simultaneous system** of M (7.6a) ordinary differential equations (7.6c) relates (standard CXII) the independent variable x to all (7.6b) dependent* *variables y_s and their derivatives up to the order $N_{m,s}$. Assume that the system (7.6a–c) can be solved explicitly (7.7d) for the highest-order derivative (7.7c) of each dependent variable (7.7a):*

$$m,r=1,\ldots,M; \quad r\neq m: \quad y_m^{(N_{m,m})}(x)=G_s\left(x;y_m,y'_m,\ldots.,y_m^{(N_{m,m-1})};\,y_r,y'_r,\ldots.,y_r^{(N_{m,r})}\right), \tag{7.7a–d}$$

with the remaining dependent variables (7.7b) and their derivatives also appearing. The corresponding autonomous system:

$$m=1,\ldots,M; \quad s_m=1,2,\ldots,N_{m,n}-1\equiv t_m: \qquad y_{m,s_m}(x)=y_m^{(s_m)}(x), \tag{7.8a–c}$$

$$N=\sum_{m=1}^{M}N_{m,m}: \qquad y'_{m,t_m}=G_m\left(x;y_m,y_{m,1},\ldots.,y_{m,t_m};y_{r,1},\ldots.,y_{r,t_r}\right) \tag{7.8d, e}$$

has: (i) extra variables (7.8a–c) for a total (7.8d); (ii) all equations (7.8c, e) have an autonomous form. For example, the pair of simultaneous ordinary differential equations of orders 2(1) explicit in the highest-order derivatives (7.9a, b):

$$y''=F(x;y,y';z), \qquad z'=G(x;z;y,y'), \tag{7.9a, b}$$

is equivalent to the autonomous system (7.10a–c) of order 3:

$$y'=y_1, \qquad y'_1=y''=F(x;y,y_1;z), \qquad z'=G(x;z;y,y_1). \tag{7.10a–c}$$

The implicit autonomous system of differential equations (7.1a, b) has a simple geometrical interpretation (section 7.2).

7.2 Tangents, Trajectories, and Paths in N-Dimensions

An autonomous system of N first-order coupled differential equations specifies a family of curves in a space of N dimensions (subsection 7.2.1), which may lie on the intersection of $M\leq N$ hypersurfaces (subsection 7.2.3). The simplest cases $N=2\,(N=3)$ are [subsection 7.2.2 (7.2.4)] plane curves (space curves specified by the intersection of two surfaces). Thus, the consideration of hypercurves (hypersurfaces) tangent (orthogonal) to a continuous N-dimensional vector field leads to an autonomous system of N differential equations [a first-order differential in N variables (notes 3.1–3.15)] that always has (may not have) a solution.

7.2.1 *N*-Dimensional Hypercurve Specified by Tangent Vectors

Denoting by (7.11b) the coordinates in an N-dimensional space (7.11a) and by a parameter such as time (t), a regular curve has parametric equations (7.11c), where the coordinates are functions of the parameter with a continuous first-order derivative, and specify a **trajectory**:

$$n, m = 1, \ldots, N: \qquad x_n(t) \in C^1(a \le t \le b): \qquad \frac{dx_n}{dt} = X_n(x_m), \tag{7.11a–c}$$

in the autonomous system of first-order differential equations (7.11c). The independent variable t does not appear explicitly, as it is designated an **implicit autonomous system** (standard CXXIII). The differentiation of the coordinates with regard to time (7.11c) specifies a continuous (7.12a) tangent vector field (Figure 7.1), not necessarily of unit length, since a metric (notes III.9.35–III.9.45) need not exist; if the dependent variables x_n are spatial coordinates and the parameter t is time, then the vector field defined by the derivatives (7.11c) is the velocity. Eliminating the parameter (7.12b) leads to a set of $(N-1)$ simultaneous ordinary differential equations (7.12c):

$$X_n(x_1, \ldots, x_N) \in C(|R^N): \qquad dt = \frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots. = \frac{dx_n}{X_n}, \tag{7.12a–c}$$

whose solution (7.13b) specifies the **path** as the intersection of $(N-1)$ hypersurfaces (7.13a):

$$m = 1, \ldots, N-1: \qquad f_m(x_1, x_2, \ldots., x_N) = C_m, \tag{7.13a, b}$$

where $C_1, \ldots, C_{N-1}$ are arbitrary constants. Note that the trajectory (7.11a–c) [path (7.13a, b)] correspond to the same curve with (without) a parameter that specifies its direction, say increasing time for the direction along the trajectory. Thus, (standard CXXIV) *an implicit autonomous system of N dimensional equations (7.11a–c) specifies a family of regular curves (7.13a, b) with* $N-1$

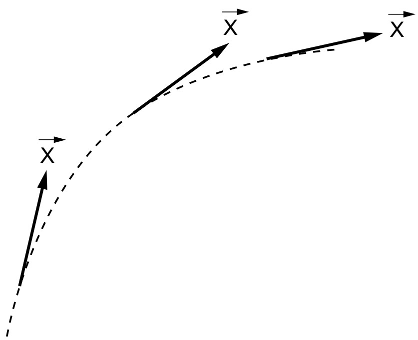


FIGURE 7.1
A continuous vector field leads to an autonomous system of ordinary differential equations whose solution is the tangent curve.

parameters $C_1,\ldots,C_{N-1}$, *which are tangent (Figure 7.1) to a given continuous vector field (7.12a)*. The simplest cases are the plane $N = 2$ (three-dimensional space $N = 3$), where a continuous vector field specifies a family of tangent curves (section 7.2.2) with one (two) parameters.

7.2.2 Families of Curves in the Plane or in Space

In the two-dimensional case, the trajectory (7.14a):

$$\left\{\frac{dx}{dt},\frac{dy}{dt}\right\} = X,Y(x,y), \qquad \frac{dy}{dx} = \frac{Y}{X}, \qquad f(x,y) = C, \tag{7.14a–b}$$

specifies a differential equation (7.14b), whose solution (7.14c) is a one-parameter family of which are curves tangent (Figure 7.1) to the vector field of components $\{X,Y\}$.

In the three-dimensional case:

$$\left\{\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right\} = X,Y,Z(x,y,z), \tag{7.15a}$$

the system of two equations (7.15b):

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z}, \qquad f,g(x,y,z) = C_1,C_2, \tag{7.15b, c}$$

specifies two families of surfaces (7.15a), whose intersection determines (Figure 7.2) a family of curves with two parameters C_1, C_2, which are tangent to the vector field of components $\{X,Y,Z\}$.

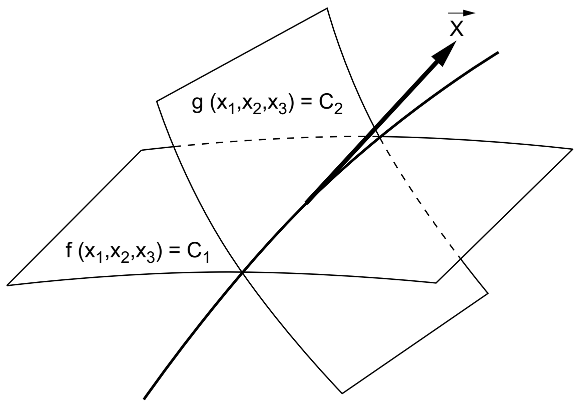


FIGURE 7.2
The tangent curve to a continuous vector field (Figure 7.1) in three dimensions may be obtained as the intersection of two surfaces.