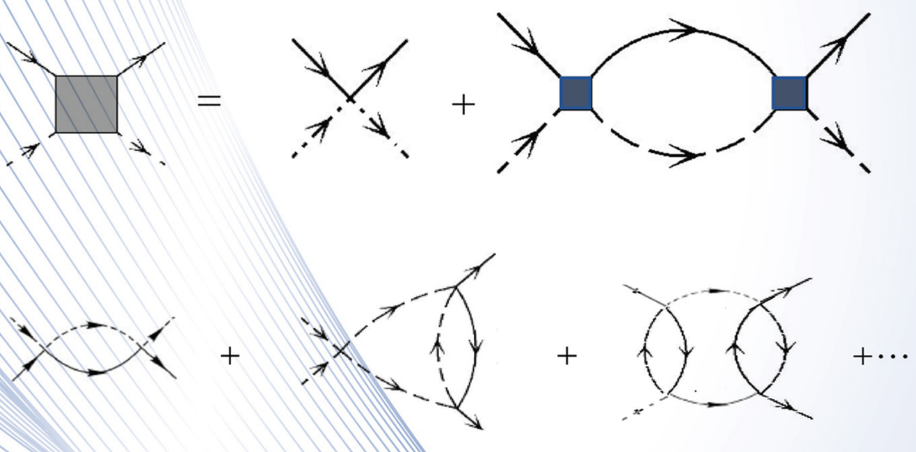
QUANTUM FIELD THEORY

Feynman Path Integrals and Diagrammatic Techniques in Condensed Matter



LUKONG CORNELIUS FAI

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Preface

This book of quantum field theory (QFT) is the continuation of Chapter 13 (Functional Integration in Statistical Physics) of the book entitled *Statistical Thermodynamics: Understanding the Properties of Macroscopic Systems* published by CRC Press in 2002. QFT is a universal tool for the quantum mechanical description of processes permitting transitions among states that differ in their particle content and has applications ranging from condensed matter physics to elementary particle physics. As the quantum mechanics of an arbitrary number of particles, QFT provides an efficient tool describing quantum statistics of the particles. This implies antisymmetrization and symmetrization of the states of identical fermions or bosons, respectively, under interchange of pairs of identical particles. QFT facilitates the treatment of spontaneously symmetrical broken states, such as superfluids as well as critical phenomena with regard to phase transitions. This book uses the strength of Feynman functional and diagrammatic techniques as a presentation foundation that comfortably applies QFT to a broad range of domains in physics and shows the universality of the techniques for a broad range of phenomena. The powerful QFT functional techniques and the renormalization group techniques applicable to equilibrium as well as nonequilibrium field theory processes are extended to treat nonequilibrium states and subsequently transport phenomena. The Green's and correlation functions and the equations derived from them are used to solve real physical problems as well as to describe processes in real physical systems—in particular quantum fluid, electron gas, electron transport, optical response, superconductivity, and superfluidity.

This book should be of interest not only to condensed matter physicists but to other physicists as well because the techniques discussed apply to high-energy as well as soft condensed matter physics. The universality of the techniques is confirmed as a unifying tool in other domains of physics. This book is written for graduate students and researchers who are not necessarily specialists in QFT. It begins with elementary concepts and a review of quantum mechanics and builds the framework of QFT, which is now applied to current problems of utmost importance in condensed matter physics. In most cases, the problem sets represent an integral part of the book and provide a means of reinforcing the explanation of QFT in real situations. The material in this book is clear, and its illustrations emphasize the subject and aid the reader in an essential understanding of the important concepts. This book should be highly recommended for all theoretical physicists in QFT.

This book is the product of lecture notes given to graduate students at the Universities of Dschang and Bamenda, Cameroon. Chapter 1 studies symmetry requirements in quantum mechanics as well as bosonic and fermionic quantum fields operating on multiparticle state space (Fock space). The chapter examines creation and annihilation operators and applies the method of second quantization, a technique that underpins the formulation of quantum many-particle theories. It also treats, in a unified manner, systems of bosons (fermions) with a fixed or variable number of particles. Chapter 2 examines bosonic and fermionic coherent states. It also studies Grassmann algebra, Berezin integration, and Gaussian integrals as well as Wick theorem for multidimensional Grassmann integrals and the trace of a physical quantity. Chapter 3 studies the path integral approach for fermions and bosons considering the

Grassmann algebra. Apart from providing a global view of the entire system, this approach has proven to be an extremely useful tool for understanding and handling quantum mechanics, quantum field theory, and statistical mechanics. This chapter also examines the Green's functions that serves as tools for describing quantum dynamics of many-body systems. In addition, noninteracting particles are also studied, with their Green's function computed via path integrals as well as via generating functionals.

In Chapter 4, perturbation theory is comfortably constructed from the average value of a functional. This chapter studies the perturbation theory in many-particle systems based on Wick theorem, which is formulated in terms of Feynman functional integral and diagrammatic techniques that are very useful for providing an insight into the physical process that they represent. This chapter also examines the cornerstone of the functional technique, which is the concept of generating functionals sufficient to derive all propagators. Chapter 5 examines the (anti)symmetrized vertex, making it simpler and more convenient to formulate perturbation theory. Discussions are facilitated by introducing fully (anti)symmetrized vertices via a uniform and compact notation for the creation and annihilation fermion operators. Chapter 6 examines connected Green's functions with one-particle (1PI) and two-particle (2PI) irreducible vertices as well as the Dyson-Schwinger equations that are most conveniently studied via path integrals and that employ the approach of generating functions in the context of the path integral. The Luttinger-Ward functional and the 2PI vertices are used to set up approximations satisfying conservation laws as well as nonperturbative approaches. Chapter 7 examines the random phase approximation via the Feynman functional integral and diagrammatic technique: screened interactions and plasmons. Here, we study a model describing electrons in a metal that considers a system of electrons interacting with each other via the instantaneous Coulomb force (Jellium model).

Chapter 8 examines the theory of phase transitions and critical phenomena as well as Ginzburg-Landau phenomenology and the connection to statistical field theory. Chapter 9 examines weakly interacting Bose gas via quantum field theory. Application to Bogoliubov theory of the weakly interacting Bose gas and superfluidity is considered as well. This chapter also studies the path integral formalism for nonideal Bose gas considering the electron-electron interaction.

Chapter 10 studies superconductivity theory via the functional integral and diagrammatic approaches, where the statistical model is built on classical field configurations. The mean-field theory is also considered as well as its applications to Cooper instability and the BCS condensate. The vertex function for small momentum transfers is considered, that is, electron-electron interaction. Chapter 11 examines the path integral approach to the BCS theory where we study an accurate theory of interacting Fermi mixtures with spin imbalance. Chapter 12 discusses Green's functions' averages over impurities and, in particular, scattering potentials and disordered systems as well as disorder diagrams, perturbation series solution via T-matrix, and quenched and disorder averages. The diagrammatic cross technique is extended to superconductors considering the Nambu-Gorkov propagators.

Chapter 13 studies in detail the classical and quantum theory of magnetism and spin wave theory spin representations as well as spin liquids. The strongly interacting system and, in particular, the Kondo problem are studied in detailed where methods of quantum statistical field theory play a central role. Chapter 14 considers the nonequilibrium quantum field theory, where we study nonequilibrium Green's functions as well as Keldysh-Schwinger diagrammatic and 2PI effective action techniques relating to nonequilibrium dynamics.

I would like to acknowledge those who have helped at various stages of the elaboration and writing through discussions, criticism, and especially, encouragement and support. I single out Prof. Nicolas Dupuis (Directeur de Recherche at CNRS Laboratoire de Physique Théorique de la Matière Condensée, CNRS UMR, Université Pierre et Marie Curie Paris, France) for allowing me to use some of his pictures. I am very thankful to my wife, Dr. Mrs. Fai Patricia Bi, for all her support and encouragement, and to my four children (Fai Fanyuy Nyuydze, Fai Fondzeyuv Nyuytari, Fai Ntumfon Tiysiy, and Fai Jinyuy Nyuydzefon) for their understanding and moral support during the writing of this book. I acknowledge with gratitude the library support received from the Abdus Salam International Centre for Theoretical Physics (ICTP), Trieste, Italy.

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Symmetry Requirements in QFT

Introduction

Quantum field theory (QFT) is a universal tool that has applications ranging from atomic, molecular, and particle physics to condensed matter and statistical physics as well as modern quantum chemistry. Recently, quantum field theory has also had an unexpected and profound impact on pure mathematics. Symmetries, which are at the heart of the universality shown by many physical systems, play a crucial role. Nowadays, quantum theory is the most complete microscopic method describing the physics of energy and matter. Considering the quantization of the electromagnetic field [1] and the representation of particles by quantized fields [2, 3] results in the development of quantum electrodynamics and quantum field theory. By convention, the original form of quantum mechanics is denoted by the first quantization, whereas quantum field theory is formulated in the language of second quantization that is an essential tool for the development of interacting many-body field theories.

The fundamental difference between classical and quantum mechanics relates to the concept of indistinguishability of identical particles. Each particle can be equipped with an identifying marker without influencing its behavior in classical mechanics. In addition, each particle follows its own continuous path in phase space. So, principally, each particle in a group of identical particles can be identified; but this is not the case in quantum mechanics. It is not possible to mark a particle without influencing its physical state. In addition, if a number of identical particles are brought to the same region in space, their wave functions will spread out rapidly and will overlap with one another. Eventually, it will be impossible to say which particle is where. One of the fundamental assumptions for n-particle systems, therefore, is that identical particles (i.e., particles characterized by the same quantum numbers such as mass, charge, and spin) are, in principle, indistinguishable.

1.1 Second Quantization

1.1.1 Fock Space

This chapter introduces and applies the method of second quantization, a technique that underpins the formulation of quantum many-particle theories. Second quantization formalism treats systems of bosons or fermions with a fixed or variable number of particles in a unified way. In this section, we review the key aspects of this formalism.

In quantum mechanics, the state of a system of n identical (indistinguishable) particles is described by a state vector belonging to a Hilbert space (a complete multiparticle system [Fock space]) as the direct sum

$$\mathrm{H} = \mathrm{H}_0 \oplus \mathrm{H}_1 \oplus \cdots = \oplus_{n=0}^{\infty} \mathrm{H}_n \tag{1}$$

The Hilbert space, H_n, corresponds to the n-particle states that are properly symmetrized (bosons) or antisymmetrized (fermions). It is a subspace of the direct tensor product

$$\bar{H}_n = H_1 \otimes H_1 \otimes \cdots H_1 \tag{2}$$

Also, H_0 corresponds to the vacuum state, $|0\rangle$, H_1 corresponds to the single-particle state, and so on.

Consider the orthogonal bases $\{|\alpha'\rangle\}$ and $\{|\alpha\rangle\}$ of H_1, where α' and α are discrete quantum numbers:

$$|\alpha\rangle = \sum_{\alpha'} |\alpha'\rangle\langle\alpha'|\alpha\rangle \tag{3}$$

Particles of the same species are completely indistinguishable in a quantum many-body system. Second quantization provides a general approach to many-body systems where the vector of state plays a minor role. Second quantization entails raising the Schrödinger vector of state to an operator that satisfies certain *canonical (anti)commutation* algebra. It is instructive to note that in first quantized physics, physical properties of a quantum particle such as density, kinetic energy, and potential energy can be expressed in terms of a one-particle vector of state. The essence of the second quantization is the elevation of each of these quantities to the status of an operator. This is done by replacing the one-particle vector of state with its corresponding field operator.

Knowing the orthonormal basis $\{|\alpha\rangle\}$ of H_1 allows us to obtain an orthonormal basis of $\bar{H}_n$ from the tensor product of the single-particle basis that is the n-particle state

$$|\alpha_1 \cdots \alpha_n) = |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \cdots \otimes |\alpha_n\rangle \tag{4}$$

where **the defined states utilize a curved bracket in the ket symbol**. The first ket on the right-hand side of this equation refers to particle 1, the second to particle 2, and so on. Then the **overlap of two vectors of the basis** is given as follows:

$$(\alpha_1' \cdots \alpha_n'|\alpha_1 \cdots \alpha_n) = \left(\langle\alpha_1'| \otimes \langle\alpha_2'| \otimes \cdots \otimes \langle\alpha_n'|\right)\left(|\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \cdots \otimes |\alpha_n\rangle\right) = \langle\alpha_1'|\alpha_1\rangle \cdots \langle\alpha_n'|\alpha_n\rangle \tag{5}$$

From here, the **orthogonality relation** may be written as follows:

$$(\alpha_1' \cdots \alpha_n'|\alpha_1 \cdots \alpha_n) = \langle\alpha_1'|\alpha_1\rangle \cdots \langle\alpha_n'|\alpha_n\rangle = \delta_{\alpha_1'\alpha_1} \cdots \delta_{\alpha_n'\alpha_n} \tag{6}$$

where the Kronecker symbol is used to include the possibility of δ-function normalization for continuous quantum numbers.

The **completeness of the basis** is obtained from the tensor product of the completeness relation for the basis $\{|\alpha\rangle\}$ and yields the **closure relation**

$$\sum_{\alpha_1 \cdots \alpha_n} |\alpha_1 \cdots \alpha_n)(\alpha_1 \cdots \alpha_n| = \hat{1} \tag{7}$$

where $\hat{1}$ is the unit operator in $\bar{H}_n$. In the case of continuous quantum numbers, integration must be used in (7) instead of a summation; or a combination of both may be used in the case of mixed spectra.

Therefore, we see from the previous information that the Hilbert space describing the n-particle system is spanned by all the n^{th}-rank tensors, such as in (4). We define the state in which the i^{th} particle is localized at a point with radius vector $\vec{r}_i$ as:

$$\langle\vec{r}_1 \cdots \vec{r}_n| = \langle\vec{r}_1| \otimes \langle\vec{r}_2| \otimes \cdots \otimes \langle\vec{r}_n| \tag{8}$$

If we multiply the ket vector in equation (4) by the bra vector in (8), this permits us to express the n-particle **wave function in coordinate space**:

$$\psi_{\alpha_1 \cdots \alpha_n}(\vec{r}_1 \cdots \vec{r}_n) = (\vec{r}_1 \cdots \vec{r}_n|\alpha_1 \cdots \alpha_n) = \langle\vec{r}_1|\alpha_1\rangle\langle\vec{r}_2|\alpha_2\rangle \cdots \langle\vec{r}_n|\alpha_n\rangle = \varphi_{\alpha_1}(\vec{r}_1)\varphi_{\alpha_2}(\vec{r}_2) \cdots \varphi_{\alpha_n}(\vec{r}_n) \tag{9}$$

This is a squarable-integrable function and represents the probability amplitude for finding particles at the n positions $\vec{r}_1 \cdots \vec{r}_n$. It satisfies the following condition:

$$\int \left|\psi_{\alpha_1 \cdots \alpha_n}(\vec{r}_1 \cdots \vec{r}_n)\right|^2 d\vec{r}_1 \cdots d\vec{r}_n < +\infty \tag{10}$$

It is useful to note that a vector of state in quantum mechanics is a scalar product on Hilbert space of the corresponding state and eigenstates of the position operator, that is,

$$\varphi_\alpha(\vec{r}) = \langle \vec{r} | \alpha \rangle \tag{11}$$

Here, $\varphi_\alpha(\vec{r})$ is a single-particle wave function in the state $|\alpha\rangle$ that forms a complete set of orthonormal functions satisfying the following **orthonormal** and **completeness relations**:

$$\int d\vec{r} \varphi^*_\alpha(\vec{r}) \varphi_{\alpha'}(\vec{r}) = \delta_{\alpha\alpha'} \quad , \quad \sum_\alpha \varphi^*_\alpha(\vec{r}\,') \varphi_\alpha(\vec{r}) = \delta(\vec{r} - \vec{r}\,') \tag{12}$$

It is physically obvious that the space $\bar{\mathrm{H}}_n$ is generated by linear combinations of products of single-particle wave functions as seen previously.

So far, in defining the Hilbert space $\bar{\mathrm{H}}_n$, the symmetry properties of the wave function have not been taken into account. **We can define mathematically only those totally (anti)symmetric states observed in nature. This is in contrast to the multitude of pure and mixed symmetry states**. We find the basis of H_n by first (anti)symmetrizing the tensor product (4):

$$|\alpha_1 \cdots \alpha_n\} = \sqrt{n!}\,\hat{\mathrm{P}}_\chi |\alpha_1 \cdots \alpha_n) = \frac{1}{\sqrt{n!}} \sum_{\mathrm{P}} \chi^{\mathrm{P}} \left|\alpha_{\mathrm{P}(1)} \cdots \alpha_{\mathrm{P}(n)}\right) \tag{13}$$

In these (anti)symmetrized states, we utilize the curly bracket in the ket symbol. Here, P runs through all permutations of n objects where $\chi = +1$ for **bosons** and $\chi = -1$ for **fermions**; the symbol χ^{P} equals unity for bosons and $(-1)^{\mathrm{P}}$ for fermions, and $\hat{\mathrm{P}}_\chi$ is the **symmetrization operator for bosons** and the **antisymmetrization operator for fermions**; $\frac{1}{\sqrt{n!}}$ is the normalization factor.

From (13), the **Pauli Exclusion Principle is automatically satisfied for antisymmetric states: Two fermions cannot occupy the same quantum state**. For example, let us say that two identical states $|\alpha_1\rangle = |\alpha_2\rangle = |\alpha\rangle$:

$$|\alpha_1, \alpha_2, \alpha_3, \cdots \alpha_n\} = \sqrt{n!}\,\hat{\mathrm{P}}_\chi |\alpha_1, \alpha_2, \alpha_3, \cdots \alpha_n) = -\sqrt{n!}\,\hat{\mathrm{P}}_\chi |\alpha_2, \alpha_1, \alpha_3, \cdots \alpha_n) = 0 \tag{14}$$

In this case, no acceptable many-fermion state exists.

We define the (anti)symmetrization operator $\hat{\mathrm{P}}$ by its action on the many-body wave function $\psi(\vec{r}_1 \cdots \vec{r}_n)$ in (9) as

$$\psi_{\alpha_1 \cdots \alpha_n}(\vec{r}_1 \cdots \vec{r}_n) = \left(\vec{r}_1 \cdots \vec{r}_n | \alpha_1 \cdots \alpha_n\right) \equiv \psi(\vec{r}_1 \cdots \vec{r}_n) \tag{15}$$

Considering (13), we have

$$\hat{\mathrm{P}}_\chi \psi(\vec{r}_1 \cdots \vec{r}_n) = \frac{1}{n!} \sum_{\mathrm{P}} \chi^{\mathrm{P}} \psi\left(\vec{r}_{\mathrm{P}(1)} \cdots \vec{r}_{\mathrm{P}(n)}\right) \tag{16}$$

Then

$$\hat{\mathrm{P}}^2_\chi \psi(\vec{r}_1 \cdots \vec{r}_n) = \frac{1}{n!} \frac{1}{n!} \sum_{\mathrm{PP}'} \chi^{\mathrm{P}'} \chi^{\mathrm{P}} \psi\left(\vec{r}_{\mathrm{P}'\mathrm{P}(1)} \cdots \vec{r}_{\mathrm{P}'\mathrm{P}(n)}\right) \tag{17}$$

Here, PP′ denotes the group composition of P′ and P. From $\chi^{P+P'} = \chi^{P'P}$, the summation over P and P′ can be swapped with the summation over M = P′P and P:

$$\hat{P}_\chi^2 \psi(\vec{r}_1 \cdots \vec{r}_n) = \frac{1}{n!}\sum_{P}\left(\frac{1}{n!}\sum_{M}\chi^{M}\psi\left(\vec{r}_{M(1)}\cdots\vec{r}_{M(n)}\right)\right) = \frac{1}{n!}\sum_{P}\hat{P}_\chi\psi(\vec{r}_1\cdots\vec{r}_n) = \hat{P}_\chi\psi(\vec{r}_1\cdots\vec{r}_n) \tag{18}$$

This equality holds for any wave function ψ, as well as for the operator itself, where the (anti)symmetrization operator is a projector.

It is easy to show that $\hat{P}_\chi^2 = \hat{P}_\chi$. For $\chi = +1$, the implication is that, for **bosons**, several particles can occupy the same one-particle state. This was shown empirically for the first time by the Indian physicist **Satyendra Nath Bose** (1894–1974) by proving the relation:

$$\hat{P}\psi(\vec{r}_1\cdots\vec{r}_n) = \psi\left(\vec{r}_{P(1)}\cdots\vec{r}_{P(n)}\right) = \psi(\vec{r}_1\cdots\vec{r}_n) \tag{19}$$

Examples of bosons are photons, pions, mesons, gluons, phonons, excitons, plasmons, magnons, cooper pair, and Helium-4 atoms. These are particles with integral spins $\{0,1,2,\cdots\}$. The wave function of n bosons is totally symmetric and satisfies the relation in (19).

From equation (19), we see that bosons are genuinely indistinguishable when enumerating the different possible states of the particles. For $\chi = -1$, the state (13) vanishes if two of α_i, s are identical. This implies that any two fermions cannot occupy the same particle state (**Pauli Exclusion Principle**). The **Pauli Exclusion Principle** was developed empirically by the German physicist **Wolfgang Pauli** (1900–1958) [4]. It is instructive to note that this principle follows directly from the symmetry requirements on vector states. The **Pauli Exclusion Principle** is a corollary to the principle of indistinguishability of particles. This principle poses a severe constraint on vector of states of many-fermion systems and limits the number of them that are physically admissible. **Fermions** take their name from the Italian physicist Enrico Fermi, who first studied the properties of fermion gases. Several important many-particle systems have fermions as their basic constituents. Examples of fermions are protons, electrons, muons, neutrinos, quarks, and helium-3 atoms. These are particles with half-integral spins $\left\{\frac{1}{2},\frac{3}{2},\cdots\right\}$. **Note that the statistics of composite particles are determined by the number of fermions. If the fermion number is odd, then the net result is a fermion. Otherwise, for energies sufficiently low compared to their binding energy, the net result is a boson**.

The states $\{\alpha_1\cdots\alpha_n\}$ constitute a basis of H_n. So, the closure relation (7) in $\bar{H}_n$ becomes a closure relation in H_n:

$$\hat{1} = \sum_{\alpha_1\cdots\alpha_n} P_\chi|\alpha_1\cdots\alpha_n)(\alpha_1\cdots\alpha_n|P_\chi = \frac{1}{n!}\sum_{\alpha_1\cdots\alpha_n}|\alpha_1\cdots\alpha_n\}\{\alpha_1\cdots\alpha_n| \tag{20}$$

The overlap between two states constructed from the same basis $|\alpha\rangle$ is given by

$$\{\alpha_1'\cdots\alpha_n'|\alpha_1\cdots\alpha_n\} = n!\left(\alpha_1'\cdots\alpha_n'|P_\chi^2|\alpha_1\cdots\alpha_n\right) = n!\left(\alpha_1'\cdots\alpha_n'|P_\chi|\alpha_1\cdots\alpha_n\right) = \sum_{P}\chi^{P}\left(\alpha_1'\cdots\alpha_n'|\alpha_{P(1)}\cdots\alpha_{P(n)}\right) \tag{21}$$

From the orthogonality of the basis $|\alpha\rangle$, the only nonvanishing terms at the right-hand side of (21) are the permutations P contributing to the given sum:

$$\alpha' = \alpha_{P(1)} \quad , \quad \cdots \quad , \quad \alpha_n' = \alpha_{P(n)} \tag{22}$$

For **fermions**, all α_i,s should be different for each one-particle state $|\alpha\rangle$. So, there is only one such permutation P that transforms $\alpha_1 \cdots \alpha_n$ into $\alpha'_1 \cdots \alpha'$. The overlap (21) reduces to one term. If the states $|\alpha_i\rangle$ are normalized, then

$$\left\{\alpha'_1 \cdots \alpha'_n \middle| \alpha_1 \cdots \alpha_n\right\} = (-1)^{\mathrm{P}} \tag{23}$$

For **bosons**, several particles can occupy the same one-particle state. So, any permutation that interchanges particles in the same state contributes to the sum in (21). The number of these permutations is $n_{\alpha_1}! \cdots n_{\alpha_n}!$, which transforms $\alpha_1 \cdots \alpha_n$ into $\alpha'_1 \cdots \alpha'_n$. Here, n_{α_i} is the number of bosons in the one-particle state $|\alpha_i\rangle$ where $\alpha_{\mathrm{P}(1)} \cdots \alpha_{\mathrm{P}(n)}$ are distinct with

$$\left\{\alpha'_1 \cdots \alpha'_n \middle| \alpha_1 \cdots \alpha_n\right\} = n_{\alpha_{\mathrm{P}(1)}}! \cdots n_{\alpha_{\mathrm{P}(n)}}! \tag{24}$$

For both fermions and bosons, the sum of the occupation numbers that counts the total number of occupied states is equal to the number of particles:

$$n = \sum_{\alpha} n_{\alpha} \tag{25}$$

For bosons, these occupation numbers are a priori not restricted, whereas for fermions, they can take only the value 0 or 1.

If we use the convention that $0! = 1$, then formulae (23) and (24) yield the equivalent single expression:

$$\left\{\alpha'_1 \cdots \alpha'_n \middle| \alpha_1 \cdots \alpha_n\right\} = \chi^{\mathrm{P}} \prod_{i=1}^{\mathrm{P}} n_{\alpha_i}! \tag{26}$$

So, the orthonormal basis for the Hilbert space H_n can be obtained by normalizing the states $|\alpha_1 \cdots \alpha_n\}$ with the help of (26):

$$|\alpha_1 \cdots \alpha_n\rangle = \frac{1}{\sqrt{\prod_{i=1}^{\mathrm{P}} n_{\alpha_i}!}} |\alpha_1 \cdots \alpha_n\} = \frac{1}{\sqrt{n! \prod_{i=1}^{\mathrm{P}} n_{\alpha_i}!}} \sum_{\mathrm{P}} \chi^{\mathrm{P}} \left|\alpha_{\mathrm{P}(1)} \cdots \alpha_{\mathrm{P}(n)}\right) \tag{27}$$

The prefactor $\frac{1}{\sqrt{n! \prod_{i=1}^{\mathrm{P}} n_{\alpha_i}!}}$ normalizes the many-body wave function. Here, n_{α_i} is the number of particles in the state α_i and, for fermions, considering the **Pauli Exclusion Principle**, $n_{\alpha_i} = 0,1$. The summation over $n!$ permutations P of $\{\alpha_1 \cdots \alpha_n\}$ is required by particle indistinguishability; the parity χ^{P} is the number of transpositions of two elements that brings permutations $(\mathrm{P}(1), \cdots, \mathrm{P}(n))$ back to ordered sequence $(1, \cdots, n)$. Note that the normalized (anti)symmetric state defined in (27) uses an angular bracket in the ket symbol in contrast to the states defined earlier in (4). Since orthonormality is used in the calculation of the normalization factor, then hereafter it is understood that whenever the symbol $|\alpha_1 \cdots \alpha_n\rangle$ is used, the basis $\{|\alpha_i\rangle\}$ is orthonormal.

The overlap of the tensor product $|\vec{r}_1 \cdots \vec{r}_n)$ and the (anti)symmetric state $|\alpha_1 \cdots \alpha_n\rangle$:

$$(\vec{r}_1 \cdots \vec{r}_n | \alpha_1 \cdots \alpha_n\rangle = \frac{1}{\sqrt{n! \prod_{i=1}^{\mathrm{P}} n_{\alpha_i}!}} \sum_{\mathrm{P}} \chi^{\mathrm{P}} \langle \vec{r}_1 | \alpha_{\mathrm{P}(1)}\rangle \cdots \langle \vec{r}_n | \alpha_{\mathrm{P}(n)}\rangle = \frac{1}{\sqrt{n! \prod_{i=1}^{\mathrm{P}} n_{\alpha_i}!}} \sum_{\mathrm{P}} \chi^{\mathrm{P}} \varphi_{\alpha_{\mathrm{P}(1)}}(\vec{r}_1) \cdots \varphi_{\alpha_{\mathrm{P}(n)}}(\vec{r}_n) \tag{28}$$

or

$$\left(\vec{r}_1\cdots\vec{r}_n\middle|\alpha_1\cdots\alpha_n\right\rangle \equiv \frac{1}{\sqrt{n!\prod_{i=1}^{\mathrm{P}} n_{\alpha_i}!}}\mathbf{S}\left(\left\langle\vec{r}_i\middle|\alpha_j\right\rangle\right) \equiv \frac{1}{\sqrt{n!\prod_{i=1}^{\mathrm{P}} n_{\alpha_i}!}}\mathbf{S}\left(\mathbf{A}_{ij}\right) \tag{29}$$

Here, $\mathbf{S}\left(\mathbf{A}_{ij}\right)$ is expressed in the following relation for fermions (bosons):

$$\left(\vec{r}_1\cdots\vec{r}_n\middle|\alpha_1\cdots\alpha_n\right\rangle = \begin{cases} \dfrac{1}{\sqrt{n!}}\det\left[\mathbf{A}_{ij}\right] = \dfrac{1}{\sqrt{n!}}\sum_{\mathrm{P}}(-1)^{\mathrm{P}}\mathbf{A}_{1\mathrm{P}(1)}\cdots\mathbf{A}_{n\mathrm{P}(n)} & ,\ \text{fermions} \\ \dfrac{1}{\sqrt{n!\prod_{i=1}^{\mathrm{P}} n_{\alpha_i}!}}\mathrm{per}\left[\mathbf{A}_{ij}\right] = \dfrac{1}{\sqrt{n!\prod_{i=1}^{\mathrm{P}} n_{\alpha_i}!}}\sum_{\mathrm{P}}\mathbf{A}_{1\mathrm{P}(1)}\cdots\mathbf{A}_{n\mathrm{P}(n)} & ,\ \text{bosons} \end{cases} \tag{30}$$

or

$$\left(\vec{r}_1\cdots\vec{r}_n\middle|\alpha_1\cdots\alpha_n\right\rangle = \begin{cases} \dfrac{1}{\sqrt{n!}}\det\left[\varphi_{\alpha_i}\left(\vec{r}_j\right)\right] & ,\ \text{fermions} \\ \dfrac{1}{\sqrt{n!\prod_{i=1}^{\mathrm{P}} n_{\alpha_i}!}}\mathrm{per}\left[\varphi_{\alpha_i}\left(\vec{r}_j\right)\right] & ,\ \text{bosons} \end{cases} \tag{31}$$

We therefore obtain a basis of **permanents for bosons (sign-less determinant)** and **Slater determinants for fermions** as seen earlier in equations (19) through (21). From (27) and the normalization in (20), we obtain the following closure relation:

$$\sum_{\alpha_1\cdots\alpha_n}\frac{\prod_{i=1}^{\mathrm{P}} n_{\alpha_i}!}{n!}\left|\alpha_1\cdots\alpha_n\right\rangle\left\langle\alpha_1\cdots\alpha_n\right| = 1 \tag{32}$$

We define an (anti)symmetrized many-particle state in the following coordinate representation via the states $\left|\vec{r},\sigma\right\rangle$ not normalized:

$$\left|\vec{r}_1,\sigma_1\cdots\vec{r}_n,\sigma_n\right\} = \frac{1}{\sqrt{n!}}\sum_{\{\mathrm{P}\}}\chi^{\mathrm{P}}\left|\vec{r}_{\mathrm{P}(1)},\sigma_{\mathrm{P}(1)}\cdots\vec{r}_{\mathrm{P}(n)},\sigma_{\mathrm{P}(n)}\right) = \frac{1}{V^{\frac{n}{2}}}\sum_{\vec{\kappa}_1,\cdots,\vec{\kappa}_n}\exp\left\{i\left(\vec{\kappa}_1\vec{r}+\cdots+\vec{\kappa}_n\vec{r}_n\right)\right\}\left|\vec{\kappa}_1,\sigma_1\cdots,\vec{\kappa}_n,\sigma_n\right\} \tag{33}$$

Here, σ denotes the spin index (as well as other discrete indices, if necessary) and $\vec{\kappa}$ the momentum.

1.1.2 Creation and Annihilation Operators

The formalism of second quantization treats systems of bosons (fermions) with a fixed or variable number of particles in a unified manner. In many physical processes, the particle number does change. Examples include electron-hole annihilations in metals or semiconductors, electron-phonon processes, and photon absorption or emission. To formulate statistical physics in terms of the grand canonical ensemble, we must deal with states having different numbers of particles. For Fock space to be a concept of interest, there must be operators connecting the different n-particle sectors; these are the creation $\hat{\psi}_\alpha^\dagger$ and annihilation $\hat{\psi}_\alpha$ operators (with each being the Hermitian adjoint of the other) that add a particle or remove a particle, respectively, in the one-particle state α thereby (anti)symmetrizing the resulting many-particle state:

$$\hat{\psi}_\alpha^\dagger|\alpha_1\cdots\alpha_n\} = |\alpha\alpha_1\cdots\alpha_n\} \quad , \quad \hat{\psi}_\alpha^\dagger|\alpha_1\cdots\alpha_n\rangle = \sqrt{n_\alpha+1}|\alpha\alpha_1\cdots\alpha_n\rangle \tag{34}$$

Here, n_α is the occupation number of the state $|\alpha\rangle$ in $|\alpha_1\cdots\alpha_n\rangle$. The annihilation operator $\hat{\psi}_\alpha$ is the adjoint of the creation operator $\hat{\psi}_\alpha^\dagger$. These operators provide a convenient representation of many-particle states and many-particle operators, generate the entire Hilbert space by their action on a single reference state, and provide a basis for the algebra of operators of the Hilbert space.

The operator $\hat{\psi}_\alpha^\dagger$ physically adds a particle in state $|\alpha\rangle$ to the state on which it operates and (anti)symmetrizes the new state. Because there can be at most one fermion in a given state, equation (34) takes the following form:

$$\hat{\psi}_\alpha^\dagger|\alpha_1\cdots\alpha_n\rangle = \begin{cases} |\alpha\alpha_1\cdots\alpha_n\rangle & , \ |\alpha\rangle \notin |\alpha_1\cdots\alpha_n\rangle \\ 0 & , \ |\alpha\rangle \in |\alpha_1\cdots\alpha_n\rangle \end{cases} \tag{35}$$

Writing $\hat{\psi}^\dagger(\vec{r})|0\rangle$ and $|\alpha\rangle = \xi_\alpha^\dagger|0\rangle$, where $|0\rangle$ is the **vacuum** state containing no particles at all and is distinguished from the zero of the Hilbert space, then we have the following relation between creation (annihilation) operators in the $\vec{r}$-basis and the α-basis:

$$\psi^\dagger(\vec{r}) = \sum_\alpha \varphi_\alpha^*(\vec{r})\xi_\alpha^\dagger \quad , \quad \psi(\vec{r}) = \sum_\alpha \varphi_\alpha(\vec{r})\xi_\alpha \tag{36}$$

The most relevant example is when $|\alpha\rangle = |\vec{\kappa},\sigma\rangle$, where $\vec{\kappa}$ and σ are the momentum and spin variables, respectively. It is important to note that like all other quantum variables, the quantum field in general is a strongly fluctuating degree of freedom. It only becomes sharp in certain special eigenstates. This function adds or subtracts particles to the system.

Since any basis vector $|\alpha_1\cdots\alpha_n\}$ or $|\alpha_1\cdots\alpha_n\rangle$ may be generated by the repeated action of a creation operator on the vacuum state

$$\hat{\psi}_\alpha^\dagger|0\rangle = |\alpha\rangle \tag{37}$$

then, generally,

$$|\alpha_1\cdots\alpha_n\} = \hat{\psi}_{\alpha_1}^\dagger\cdots\hat{\psi}_{\alpha_n}^\dagger|0\rangle \quad , \quad |\alpha_1\cdots\alpha_n\rangle = \frac{1}{\sqrt{\prod_{i=1}^{P} n_{\alpha_i}!}}|\alpha_1\cdots\alpha_n\} \tag{38}$$

We see that the creation operators generate the entire Fock space by repeated action on the vacuum state. From relation (34), we have

$$\{\alpha_1\cdots\alpha_n|\hat{\psi}_\alpha|\alpha_1'\cdots\alpha_m'\} = \left[\{\alpha_1'\cdots\alpha_m'|\hat{\psi}_\alpha^\dagger|\alpha_1\cdots\alpha_n\}\right]^* = \{\alpha\alpha_1\cdots\alpha_n|\alpha_1'\cdots\alpha_m'\} \tag{39}$$

This result can only be finite when $n = m-1$, so that $\hat{\psi}_\alpha$ removes a particle from the state on which it acts. In the case of a vacuum, this implies that $\hat{\psi}_\alpha|0\rangle = \langle 0|\hat{\psi}_\alpha^\dagger = 0$ for any state $|\alpha\rangle$. This is evidence that the vacuum is the kernel of the annihilation operators.

From the closure relation in the Fock space then

$$\sum_{n=0}^{\infty}\sum_{\alpha_1\cdots\alpha_n}\frac{1}{n!}\left|\alpha_1\cdots\alpha_n\right\}\left\{\alpha_1\cdots\alpha_n\right| = 1 \tag{40}$$

Using (21), we have

$$\begin{aligned}\hat{\psi}_\alpha\left|\alpha_1'\cdots\alpha_m'\right\} &= \sum_{n-0}^{\infty}\sum_{\alpha_1\cdots\alpha_n}\frac{1}{n!}\left\{\alpha_1\cdots\alpha_n\right|\hat{\psi}_\alpha\left|\alpha_1'\cdots\alpha_m'\right\}\left|\alpha_1\cdots\alpha_n\right\} \\ &= \frac{1}{(m-1)!}\sum_{\alpha_1\cdots\alpha_{m-1}}\left\{\alpha\alpha_1\cdots\alpha_{m-1}\middle|\alpha_1'\cdots\alpha_m'\right\}\left|\alpha_1\cdots\alpha_{m-1}\right\}\end{aligned} \tag{41}$$

With the help of (6) and (21), we now have

$$\hat{\psi}_\alpha\left|\alpha_1'\cdots\alpha_m'\right\} = \frac{1}{(m-1)!}\sum_{\mathrm{P}}\chi^{\mathrm{P}}\left\langle\alpha\middle|\alpha_{\mathrm{P}(1)}'\right\rangle\left|\alpha_{\mathrm{P}(2)}'\cdots\alpha_{\mathrm{P}(m)}'\right\} \tag{42}$$

Because the permutation $\left(\mathrm{P}(2)\cdots\mathrm{P}(m)\right)\to\left(1,\cdots,\mathrm{P}(1)-1,\mathrm{P}(1)+1,\cdots,m\right)$ has the signature $\chi^{\mathrm{P}+\mathrm{P}(1)-1}$, this allows us to arrive at

$$\hat{\psi}_\alpha\left|\alpha_1'\cdots\alpha_m'\right\} = \frac{1}{(m-1)!}\sum_{\mathrm{P}}\chi^{\mathrm{P}(1)-1}\delta_{\alpha,\alpha_{\mathrm{P}(1)}'}\left|\alpha_1'\cdots\alpha_{\mathrm{P}(1)-1}',\alpha_{\mathrm{P}(1)+1}'\cdots\alpha_m'\right\} = \sum_{i=1}^{m}\chi^{i-1}\delta_{\alpha,\alpha_i'}\left|\alpha_1'\cdots\hat{\alpha}_i'\cdots\alpha_m'\right\} \tag{43}$$

Here, $\hat{\alpha}_i'$ indicates that α_i' is removed from the many-particle state $\left|\alpha_1'\cdots\alpha_m'\right\}$. For a similar result for the normalized state $\left|\alpha_1'\cdots\alpha_m'\right\rangle$ with occupation number n_α for the state α, we have

$$\hat{\psi}_\alpha\left|\alpha_1'\cdots\alpha_m'\right\rangle = \frac{1}{\sqrt{n_\alpha}}\sum_{i=1}^{m}\chi^{i-1}\delta_{\alpha,\alpha_i'}\left|\alpha_1'\cdots\hat{\alpha}_i'\cdots\alpha_m'\right\rangle \tag{44}$$

From here, we observe that the effect of $\hat{\psi}_\alpha$ acting on any state is to annihilate one particle in the state α from a given state. In the case of bosons, the general result is conveniently expressed in occupation number representation:

$$\left|n_{\alpha_1}\cdots n_{\alpha_{\mathrm{P}}}\right\rangle = \frac{1}{\sqrt{\prod_{i=1}^{\mathrm{P}} n_{\alpha_i}!}}\left(\hat{\psi}_{\alpha_1}^{\dagger}\right)^{n_{\alpha_1}}\cdots\left(\hat{\psi}_{\alpha_{\mathrm{P}}}^{\dagger}\right)^{n_{\alpha_{\mathrm{P}}}}\left|0\right\rangle \tag{45}$$

or

$$\left|n_{\alpha_1}\cdots n_{\alpha_{\mathrm{P}}}\right\rangle = \prod_{i=1}^{\mathrm{P}}\frac{1}{\sqrt{n_{\alpha_i}!}}\left(\hat{\psi}_{\alpha_i}^{\dagger}\right)^{n_{\alpha_i}}\left|0\right\rangle \tag{46}$$

Here, n_{α_i} is the occupation number of the one-particle state α_i. From (46), an arbitrary state in the Fock space can be obtained by acting on $\left|0\right\rangle$ with some polynomial of creation operators $\hat{\psi}_{\alpha_i}^{\dagger}$. This also follows that a fundamental property of creation (annihilation) operators is that they provide a basis for all operators in the Fock space. So, any operator can be expressed as a linear combination of the set of all products of the operators $\left\{\hat{\psi}_\alpha^{\dagger},\hat{\psi}_\alpha\right\}$.

The states (46) form an orthonormal basis for the complete multiparticle system and satisfy the following closure relation:

$$\sum_{\{\alpha_i\}} \left| n_{\alpha_1} \cdots n_{\alpha_i} \cdots \right\rangle \left\langle n_{\alpha_1} \cdots n_{\alpha_i} \cdots \right| = 1 \tag{47}$$

Let us now define the creation and annihilation operators $\hat{\psi}_\alpha^\dagger$ and $\hat{\psi}_\alpha$ for the α^{th} type boson such that

$$\hat{\psi}_\alpha^\dagger \left| n_{\alpha_1} \cdots n_{\alpha_P} \right\rangle = \sqrt{n_\alpha + 1} \left| n_{\alpha_1} \cdots (n_\alpha + 1) \cdots n_{\alpha_P} \right\rangle \quad , \quad \hat{\psi}_\alpha \left| n_{\alpha_1} \cdots n_{\alpha_P} \right\rangle = \sum_{i=1}^{P} \delta_{\alpha,\alpha_i} \sqrt{n_\alpha} \left| n_{\alpha_1} \cdots (n_\alpha - 1) \cdots n_{\alpha_P} \right\rangle \tag{48}$$

Here, $\hat{\psi}_\alpha^\dagger$ increases, by one, the number of particles in the α^{th} eigenstates; its adjoint $\hat{\psi}_\alpha$ reduces, by one, the number of particles. The operators (48) are therefore eigenstates of the operator $\hat{n}_\alpha = \hat{\psi}_\alpha^\dagger \hat{\psi}_\alpha$, which measures the number of particles in the one-body state $|\alpha\rangle$:

$$\hat{n}_\alpha \left| n_{\alpha_1} \cdots n_{\alpha_P} \right\rangle = \sum_{i=1}^{P} \delta_{\alpha,\alpha_i} n_\alpha \left| n_{\alpha_1} \cdots n_{\alpha_P} \right\rangle \tag{49}$$

1.1.3 (Anti)Commutation Relations

We now examine the (anti)commutation of creation and annihilation operators. The (anti)symmetry properties of the many-particle states impose (anti)commutation relations among the creation operators. We consider the case in which any two single-particle states $|\alpha\rangle$ and $|\alpha'\rangle$ belong to the orthonormal basis $\{|\alpha\rangle\}$ for any state $|\alpha_1 \cdots \alpha_n\}$, then

$$\hat{\psi}_\alpha^\dagger \hat{\psi}_{\alpha'}^\dagger |\alpha_1 \cdots \alpha_n\} = |\alpha\alpha'\alpha_1 \cdots \alpha_n\} = \chi |\alpha'\alpha\alpha_1 \cdots \alpha_n\} = \chi \hat{\psi}_{\alpha'}^\dagger \hat{\psi}_\alpha^\dagger |\alpha_1 \cdots \alpha_n\} \tag{50}$$

From here,

$$\left(\hat{\psi}_\alpha^\dagger \hat{\psi}_{\alpha'}^\dagger - \chi \hat{\psi}_{\alpha'}^\dagger \hat{\psi}_\alpha^\dagger \right) |\alpha_1 \cdots \alpha_n\} = 0 \tag{51}$$

So,

$$\hat{\psi}_\alpha^\dagger \hat{\psi}_{\alpha'}^\dagger - \chi \hat{\psi}_{\alpha'}^\dagger \hat{\psi}_\alpha^\dagger = 0 \tag{52}$$

and taking the adjoint of (52), then

$$\hat{\psi}_\alpha \hat{\psi}_{\alpha'} - \chi \hat{\psi}_{\alpha'} \hat{\psi}_\alpha = 0 \tag{53}$$

We may also write

$$\hat{\psi}_\alpha^\dagger \hat{\psi}_{\alpha'}^\dagger - \chi \hat{\psi}_{\alpha'}^\dagger \hat{\psi}_\alpha^\dagger = \left[\hat{\psi}_\alpha^\dagger, \hat{\psi}_{\alpha'}^\dagger \right]_{-\chi} = \hat{\psi}_\alpha \hat{\psi}_{\alpha'} - \chi \hat{\psi}_{\alpha'} \hat{\psi}_\alpha = \left[\hat{\psi}_\alpha, \hat{\psi}_{\alpha'} \right]_{-\chi} = 0 \tag{54}$$

In equations (52) and (53), for the case of bosons $\chi = +1$, we have the commutators

$$\hat{\psi}_\alpha^\dagger \hat{\psi}_{\alpha'}^\dagger - \hat{\psi}_{\alpha'}^\dagger \hat{\psi}_\alpha^\dagger = \hat{\psi}_\alpha \hat{\psi}_{\alpha'} - \hat{\psi}_{\alpha'} \hat{\psi}_\alpha = 0 \tag{55}$$

and for the case of fermions, we have the anticommutators

$$\hat{\psi}^\dagger_\alpha \hat{\psi}^\dagger_{\alpha'} + \hat{\psi}^\dagger_{\alpha'} \hat{\psi}^\dagger_\alpha = \hat{\psi}_\alpha \hat{\psi}_{\alpha'} + \hat{\psi}_{\alpha'} \hat{\psi}_\alpha = 0 \tag{56}$$

Comparing

$$\hat{\psi}_\alpha \hat{\psi}^\dagger_{\alpha'} \left|\alpha_1 \cdots \alpha_n\right\} = \hat{\psi}_\alpha \left|\alpha' \alpha_1 \cdots \alpha_n\right\} = \delta_{\alpha\alpha'} \left|\alpha_1 \cdots \alpha_n\right\} + \sum_{i=1}^{n} \chi^i \delta_{\alpha\alpha_i} \left|\alpha' \alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_n\right\} \tag{57}$$

and

$$\hat{\psi}^\dagger_{\alpha'} \hat{\psi}_\alpha \left|\alpha_1 \cdots \alpha_n\right\} = \hat{\psi}^\dagger_{\alpha'} \sum_{i=1}^{n} \chi^{i-1} \delta_{\alpha\alpha_i} \left|\alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_n\right\} = \sum_{i=1}^{n} \chi^{i-1} \delta_{\alpha\alpha_i} \left|\alpha' \alpha_1 \cdots \hat{\alpha}_i \cdots \alpha_n\right\} \tag{58}$$

then also

$$\hat{\psi}_\alpha \hat{\psi}^\dagger_{\alpha'} - \chi \hat{\psi}^\dagger_{\alpha'} \hat{\psi}_\alpha = \left[\hat{\psi}_\alpha, \hat{\psi}^\dagger_{\alpha'}\right]_\chi = \delta_{\alpha\alpha'} \tag{59}$$

We consider relation (36) and calculate the following anti(commutation) relation

$$\left[\psi(\vec{r}\,'), \psi^\dagger(\vec{r})\right]_\chi = \sum_{\alpha\alpha'} \varphi^*_\alpha(\vec{r}) \varphi_{\alpha'}(\vec{r}\,') \left\{\psi_{\alpha'} \psi^\dagger_\alpha - \chi \psi^\dagger_\alpha \psi_{\alpha'}\right\} = \sum_{\alpha\alpha'} \varphi^*_\alpha(\vec{r}) \varphi_{\alpha'}(\vec{r}\,') \delta_{\alpha\alpha'} = \sum_\alpha \varphi^*_\alpha(\vec{r}) \varphi_\alpha(\vec{r}\,') \tag{60}$$

So,

$$\psi(\vec{r}\,') \psi^\dagger(\vec{r}) - \chi \psi^\dagger(\vec{r}) \psi(\vec{r}\,') = \sum_{\alpha\alpha'} \varphi^*_\alpha(\vec{r}) \varphi_{\alpha'}(\vec{r}\,') \delta_{\alpha\alpha'} = \sum_\alpha \varphi^*_\alpha(\vec{r}) \varphi_\alpha(\vec{r}\,') = \delta(\vec{r} - \vec{r}\,') \tag{61}$$

This means that the operator function $\psi^\dagger(\vec{r})$ creates a particle with coordinate $\vec{r}$. It is instructive to note that if $\psi_\alpha, \psi^\dagger_\alpha$ were not operators but the c-numbers, then $\psi(\vec{r})$ and $\psi^\dagger(\vec{r})$ in (36) would be considered as one-particle conjugated wave functions. Generally, to introduce wave functions for a particle, it is necessary to change from classical to quantum mechanics by performing a so-called first quantization. Therefore, in the transformation from wave functions to operator functions $\psi(\vec{r})$ and $\psi^\dagger(\vec{r})$, with the help of (61), we perform a so-called second quantization.

1.1.4 Change of Basis in Second Quantization

Different quantum operators are expressed most naturally in different representations, which makes basis changes an essential issue in quantum physics. Next, we characterize the Fock space bases introduced earlier to a full reformulation of many-body quantum mechanics and then introduce general transformation rules that will be exploited further in this book. To find out how changes from one ordered single-particle basis $\{|\alpha\rangle\}$ to another $\{|\tilde{\alpha}\rangle\}$ affect the operator algebra $\{\hat{\psi}_\alpha\}$:

$$|\tilde{\alpha}\rangle = \sum_\alpha |\alpha\rangle\langle\alpha|\tilde{\alpha}\rangle = \sum_\alpha \langle\tilde{\alpha}|\alpha\rangle^* |\alpha\rangle \tag{62}$$

For single-particle systems, we conveniently define creation operators $\hat{\psi}^\dagger_{\tilde{\alpha}}$ and $\hat{\psi}^\dagger_\alpha$, which correspond to the two basis sets $\{|\tilde{\alpha}\rangle\}$, to another $\{|\alpha\rangle\}$:

$$\hat{\psi}^\dagger_{\tilde{\alpha}} \left|\alpha_1 \cdots \alpha_n\right\} = \left|\tilde{\alpha} \alpha_1 \cdots \alpha_n\right\} = \sum_\alpha \langle\alpha|\tilde{\alpha}\rangle \left|\alpha \alpha_1 \cdots \alpha_n\right\rangle = \sum_\alpha \langle\alpha|\tilde{\alpha}\rangle \hat{\psi}^\dagger_\alpha \left|\alpha_1 \cdots \alpha_n\right\} \tag{63}$$

Therefore, the transformation rules for creation (annihilation) field operators

$$\hat{\psi}^{\dagger}_{\tilde{\alpha}} = \sum_{\alpha} \langle \alpha | \tilde{\alpha} \rangle \hat{\psi}^{\dagger}_{\alpha} \quad , \quad \hat{\psi}_{\tilde{\alpha}} = \sum_{\alpha} \langle \tilde{\alpha} | \alpha \rangle \hat{\psi}_{\alpha} \tag{64}$$

The general validity of equation (64) stems from applying the first quantization single-particle result in equation (62) to the n-particle first quantized basis states $|\alpha_1 \cdots \alpha_n\}$. This yields

$$\hat{\psi}^{\dagger}_{\tilde{\alpha}_1} \cdots \hat{\psi}^{\dagger}_{\tilde{\alpha}_n} |0\rangle = \left(\sum_{\alpha_1} \langle \alpha_1 | \tilde{\alpha}_1 \rangle \hat{\psi}^{\dagger}_{\alpha_1} \right) \cdots \left(\sum_{\alpha_n} \langle \alpha_n | \tilde{\alpha}_n \rangle \hat{\psi}^{\dagger}_{\alpha_n} \right) |0\rangle \tag{65}$$

The transformation rules (64) lead to two important results:

1. The basis transformation preserves the bosonic or fermionic particle statistics

$$\left[\hat{\psi}_{\tilde{\alpha}}, \hat{\psi}^{\dagger}_{\tilde{\alpha}'} \right]_{\chi} = \sum_{\alpha\alpha'} \langle \tilde{\alpha} | \alpha \rangle \langle \alpha' | \tilde{\alpha}' \rangle \left\{ \hat{\psi}_{\alpha} \hat{\psi}^{\dagger}_{\alpha'} - \chi \hat{\psi}^{\dagger}_{\alpha'} \hat{\psi}_{\alpha} \right\} = \sum_{\alpha\alpha'} \langle \tilde{\alpha} | \alpha \rangle \langle \alpha' | \tilde{\alpha}' \rangle \delta_{\alpha\alpha'} = \delta_{\tilde{\alpha}\tilde{\alpha}'} \tag{66}$$

2. The basis transformation leaves the total number of particles invariant

$$\sum_{\tilde{\alpha}} \hat{\psi}^{\dagger}_{\tilde{\alpha}} \hat{\psi}_{\tilde{\alpha}} = \sum_{\tilde{\alpha}\alpha\alpha'} \langle \alpha | \tilde{\alpha} \rangle \langle \tilde{\alpha} | \alpha' \rangle \hat{\psi}^{\dagger}_{\alpha} \hat{\psi}_{\alpha'} = \sum_{\alpha\alpha'} \langle \alpha | \alpha' \rangle \hat{\psi}^{\dagger}_{\alpha} \hat{\psi}_{\alpha'} = \sum_{\alpha\alpha'} \delta_{\alpha\alpha'} \hat{\psi}^{\dagger}_{\alpha} \hat{\psi}_{\alpha'} = \sum_{\alpha} \hat{\psi}^{\dagger}_{\alpha} \hat{\psi}_{\alpha} \tag{67}$$

When the new basis is orthonormal, the (anti)commutation relations are preserved and the following transformation is unitary:

$$\left\{ \hat{\psi}^{\dagger}_{\alpha}, \hat{\psi}_{\alpha} \right\} \to \left\{ \hat{\psi}^{\dagger}_{\tilde{\alpha}}, \hat{\psi}_{\tilde{\alpha}} \right\} \tag{68}$$

1.1.5 Quantum Field Operators

For many applications, the coordinate representation turns out to be suitable, which leads to the definition of quantum field operators. We define the **second-quantized field operators** at every point in space as follows

$$\begin{aligned} \hat{\psi}^{\dagger}_{\sigma}(\vec{r}) &= \sum_{\alpha} \langle \alpha | \vec{r}, \sigma \rangle \hat{\psi}^{\dagger}_{\alpha} = \frac{1}{\sqrt{V}} \sum_{\vec{\kappa}} \exp\{-i\vec{\kappa}\vec{r}\} \hat{\psi}^{\dagger}_{\sigma}(\vec{\kappa}) \quad , \quad \hat{\psi}_{\sigma}(\vec{r}) = \sum_{\alpha} \langle \vec{r}, \sigma | \alpha \rangle \hat{\psi}_{\alpha} \\ &= \frac{1}{\sqrt{V}} \sum_{\vec{\kappa}} \exp\{i\vec{\kappa}\vec{r}\} \hat{\psi}_{\sigma}(\vec{\kappa}) \end{aligned} \tag{69}$$

Here, the sum extends over all states α of the orthonormal basis. In (69), the last relations are achieved by selecting the momentum-representation basis $\{|\vec{\kappa}, \alpha\rangle\}$. The field operators satisfy the following (anti)commutation relations

$$\left[\hat{\psi}^{\dagger}_{\sigma}(\vec{r}), \hat{\psi}^{\dagger}_{\sigma'}(\vec{r}') \right]_{-\chi} = \left[\hat{\psi}_{\sigma}(\vec{r}), \hat{\psi}_{\sigma'}(\vec{r}') \right]_{-\chi} = 0 \quad , \quad \left[\hat{\psi}_{\sigma}(\vec{r}), \hat{\psi}^{\dagger}_{\sigma'}(\vec{r}') \right]_{-\chi} = \delta_{\sigma\sigma'} \delta(\vec{r} - \vec{r}') \tag{70}$$

In a way, the quantum field operators express the essence of the wave/particle duality in quantum physics. On the one hand, they are defined as fields. This implies some type of waves. However, on the other hand, they exhibit the commutator properties associated with particles.

Considering equations (34) and (43), we have

$$\hat{\psi}_{\sigma}^{\dagger}(\vec{r})\left|\vec{r}_1,\sigma_1,\cdots,\vec{r}_n,\sigma_n\right\} = \left|\vec{r},\sigma,\vec{r}_1,\sigma_1,\cdots,\vec{r}_n,\sigma_n\right\} \tag{71}$$

$$\hat{\psi}_{\sigma}(\vec{r})\left|\vec{r}_1,\sigma_1,\cdots,\vec{r}_n,\sigma_n\right\} = \sum_{i=1}^{n}\chi^{i-1}\delta(\vec{r}-\vec{r}_i)\left|\vec{r}_1,\sigma_1,\cdots,\widehat{\vec{r}_i,\sigma_i},\cdots\vec{r}_n,\sigma_n\right\} \tag{72}$$

where $\widehat{\vec{r}_i,\sigma_i}$ implies that $\vec{r}_i,\sigma_i$ is omitted. So, the field operator $\hat{\psi}_{\sigma}^{\dagger}(\vec{r})$ adds a spin-σ particle at point $\vec{r}$ and (anti)symmetrizes the resultant many-body state.

1.1.6 Operators in Second-Quantized Form

Second quantization provides a natural formalism for describing many-particle systems. In this section, we examine the case of a system of n interacting particles. We note that, in reality, particles do interact with one another. We present a general theory in which the particles not only interact with the external potential, say the operator $\hat{v}^{(1)}$, but also interact with each other via the potential, say $\hat{v}^{(2)}$. The state operators describing physical states should be (anti)symmetric under the exchange of two particles. This depends on the statistics of the particles and whether they are fermions or bosons. We note that any operator acting within the Fock space may be written in second quantization. When all operators are expressed in terms of the fundamental creation and annihilation operators, we consider the example of one-, two-, and n-body operators.

1.1.6.1 One-Body Operator

The operator $\hat{v}^{(1)}$ is a one-body operator that acts on each particle separately:

$$\hat{v}^{(1)}\left|\alpha_1\cdots\alpha_n\right) = \sum_{i=1}^{n}\hat{v}_i\left|\alpha_1\cdots\alpha_n\right) \quad , \quad \hat{v}^{(1)}\left|\alpha_1\cdots\alpha_n\right\} = \frac{1}{\sqrt{n!}}\sum_{\mathrm{P}}\chi^{\mathrm{P}}\sum_{i=1}^{n}\hat{v}_i\left|\alpha_{\mathrm{P}(1)}\cdots\alpha_{\mathrm{P}(n)}\right) \tag{73}$$

where $\hat{v}_i$ operates only on the i^{th} particle. For example, say

$$\hat{v}_1\left|\alpha_1\cdots\alpha_n\right) = \left(\hat{v}\left|\alpha_1\right\rangle\right)\otimes\left|\alpha_2\right\rangle\otimes\cdots\otimes\left|\alpha_n\right\rangle \tag{74}$$

Suppose we first choose a basis where the operator $\hat{v}$ is diagonal:

$$\hat{v}\left|\alpha\right\rangle = \left\langle\alpha\right|\hat{v}\left|\alpha\right\rangle\left|\alpha\right\rangle \equiv v_{\alpha}\left|\alpha\right\rangle \tag{75}$$

So,

$$\hat{v}^{(1)}\left|\alpha_1\cdots\alpha_n\right\} = \frac{1}{\sqrt{n!}}\sum_{\mathrm{P}}\chi^{\mathrm{P}}\sum_{i=1}^{n}v_{\alpha_{\mathrm{P}(i)}}\left|\alpha_{\mathrm{P}(1)}\cdots\alpha_{\mathrm{P}(n)}\right) = \sum_{i=1}^{n}v_{\alpha_i}\left|\alpha_1\cdots\alpha_n\right\} = \sum_{\alpha}v_{\alpha}\hat{n}_{\alpha}\left|\alpha_1\cdots\alpha_n\right\} \tag{76}$$

Here, the sum extends over the complete set of one-body states α and the number operator $\hat{n}_{\alpha} = \hat{\psi}_{\alpha}^{\dagger}\hat{\psi}_{\alpha}$. From the aforementioned, we see that

$$\hat{v}^{(1)} = \sum_{\alpha}\left\langle\alpha\right|\hat{v}\left|\alpha\right\rangle\hat{n}_{\alpha} \tag{77}$$

To obtain the action of $\hat{v}^{(1)}$, we must sum over all states $\left|\alpha\right\rangle$, multiplying v_{α} by the number of particles in state $\left|\alpha\right\rangle$.

From equation (77), we arrive at the general expression for one-body operators that is valid in any complete basis in terms of the field operators:

$$\hat{v}^{(1)} = \sum_{\alpha\alpha'} \langle\alpha|\hat{v}|\alpha'\rangle \hat{\psi}_{\alpha}^{\dagger}\hat{\psi}_{\alpha'} \tag{78}$$

Formula (78) is precisely the expression of $\hat{v}^{(1)}$ in the **second quantization** that can be rewritten as follows

$$\hat{v}^{(1)} = \int d^d r d^d r' \sum_{\alpha\alpha'} \langle\vec{r},\alpha|\hat{v}|\vec{r}',\alpha'\rangle \hat{\psi}_{\alpha}^{\dagger}(\vec{r})\hat{\psi}_{\alpha'}(\vec{r}') \tag{79}$$

In order to obtain the action of $\hat{v}^{(1)}$, we have to sum over all states $|\alpha\rangle$ and multiplying v_{α} by the number of particles in state $|\alpha\rangle$. It is instructive to note that expressions (78) and (79) make no reference to the total number of particles actually present in the system.

We may compute another operator of basic importance—particle density at the point $\vec{r}$:

$$\hat{n}(\vec{r}) = \int d^d r_1 d^d r_2 \sum_{\alpha\alpha'} \langle\vec{r}_1,\alpha|\delta\left(\vec{r}-\hat{\vec{r}}\right)|\vec{r}_2,\alpha'\rangle \hat{\psi}_{\alpha}^{\dagger}(\vec{r}_1)\hat{\psi}_{\alpha'}(\vec{r}_2) = \sum_{\alpha} \hat{\psi}_{\alpha}^{\dagger}(\vec{r})\hat{\psi}_{\alpha}(\vec{r}) \tag{80}$$

The particle number operator:

$$\hat{N} = \int d^d r \hat{n}(\vec{r}) = \sum_{\alpha} \hat{\psi}_{\alpha}^{\dagger}\hat{\psi}_{\alpha} \tag{81}$$

This applies for any complete basis of the one-body states $|\alpha\rangle$.

Considering the one-body potential

$$\hat{v}^{(1)} = \sum_{i=1}^{n} v\left(\hat{\vec{r}}_i\right) \tag{82}$$

then

$$v^{(1)} = \int d^d r_1 d^d r_2 \sum_{\alpha\alpha'} \langle\vec{r}_1,\alpha|v\left(\hat{\vec{r}}\right)|\vec{r}_2,\alpha'\rangle \hat{\psi}_{\alpha}^{\dagger}(\vec{r}_1)\hat{\psi}_{\alpha'}(\vec{r}_2) = \int d^d r v(\vec{r}) \sum_{\alpha} \hat{\psi}_{\alpha}^{\dagger}(\vec{r})\hat{\psi}_{\alpha}(\vec{r}) = \int d^d r v(\vec{r})\hat{n}(\vec{r}) \tag{83}$$

Considering that $\hat{\mathrm{T}} = \sum_{i=1}^{n} \frac{\hat{p}_i^2}{2m}$ is the kinetic energy operator, then in the second-quantized form, one obtains the following:

$$\begin{aligned} \hat{\mathrm{T}} &= \sum_{\vec{\kappa}\vec{\kappa}'\alpha\alpha'} \langle\vec{\kappa},\alpha|\frac{\hat{p}^2}{2m}|\vec{\kappa}',\alpha'\rangle \hat{\psi}_{\alpha}^{\dagger}(\vec{\kappa})\hat{\psi}_{\alpha'}(\vec{\kappa}') = \sum_{\vec{\kappa}\alpha} \frac{\kappa^2}{2m} \hat{\psi}_{\alpha}^{\dagger}(\vec{\kappa})\hat{\psi}_{\alpha}(\vec{\kappa}) \\ &= \sum_{\vec{\kappa}\alpha} \epsilon(\vec{\kappa})\hat{\psi}_{\alpha}^{\dagger}(\vec{\kappa})\hat{\psi}_{\alpha}(\vec{\kappa}) \quad , \quad \epsilon(\vec{\kappa}) = \frac{\kappa^2}{2m} \end{aligned} \tag{84}$$

This expression is simple and intuitive because the underlying basis diagonalizes the kinetic energy.

1.1.6.2 Two-Body Operator

In this section, we introduce a two-body operator $\hat{v}^{(2)}$ acting on the state $|\alpha_1 \cdots \alpha_n)$ of n particles as the sum of the $\hat{v}^{(2)}$ on all distinct pairs of particles:

$$\hat{v}^{(2)}|\alpha_1 \cdots \alpha_n) = \sum_{\substack{i,j=1\\(i<j)}}^{n} \hat{v}_{ij}|\alpha_1 \cdots \alpha_n) \quad , \quad \hat{v}^{(2)}|\alpha_1 \cdots \alpha_n\} = \frac{1}{\sqrt{n!}} \sum_{\mathrm{P}} \chi^{\mathrm{P}} \sum_{\substack{i,j=1\\(i<j)}}^{n} \hat{v}_{ij} \left|\alpha_{\mathrm{P}(1)} \cdots \alpha_{\mathrm{P}(n)}\right) \tag{85}$$

Here, $\hat{v}_{ij}$ acts only on the i and j particles. The restriction $i < j$ results in a summation over distinct pairs. Considering the basis $\{|\alpha\rangle\}$ where $\hat{v}$ is diagonal, then

$$\hat{v}|\alpha\beta) = \left(\alpha\beta|\hat{v}|\alpha\beta\right)|\alpha\beta) \equiv v_{\alpha\beta}|\alpha\beta) \tag{86}$$

and

$$\begin{aligned} \hat{v}^{(2)}|\alpha_1 \cdots \alpha_n\} &= \frac{1}{\sqrt{n!}} \sum_{\mathrm{P}} \chi^{\mathrm{P}} \sum_{\substack{i,j=1\\(i<j)}}^{n} \hat{v}_{\alpha_{\mathrm{P}(i)}\alpha_{\mathrm{P}(j)}} \left|\alpha_{\mathrm{P}(1)} \cdots \alpha_{\mathrm{P}(n)}\right) \\ &= \sum_{\substack{i,j=1\\(i<j)}}^{n} \hat{v}_{\alpha_i\alpha_j}|\alpha_1 \cdots \alpha_n\} = \frac{1}{2}\sum_{i,j=1}^{n} \left(\hat{v}_{\alpha_i\alpha_j} - \delta_{ij}\hat{v}_{\alpha_i\alpha_i}\right)|\alpha_1 \cdots \alpha_n\} \end{aligned} \tag{87}$$

or

$$\hat{v}^{(2)}|\alpha_1 \cdots \alpha_n\} = \frac{1}{2}\sum_{\alpha\beta} v_{\alpha\beta}\left(\hat{n}_\alpha \hat{n}_\beta - \delta_{\alpha\beta}\hat{n}_\alpha\right)|\alpha_1 \cdots \alpha_n\} \tag{88}$$

Here, the sum extends over all states α,β of the complete basis of the one-body states. From the (anti) commutation of the field operators, we find the operator $\hat{\wp}_{\alpha\beta}$ that counts the number of pairs in the states $|\alpha\rangle$ and $|\beta\rangle$. If $|\alpha\rangle$ and $|\beta\rangle$ are different, then the number of pairs is $n_\alpha n_\beta$; if $|\alpha\rangle = |\beta\rangle$, the number of pairs is $n_\alpha(n_\alpha - 1)$. Hence, the operator counting pairs may be written

$$\hat{\wp}_{\alpha\beta} = \hat{n}_\alpha \hat{n}_\beta - \delta_{\alpha\beta}\hat{n}_\alpha = \psi^\dagger_\alpha \psi_\alpha \psi^\dagger_\beta \psi_\beta - \delta_{\alpha\beta}\psi^\dagger_\alpha \psi_\alpha = \psi^\dagger_\alpha \chi \psi^\dagger_\beta \psi_\alpha \psi_\beta = \psi^\dagger_\alpha \psi^\dagger_\beta \psi_\beta \psi_\alpha \tag{89}$$

then

$$\hat{v}^{(2)} = \frac{1}{2}\sum_{\alpha\beta}\left(\alpha\beta|\hat{v}|\alpha\beta\right)\psi^\dagger_\alpha \psi^\dagger_\beta \psi_\beta \psi_\alpha \tag{90}$$

Similar to the case of the one-body operator, the action of a two-body operator is obtained by summation over pairs of single-particle states $|\alpha\rangle$ and $|\beta\rangle$. Then, we multiply the matrix element $\left(\alpha\beta|\hat{v}|\alpha\beta\right)$ by the number of pairs of such particles present in the physical state.

Transforming from the diagonal representation to an arbitrary basis, we find the general expression of a two-body operator:

$$\hat{v}^{(2)} = \frac{1}{2}\sum_{\alpha_1 \cdots \alpha_2'}\left(\alpha_1\alpha_2|\hat{v}|\alpha_1'\alpha_2'\right)\psi^\dagger_{\alpha_1}\psi^\dagger_{\alpha_2}\psi_{\alpha_2'}\psi_{\alpha_1'} \tag{91}$$

or

$$\hat{v}^{(2)} = \frac{1}{2}\int d^d r_1 d^d r_2' \sum_{\alpha_1 \cdots \alpha_2'} \left(\vec{r}_1, \alpha_1, \vec{r}_2, \alpha_2 \middle| v \middle| \vec{r}_1', \alpha_1', \vec{r}_2', \alpha_2'\right) \hat{\psi}^{\dagger}_{\alpha_1}(\vec{r}_1)\hat{\psi}^{\dagger}_{\alpha_2}(\vec{r}_2)\hat{\psi}_{\alpha_2'}(\vec{r}_2')\hat{\psi}_{\alpha_1'}(\vec{r}_1') \tag{92}$$

Simplicity and convenience are among the great virtues of representing operators via creation and annihilation operators that concisely handle all the bookkeeping for Fermi and Bose statistics. Sometimes, it is convenient to use (92) where the matrix element is indeed (anti)symmetrized:

$$\hat{v}^{(2)} = \frac{1}{4}\sum_{\alpha_1 \cdots \alpha_2'} \left\{\alpha_1\alpha_2 \middle| \hat{v} \middle| \alpha_1'\alpha_2'\right\} \psi^{\dagger}_{\alpha_1}\psi^{\dagger}_{\alpha_2}\psi_{\alpha_2'}\psi_{\alpha_1'} \tag{93}$$

with

$$\left\{\alpha_1\alpha_2 \middle| \hat{v} \middle| \alpha_1'\alpha_2'\right\} = \left(\alpha_1\alpha_2 \middle| \hat{v} \middle| \alpha_1'\alpha_2'\right) + \chi\left(\alpha_1\alpha_2 \middle| \hat{v} \middle| \alpha_2'\alpha_1'\right) \tag{94}$$

We examine the example of a two-body potential

$$\hat{v}^{(2)} = \sum_{(i<j)}^{n} \hat{v}\left(\hat{\vec{r}}_i - \hat{\vec{r}}_j\right) \tag{95}$$

such that

$$\hat{v}^{(2)} = \frac{1}{2}\int d^d r_1 d^d r_2' \sum_{\alpha_1 \cdots \alpha_2'} \left(\vec{r}_1, \alpha_1, \vec{r}_2, \alpha_2 \middle| v\left(\hat{\vec{r}} - \hat{\vec{r}}'\right) \middle| \vec{r}_1', \alpha_1', \vec{r}_2', \alpha_2'\right) \hat{\psi}^{\dagger}_{\alpha_1}(\vec{r}_1)\hat{\psi}^{\dagger}_{\alpha_2}(\vec{r}_2)\hat{\psi}_{\alpha_2'}(\vec{r}_2')\hat{\psi}_{\alpha_1'}(\vec{r}_1') \tag{96}$$

or

$$\hat{v}^{(2)} = \frac{1}{2}\int d^d r_1 d^d r_2 \sum_{\alpha_1\alpha_2} v(\vec{r}_1 - \vec{r}_2)\hat{\psi}^{\dagger}_{\alpha_1}(\vec{r}_1)\hat{\psi}^{\dagger}_{\alpha_2}(\vec{r}_2)\hat{\psi}_{\alpha_2}(\vec{r}_2)\hat{\psi}_{\alpha_1}(\vec{r}_1) \tag{97}$$

Here, we consider the fact that $\hat{v}(\vec{r}_1 - \vec{r}_2)$ is diagonal in the coordinate representation:

$$v\left(\hat{\vec{r}} - \hat{\vec{r}}'\right)\left|\vec{r}_1, \alpha_1, \vec{r}_2, \alpha_2\right) = v(\vec{r}_1 - \vec{r}_2)\left|\vec{r}_1, \alpha_1, \vec{r}_2, \alpha_2\right) \tag{98}$$

If we consider the Fourier space, then equation (97) yields the second-quantized form of the two-body potential in the momentum space:

$$\hat{v}^{(2)} = \frac{1}{2V}\sum_{\vec{\kappa},\vec{\kappa}',\vec{q},\alpha,\alpha'} v(\vec{q})\hat{\psi}^{\dagger}_{\alpha}(\vec{\kappa}+\vec{q})\hat{\psi}^{\dagger}_{\alpha'}(\vec{\kappa}'-\vec{q})\hat{\psi}_{\alpha'}(\vec{\kappa}')\hat{\psi}_{\alpha}(\vec{\kappa}) \tag{99}$$

Here, $v(\vec{q})$ is the Fourier transformed of the interaction potential $v(\vec{r})$. This may be thought of as one particle with initial momentum $\vec{\kappa}'$ interacting with another particle with initial momentum $\vec{\kappa}$ by exchanging a momentum $\vec{q}$; $v(\vec{q})$ is then the matrix element of such a process.

We may make a generalization for the case of an n-body operator in second-quantized form:

$$\hat{v}^{(n)} = \frac{1}{n!} \sum_{\{\alpha_1, \alpha_1'\}} \left(\alpha_1 \cdots \alpha_n | \hat{v} | \alpha_1' \cdots \alpha_n'\right) \hat{\psi}^\dagger_{\alpha_1} \cdots \hat{\psi}^\dagger_{\alpha_n} \hat{\psi}_{\alpha_n'} \cdots \hat{\psi}_{\alpha_1'}$$
$$= \frac{1}{(n!)^2} \sum_{\{\alpha_1, \alpha_1'\}} \left\{\alpha_1 \cdots \alpha_n | \hat{v} | \alpha_1' \cdots \alpha_n'\right\} \hat{\psi}^\dagger_{\alpha_1} \cdots \hat{\psi}^\dagger_{\alpha_n} \hat{\psi}_{\alpha_n'} \cdots \hat{\psi}_{\alpha_1'} \tag{100}$$

Next, it will be useful to examine the normal ordering for a many-particle operator: **An operator is normal-ordered if all the creation operators are to the left of all the annihilation operators**. As an example, the right-hand side of (89) is in normal order. Regardless, any expression not in normal order may be brought into normal order by a sequence of applications of (anti)commutation relations for creation and annihilation operators.

2

Coherent States

Introduction

The *minimum uncertainty wave packets* for the harmonic oscillator were published as *coherent states* by Schrödinger [5]. Recently, coherent states have played an important role in many branches of physics and, in particular, quantum field theory and quantum optics. We have seen earlier that second quantization is the natural formalism for studying many-particle systems. In this chapter, we show how to construct path integral representation using a closure relation based on the eigenstates of the creation or the annihilation operators. We show, in particular, how coherent states are used to obtain a path integral representation of the partition function as well as calculating directly the trace defining the partition function. Quantum field theory combines classical field theory with quantum mechanics and provides analytical tools to understand many-body and relativistic quantum systems. Recently, there have been many advances in controlled fabrication of phase coherent electron devices on the nanoscale and in the realization that ultracold atomic gases exhibit strong interaction and condensation phenomena in Fermi and Bose systems. These advances, along with many others, have resulted in new perspectives on quantum physics of many-particle systems.

This book aims to introduce the ideas and techniques of quantum field theory for the many-particle system. This begins with the introduction of path integrals that provide a description of quantum mechanical time evolution in terms of trajectories. Further, perturbation theory and Feynman diagrams, which provide powerful techniques to approximately evaluate path integrals of more complicated systems, are also introduced to further generalize path integral formalism to many-particle systems. Particular attention is paid to the treatment of fermionic many-particle systems because the corresponding path integral has to be formulated in terms of anticommuting (Grassmann) variables. This also aids us in examining the concept of supersymmetry. To begin, we look at bosonic coherent states and then generalize to the fermionic case.

2.1 Coherent States for Bosons

In the preceding chapter, we used permanents or Slater determinants as a natural basis for the Fock space

$$\mathrm{H} = \mathrm{H}_0 \oplus \mathrm{H}_1 \oplus \cdots = \oplus_{n=0}^{\infty} \mathrm{H}_n \tag{101}$$

Another useful basis of the Fock space is that of coherent states, which are an analog to the basis of position eigenstates in quantum mechanics. Though it is not an orthonormal basis, it spans the entire Fock space. Position states $|\vec{r}\rangle$ are defined as eigenstates of $\hat{\vec{r}}$, while coherent states are defined as eigenstates of the annihilation operators. To see why annihilation operators are selected rather than creation operators, we denote by $|\xi\rangle$ a general vector of the Fock space:

$$|\xi\rangle = \sum_{n=0}^{\infty} \sum_{\alpha_1\cdots\alpha_n} \xi_{\alpha_1\cdots\alpha_n} |\alpha_1\cdots\alpha_n\rangle \tag{102}$$

So, $|\xi\rangle$ has a component with a minimum number of particles. Applying any **creation operator** $\hat{\psi}^{\dagger}$ to $|\xi\rangle$, the minimum number of particles in $|\xi\rangle$ is observed to be increased by one. Hence, the resulting state cannot be a multiple of the initial state, and a creation operator $\hat{\psi}^{\dagger}$ cannot have an eigenstate. Applying an **annihilation operator** $\hat{\psi}$ to $|\xi\rangle$ decreases the maximum number of particles in $|\xi\rangle$ by one. Since $|\xi\rangle$ may contain components with all particle numbers, nothing a priori prohibits $|\xi\rangle$ from having eigenstates. Our goal is to find eigenstates of the (non-Hermitian) Fock space operators $\hat{\psi}^{\dagger}$ and $\hat{\psi}$.

2.2 Coherent States and Overcompleteness

Coherent States

It is useful to note that an important property needed for setting up path integration is the completeness of the states. We now examine coherent states and define eigenstates of the operator $\hat{\psi}$. The physical meaning of the bosonic coherent states can be understood from a study of the system of harmonic oscillators described by the following Hamiltonian equation:

$$\hat{H} = \sum_{\alpha_k} \hat{H}_{\alpha_k} \quad , \quad \hat{H}_{\alpha_k} = \frac{\hat{P}_{\alpha_k}^2}{2m_k} + \frac{m_k\omega_{\alpha_k}^2 \hat{Q}_{\alpha_k}^2}{2} \tag{103}$$

with

$$\left[\hat{Q}_{\alpha_k}, \hat{P}_{\alpha_l}\right] = i\delta_{\alpha_k\alpha_l} \tag{104}$$

Here, m_k, ω_{α_k}, $\hat{Q}_{\alpha_k}$, and $\hat{P}_{\alpha_k}$ are the mass, frequency, position, and momentum operators, respectively, of the oscillator. The ladder operators

$$\hat{\psi}_{\alpha_k} = \sqrt{\frac{m\omega_{\alpha_k}}{2}}\left(\hat{Q}_{\alpha_k} + i\frac{\hat{P}_{\alpha_k}}{m\omega_{\alpha_k}}\right) \quad , \quad \hat{\psi}_{\alpha_k}^{\dagger} = \sqrt{\frac{m\omega_{\alpha_k}}{2}}\left(\hat{Q}_{\alpha_k} - i\frac{\hat{P}_{\alpha_k}}{m\omega_{\alpha_k}}\right) \quad , \quad \hat{P}_{\alpha_k} = -i\frac{\partial}{\partial Q_{\alpha_k}} \tag{105}$$

satisfy the canonical bosonic commutation relations

$$\left[\hat{\psi}_{\alpha_k}, \hat{\psi}_{\alpha_{k'}}\right] = \left[\hat{\psi}_{\alpha_k}^{\dagger}, \hat{\psi}_{\alpha_{k'}}^{\dagger}\right] = 0 \quad , \quad \left[\hat{\psi}_{\alpha_k}, \hat{\psi}_{\alpha_{k'}}^{\dagger}\right] = \delta_{\alpha_k\alpha_{k'}} \tag{106}$$

This permits us to rewrite the Hamiltonian equation as follows:

$$H = \sum_{\alpha_k} \omega_{\alpha_k}\left(\hat{\psi}_{\alpha_k}^{\dagger}\hat{\psi}_{\alpha_k} + \frac{1}{2}\right) \tag{107}$$

Singling out the ground state $|0\rangle$ then

$$\hat{\psi}_{\alpha_k}|0\rangle = 0 \quad , \quad \langle 0|\hat{P}_{\alpha_k}|0\rangle = \langle 0|\hat{Q}_{\alpha_k}|0\rangle = 0 \tag{108}$$

The eigenstates of the Hamiltonian equation (107) can be obtained from the tensor product of the states $|n_{\alpha_k}\rangle$:

$$|n_{\alpha_1}\cdots n_{\alpha_n}\rangle = |n_{\alpha_1}\rangle\otimes\cdots\otimes|n_{\alpha_n}\rangle = \frac{\left(\hat{\psi}^{\dagger}_{\alpha_1}\right)^{n_1}}{\sqrt{n_{\alpha_1}!}}\cdots\frac{\left(\hat{\psi}^{\dagger}_{\alpha_n}\right)^{n_{\alpha n}}}{\sqrt{n_{\alpha_n}!}}|0\rangle \quad , \quad \hat{\mathrm{H}}|n_{\alpha_1}\cdots n_{\alpha_n}\rangle = \sum_{\alpha_k}\omega_{\alpha_k}\left(\hat{n}_{\alpha_k}+\frac{1}{2}\right)|n_{\alpha_1}\cdots n_{\alpha_n}\rangle \tag{109}$$

It is instructive to note that one of the drawbacks of the given states is that they do not serve as eigenstates of either the position $\hat{\mathrm{Q}}$ or the momentum $\hat{\mathrm{P}}$ operator. Moreover, the commutation relation (104) prevents us from searching eigenstates for both operators. Notwithstanding, it is possible to define a **so-called coherent state** $|p_i,q_j\rangle$ having average position and momentum given by some classical value (p_i,q_j):

$$\langle p,q|\hat{\mathrm{P}}_{\alpha_i}|p,q\rangle = p_{\alpha_i} \quad , \quad \langle p,q|\hat{\mathrm{Q}}_{\alpha_i}|p,q\rangle = q_{\alpha_i} \tag{110}$$

We find such a state by considering

$$|p,q\rangle = \exp\left\{-\hat{\mathrm{A}}\right\}|\xi\rangle \tag{111}$$

and the identity

$$\exp\left\{\hat{\mathrm{A}}\right\}\hat{\mathrm{B}}\exp\left\{-\hat{\mathrm{A}}\right\} = \hat{\mathrm{B}}+\frac{1}{1!}\left[\hat{\mathrm{A}},\hat{\mathrm{B}}\right]+\frac{1}{2!}\left[\hat{\mathrm{A}},\left[\hat{\mathrm{A}},\hat{\mathrm{B}}\right]\right]+\cdots \tag{112}$$

that ends at the second term when $\left[\hat{\mathrm{A}},\hat{\mathrm{B}}\right]$ is the c-number. So (110) sets the condition

$$|\xi\rangle \equiv |0\rangle \tag{113}$$

and

$$\hat{\mathrm{A}} = -\sum_{\alpha_k} i\left(p_{\alpha_k}\hat{\mathrm{Q}}_{\alpha_k} - q_{\alpha_k}\hat{\mathrm{P}}_{\alpha_k}\right) \tag{114}$$

with

$$|p,q\rangle = \exp\left\{-\hat{\mathrm{A}}\right\}|0\rangle \tag{115}$$

Then, from (105) and

$$\xi_{\alpha_k} = \sqrt{\frac{m\omega}{2}}\left(q_{\alpha_k}+i\frac{p_{\alpha_k}}{m\omega}\right) \quad , \quad \xi^{\dagger}_{\alpha_k} = \sqrt{\frac{m\omega}{2}}\left(q_{\alpha_k}-i\frac{p_{\alpha_k}}{m\omega}\right) \tag{116}$$

we have

$$|p,q\rangle = \exp\left\{\sum_{\alpha_k}\left(\xi_{\alpha_k}\hat{\psi}^{\dagger}_{\alpha_k} - \xi^{*}_{\alpha_k}\hat{\psi}_{\alpha_k}\right)\right\}|0\rangle = \exp\left\{\hat{\psi}^{\dagger}\xi - \xi^{\dagger}\hat{\psi}\right\}|0\rangle \tag{117}$$

Here,

$$\hat{\psi} = \begin{bmatrix} \hat{\psi}_1 \\ \vdots \\ \hat{\psi}_{\alpha_P} \end{bmatrix}, \quad \xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_{\alpha_P} \end{bmatrix} \tag{118}$$

It can be easily verified that

$$\langle \xi | \left(\hat{P}_{\alpha_k} - p_{\alpha_k} \right)^2 | \xi \rangle = \langle \xi | \left(\hat{Q}_{\alpha_k} - q_{\alpha_k} \right)^2 | \xi \rangle = \frac{1}{2} \tag{119}$$

Therefore, $|\xi\rangle$ is as near as possible to a classical state. Considering (112) and (115), then we rewrite (117) as follows:

$$|\xi\rangle = \exp\left\{ \hat{\psi}^\dagger \xi \right\} |0\rangle = \sum_{\alpha_1 \cdots \alpha_n} \frac{\left(\xi_{\alpha_1} \hat{\psi}^\dagger_{\alpha_1} \right)^{n_{\alpha_1}}}{n_{\alpha_1}!} \cdots \frac{\left(\xi_{\alpha_n} \hat{\psi}^\dagger_{\alpha_n} \right)^{n_{\alpha_n}}}{n_{\alpha_n}!} |0\rangle \tag{120}$$

Since

$$|n_{\alpha_1} \cdots n_{\alpha_n}\rangle = \frac{\left(\hat{\psi}^\dagger_{\alpha_1} \right)^{n_1}}{\sqrt{n_{\alpha_1}!}} \cdots \frac{\left(\hat{\psi}^\dagger_{\alpha_n} \right)^{n_{\alpha_n}}}{\sqrt{n_{\alpha_n}!}} |0\rangle \tag{121}$$

then the given coherent state is defined by the following eigenvector [6]:

$$|\xi\rangle = \sum_{\alpha_1 \cdots \alpha_n} \frac{\left(\xi_{\alpha_1} \right)^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}} \cdots \frac{\left(\xi_{\alpha_P} \right)^{n_{\alpha n}}}{\sqrt{n_{\alpha_n}!}} |n_{\alpha_1} \cdots n_{\alpha_n}\rangle \tag{122}$$

The eigenstate $|n_{\alpha_1} \cdots n_{\alpha_n}\rangle$, as seen earlier in equation (25), has a total number of particles, $n = \sum_{\alpha_k} n_{\alpha_k}$, while the Hilbert space (generically referred to as the Fock space) is written as the direct sum, such as in equation (1).

2.2.1 Overcompleteness of Coherent States

Assume we have constructed an eigenstate $|\xi\rangle$ of the annihilation operators $\hat{\psi}_\alpha$; then the eigenstates and eigenvalues of the bosonic operator can be obtained via the eigenvalue equation:

$$\hat{\psi}_\alpha |\xi\rangle = \hat{\psi}_\alpha \exp\left\{ \hat{\psi}^\dagger \xi_\alpha \right\} |0\rangle = \hat{\psi}_\alpha \sum_{n=0}^{\infty} \frac{\xi_\alpha^n}{n!} \left(\hat{\psi}^\dagger \right)^n |0\rangle = \hat{\psi}_\alpha \sum_{n=0}^{\infty} \frac{\xi_\alpha^n}{\sqrt{n!}} |n\rangle = \sum_{n=0}^{\infty} \frac{\xi_\alpha^n}{\sqrt{n!}} \hat{\psi}_\alpha |n\rangle = \sum_{n=1}^{\infty} \frac{\xi_\alpha^n}{\sqrt{n!}} \sqrt{n} |n-1\rangle$$

$$= \sum_{n=1}^{\infty} \frac{\xi_\alpha^n}{\sqrt{(n-1)!}} |n-1\rangle \underset{n'=n-1}{=} \sum_{n'=0}^{\infty} \frac{\xi_\alpha^{n'+1}}{\sqrt{n'!}} |n'\rangle = \xi_\alpha \sum_{n=0}^{\infty} \frac{\xi_\alpha^n}{\sqrt{n!}} |n\rangle = \xi_\alpha \exp\left\{ \hat{\psi}^\dagger \xi_\alpha \right\} |0\rangle = \xi_\alpha |\xi\rangle \tag{123}$$

Similarly, the adjoint

$$\langle \xi | \hat{\psi}^\dagger_\alpha = \langle \xi | \xi^*_\alpha \tag{124}$$

The action of a creation (annihilation) operator $\hat{\psi}_\alpha^\dagger\left(\hat{\psi}_\alpha\right)$ on a coherent state is obtained as

$$\hat{\psi}_\alpha^\dagger|\xi\rangle = \hat{\psi}_\alpha^\dagger \exp\left\{\sum_{\alpha'}\hat{\psi}_{\alpha'}^\dagger\xi_{\alpha'}\right\}|0\rangle = \frac{\partial}{\partial\xi_\alpha}|\xi\rangle \quad , \quad \langle\xi|\hat{\psi}_\alpha = \frac{\partial}{\partial\xi_\alpha^*}\langle\xi| \tag{125}$$

It is useful to show that the operators $\hat{\psi}_\alpha^\dagger$ and $\hat{\psi}_\alpha$ act in the coherent state representation in the same way as the operators $\hat{\vec{r}}$ and $\hat{\vec{p}}$ act in coordinate representation. From equations (123), (124), and (125), then

$$\langle\xi|\hat{\psi}_\alpha|\phi\rangle = \frac{\partial}{\partial\xi_\alpha^*}\phi\left(\xi^*\right) \quad , \quad \langle\xi|\hat{\psi}_\alpha^\dagger|\phi\rangle = \xi_\alpha^*\phi\left(\xi^*\right) \tag{126}$$

So, symbolically, we can write

$$\hat{\psi}_\alpha = \frac{\partial}{\partial\xi_\alpha^*} \quad , \quad \hat{\psi}_\alpha^\dagger = \xi_\alpha^* \tag{127}$$

This is consistent with the bosonic commutation rules:

$$\left[\xi_\alpha^*,\xi_{\alpha'}^*\right] = \left[\frac{\partial}{\partial\xi_\alpha^*},\frac{\partial}{\partial\xi_\alpha^*}\right] = 0 \quad , \quad \left[\frac{\partial}{\partial\xi_\alpha^*},\xi_{\alpha'}^*\right] = \delta_{\alpha\alpha'} \tag{128}$$

We observe that the behavior of $\hat{\psi}_\alpha$ an $\hat{\psi}_\alpha^\dagger$ in the coherent state representation is thus analogous to that of $\hat{\vec{r}}$ and $\hat{\vec{p}}$ in coordinate representation.

It is important to note that the coherent state with $\xi = 0$ is identical to the Fock vacuum $|0\rangle$. The eigenvalue ξ_α of the bosonic annihilation operator $\hat{\psi}_\alpha$ may be any real or complex number. What is unusual about this definition is that $\hat{\psi}_\alpha$ is not a Hermitian operator (and so is not observable in the usual sense). Nevertheless, the states $|\xi\rangle$ defined in such a way form **a complete set**—indeed **an overcomplete set**—and define a new representation, the **coherent state representation**. Introducing the **overcomplete base** of coherent states widens the concept of the **path integral formalism** in areas of many-particle systems [3, 7].

Because $|\xi\rangle$ is an eigenstate of the annihilation operators $\hat{\psi}_\alpha$, then the (anti)commutation relations

$$\left[\hat{\psi}_\alpha,\hat{\psi}_{\alpha'}\right]_{-\chi} = 0 \tag{129}$$

This implies that

$$\hat{\psi}_\alpha\hat{\psi}_{\alpha'}|\xi\rangle = \hat{\psi}_\alpha\xi_{\alpha'}|\xi\rangle = \xi_\alpha\xi_{\alpha'}|\xi\rangle = \chi\hat{\psi}_{\alpha'}\hat{\psi}_\alpha|\xi\rangle = \chi\hat{\psi}_{\alpha'}\xi_\alpha|\xi\rangle = \chi\xi_{\alpha'}\xi_\alpha|\xi\rangle \tag{130}$$

In the case of fermions, the main difficulty encountered is the lack of a classical limit for the eigenvalue ξ_α. This implies that c-numbers cannot reflect the anticommuting character of fermions. As a way of introducing anticommuting objects, we present anticommuting variables called **Grassmann numbers**, which follow the same concepts used in constructing coherent states for bosons.

2.2.2 Overlap of Two Coherent States

We examine the following inner product or overlap of two coherent states given by:

$$\langle\xi|\xi'\rangle = \sum_{n_{\alpha_1}\cdots n_{\alpha_P}}\sum_{n'_{\alpha_1}\cdots n'_{\alpha_P}}\frac{\left(\xi_{\alpha_1}^*\right)^{n_{\alpha_1}}}{\sqrt{n_{\alpha_1}!}}\cdots\frac{\left(\xi_{\alpha_P}^*\right)^{n_{\alpha_P}}}{\sqrt{n_{\alpha_P}!}}\frac{\left(\xi'_{\alpha_1}\right)^{n'_{\alpha_1}}}{\sqrt{n'_{\alpha_1}!}}\cdots\frac{\left(\xi'_{\alpha_P}\right)^{n'_{\alpha_P}}}{\sqrt{n'_{\alpha_P}!}}\langle n_{\alpha_1}\cdots n_{\alpha_P}|n'_{\alpha_1}\cdots n'_{\alpha_P}\rangle \tag{131}$$

Because the basis $|\alpha\rangle$ is orthonormal, the scalar product

$$\langle n_{\alpha_1}\cdots n_{\alpha_P}|n'_{\alpha_1}\cdots n'_{\alpha_P}\rangle = \delta_{n_{\alpha_1}n'_{\alpha_1}}\cdots\delta_{n_{\alpha_P}n'_{\alpha_P}} \tag{132}$$

and, so,

$$\langle\xi|\xi'\rangle = \exp\left\{\sum_\alpha \xi^*_\alpha \xi'_\alpha\right\} \tag{133}$$

It is instructive to note that the coherent states corresponding to two different values of ξ are not orthogonal states because they do not form a proper basis and are eigenvectors of a non-Hermitian operator. The overlap quickly falls off exponentially with the distance between the two points and gives a measure of the intrinsic uncertainty of the coherent state as probability amplitude in phase space.

From their definition, it is obvious that coherent states do not have a fixed number of particles but that the occupation numbers n_α in the coherent state $|\xi\rangle$ are Poisson distributed with mean values $|\xi_\alpha|^2$:

$$\left|\langle n_{\alpha_1}\cdots n_{\alpha_P}|\xi\rangle\right|^2 = \prod_\alpha \frac{|\xi_\alpha|^{2n_\alpha}}{n_\alpha!} \tag{134}$$

So, the distribution of the total particle number has the average value

$$n = \langle\hat{n}\rangle = \frac{\langle\xi|\hat{n}|\xi\rangle}{\langle\xi|\xi\rangle} = \frac{\langle\xi|\hat{\psi}^\dagger_\alpha\hat{\psi}_\alpha|\xi\rangle}{\langle\xi|\xi\rangle} = \sum_\alpha \xi^*_\alpha\xi_\alpha \tag{135}$$

with the variance

$$\sigma^2 = \left\langle\left(\hat{n} - \langle\hat{n}\rangle\right)^2\right\rangle = \sum_\alpha \xi^*_\alpha\xi_\alpha = \langle\hat{n}\rangle \tag{136}$$

In the thermodynamic limit, where $n \to \infty$, the relative width

$$\frac{\sigma}{n} = \frac{1}{\sqrt{n}} \to 0 \tag{137}$$

In this case, the particle number distribution becomes sharply peaked around n. This indicates that a product of Poisson distributions approaches a normal distribution.

2.2.3 Overcompleteness Condition

The most important property of coherent states is their **overcompleteness** in the Fock space, which implies that any vector of the Fock space can be expanded in terms of coherent states. To obtain a path integral representation, a closure relation for the coherent states is needed. We examine the closure relation (resolution identity) for the bosonic coherent states, which is defined as:

$$\int\prod_\alpha \frac{d\xi^*_\alpha d\xi_\alpha}{2\pi i}\exp\left\{-\sum_\alpha \xi^*_\alpha\xi_\alpha\right\}|\xi\rangle\langle\xi| = \hat{1} \tag{138}$$

In (138), $\hat{1}$ is the Fock space identity operator, and the measure is given by

$$\frac{d\xi^*_\alpha d\xi_\alpha}{2\pi i} = \frac{d(\mathrm{Re}\,\xi_\alpha)\,d(\mathrm{Im}\,\xi_\alpha)}{\pi} \tag{139}$$

As shown in (133), because the coherent states are in general not orthogonal, the set of coherent states is **overcomplete**, while formula (138) shows that the coherent states form a basis in the Fock space. Nonetheless the coherent states are very useful, particularly for deriving path integrals. In this book, they are important as an example of states where the creation and annihilation operators have nonvanishing expectation values. This factor will be essential in the discussion on Bose-Einstein condensates.

2.2.4 Closure Relation via Schur's Lemma

We prove the closure relation via **Schur's lemma**, which in this case states: **If an operator commutes with all creation and annihilation operators, then it is proportional to the unit operator in the Fock space**.

From equation (125), then

$$\left[\hat{\psi}_\alpha, |\xi\rangle\langle\xi|\right] = \left(\xi_\alpha - \frac{\partial}{\partial \xi_\alpha^*}\right)|\xi\rangle\langle\xi| \tag{140}$$

And, by evaluating the commutator (138) and performing integration by parts, we have

$$\left[\hat{\psi}_\alpha, \int\prod_{\alpha'}\frac{d\xi_{\alpha'}^* d\xi_{\alpha'}}{2\pi i}\exp\left\{-\sum_{\alpha'}\xi_{\alpha'}^*\xi_{\alpha'}\right\}|\xi\rangle\langle\xi|\right] = \int\prod_{\alpha'}\frac{d\xi_{\alpha'}^* d\xi_{\alpha'}}{2\pi i}\exp\left\{-\sum_{\alpha'}\xi_{\alpha'}^*\xi_{\alpha'}\right\}\left(\xi_\alpha - \frac{\partial}{\partial \xi_\alpha^*}\right)|\xi\rangle\langle\xi| = 0 \tag{141}$$

If we look at the adjoint of (141), we observe that the left-hand side of (138) commutes with all of the creation as well as the annihilation operators. Therefore, it must be proportional to the unit operator. We can calculate the proportionality factor by taking the expectation value of the left-hand side of (138) in the vacuum:

$$\int\prod_{\alpha}\frac{d\xi_\alpha^* d\xi_\alpha}{2\pi i}\exp\left\{-\sum_{\alpha}\xi_\alpha^*\xi_\alpha\right\}\langle 0|\xi\rangle\langle\xi|0\rangle = \int\prod_{\alpha}\frac{d\xi_\alpha^* d\xi_\alpha}{2\pi i}\exp\left\{-\sum_{\alpha}\xi_\alpha^*\xi_\alpha\right\}\langle 0|\xi\rangle\langle\xi|0\rangle = \hat{1} \tag{142}$$

Therefore, we prove the closure relation in (138).

For a single degree of freedom, we write ξ in polar form

$$\xi = |\xi|\exp\{i\varphi\} \quad , \quad \xi^* = |\xi|\exp\{-i\varphi\} \tag{143}$$

then this changes the variables from (ξ, ξ^*) to $(|\xi|, \varphi)$. Considering that the Jacobian determinant of this variable transformation is $2i|\xi|$, then the measure

$$\frac{d\xi d\xi^*}{2\pi i} = \frac{2i|\xi|\,d\xi d\varphi}{2\pi i} \tag{144}$$

Hence,

$$\int\frac{d\xi d\xi^*}{2\pi i}\exp\left\{-|\xi|^2\right\}|\xi\rangle\langle\xi| = \int_0^\infty\frac{2i|\xi|\,d\xi}{2\pi i}\sum_{n,m=0}^{\infty}\frac{|\xi|^{n+m}}{\sqrt{n!m!}}\exp\left\{-|\xi|^2\right\}|n\rangle\langle m|\int_0^{2\pi}\exp\{i(n-m)\varphi\}d\varphi \tag{145}$$

or

$$\int_0^\infty \frac{2i|\xi|d\xi}{2\pi i}\sum_{n,m=0}^{\infty}\frac{|\xi|^{n+m}}{\sqrt{n!m!}}\exp\left\{-|\xi|^2\right\}|n\rangle\langle m|2\pi\delta_{nm} = 2\pi i\int_0^\infty \frac{2|\xi|d\xi}{2\pi i}\sum_{n=0}^{\infty}\frac{|\xi|^{2n}}{n!}\exp\left\{-|\xi|^2\right\}|n\rangle\langle n| \tag{146}$$

Changing variables again, $z=|\xi|^2$, and using the definition of the Gamma function,

$$\int_0^\infty dz\exp\{-z\}z^n = \Gamma(n+1) = n! \tag{147}$$

then

$$2\int_0^\infty |\xi|d\xi\sum_{n=0}^{\infty}\frac{|\xi|^{2n}}{n!}\exp\left\{-|\xi|^2\right\}|n\rangle\langle n| = \int_0^\infty dz\sum_{n=0}^{\infty}\frac{z^n}{n!}\exp\{-z\}|n\rangle\langle n| = \sum_{n=0}^{\infty}|n\rangle\langle n| = 1 \tag{148}$$

For one degree of freedom and considering the position eigenstates $|q\rangle$ and the coherent states $|\xi\rangle$, then it makes sense to compute their inner product:

$$\langle q|\xi\rangle = \sum_{n=0}^{\infty}\frac{\xi^n}{\sqrt{n!}}\langle q|n\rangle \tag{149}$$

In this case, the wave function $\langle q|n\rangle$ of the n^{th} excited state of the harmonic oscillator:

$$\langle q|n\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\frac{\exp\left\{-\frac{m\omega q^2}{2\hbar}\right\}}{\sqrt{2^n n!}}\mathrm{H}_n\left(\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}q\right) \tag{150}$$

or

$$\langle q|n\rangle = \frac{\sqrt{\alpha}\exp\left\{-\frac{(\alpha q)^2}{2}\right\}}{\sqrt{2^n n!\sqrt{\pi}}}\mathrm{H}_n(\alpha q) \quad , \quad \alpha = \sqrt{\frac{m\omega}{\hbar}} \tag{151}$$

Here $\mathrm{H}_n(x)$ is the Hermite polynomial with argument x. The generating function for the Hermite polynomials $\mathrm{H}_n(x)$:

$$\exp\left\{2xy - y^2\right\} = \sum_{n=0}^{\infty}\mathrm{H}_n(x)\frac{y^n}{n!} \tag{152}$$

Therefore, from the aforementioned,

$$\langle q|\xi\rangle = \sum_{n=0}^{\infty}\frac{\xi^n}{\sqrt{n!}}\langle q|n\rangle = \sum_{n=0}^{\infty}\frac{\xi^n}{\sqrt{n!}}\frac{\sqrt{\alpha}\exp\left\{-\frac{(\alpha q)^2}{2}\right\}}{\sqrt{2^n n!\sqrt{\pi}}}\mathrm{H}_n(\alpha q) \tag{153}$$

or

$$\langle q|\xi\rangle = \sqrt{\frac{\alpha}{\sqrt{\pi}}}\exp\left\{-\frac{(\alpha q)^2}{2}\right\}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\xi}{\sqrt{2}}\right)^n \mathrm{H}_n(\alpha q) = \sqrt{\frac{\alpha}{\sqrt{\pi}}}\exp\left\{-\frac{(\alpha q)^2}{2}\right\}\exp\left\{\sqrt{2}\alpha\xi q - \frac{\xi^*\xi}{2}\right\} \tag{154}$$

or

$$\langle q|\xi\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left\{-\frac{\xi^*\xi}{2} - \frac{m\omega}{2\hbar}q^2 + \sqrt{\frac{2m\omega}{\hbar}}q\xi\right\} \tag{155}$$

2.2.5 Normal-Ordered Operators

In this section, we will examine one property of coherent states—the simple form of matrix elements of **normal-ordered** operators between coherent states. **An operator $\mathbf{A}\left(\hat{\psi}^\dagger_\alpha, \hat{\psi}_\alpha\right)$ is said to be normal-ordered when all creation operators stand to the left of the annihilation operators** $\hat{\psi}_\alpha$. The matrix element between coherent states of such an operator takes the form

$$\langle\xi|\mathbf{A}\left(\hat{\psi}^\dagger_\alpha, \hat{\psi}_\alpha\right)|\xi'\rangle = \exp\left\{\sum_\alpha \xi^*_\alpha\xi'_\alpha\right\}\mathbf{A}\left(\xi^*_\alpha, \xi'_\alpha\right) \tag{156}$$

One example is a two-body potential:

$$\langle\xi|\hat{v}^{(2)}|\xi'\rangle = \frac{1}{2}\sum_{\alpha\alpha'\beta\beta'}(\alpha\alpha'|v|\beta\beta')\langle\xi|\hat{\psi}^\dagger_\alpha\hat{\psi}^\dagger_{\alpha'}\hat{\psi}_{\beta'}\hat{\psi}_\beta|\xi'\rangle \tag{157}$$

or

$$\langle\xi|\hat{v}^{(2)}|\xi'\rangle = \frac{1}{2}\sum_{\alpha\alpha'\beta\beta'}(\alpha\alpha'|v|\beta\beta')\langle\xi|\xi^*_\alpha\xi^*_{\alpha'}\xi'_{\beta'}\xi'_\beta|\xi'\rangle\exp\left\{\sum_\alpha \xi^*_\alpha\xi'_\alpha\right\} \tag{158}$$

2.2.6 The Trace of an Operator

The overcompleteness relation can be used to represent a state of the extended Fock space in terms of coherent states. The completeness relation provides a useful expression for the trace of any operator **A**. Denoting $\{|n\rangle\}$ as a complete set of states, then

$$\begin{aligned}\mathrm{Tr}\,\mathbf{A} &= \sum_n \langle n|\mathbf{A}|n\rangle = \int\prod_\alpha \frac{d\xi^*_\alpha d\xi_\alpha}{2\pi i}\exp\left\{-\sum_\alpha \xi^*_\alpha\xi_\alpha\right\}\sum_n\langle n|\xi\rangle\langle\xi|\mathbf{A}|n\rangle = \\ &= \int\prod_\alpha \frac{d\xi^*_\alpha d\xi_\alpha}{2\pi i}\exp\left\{-\sum_\alpha \xi^*_\alpha\xi_\alpha\right\}\sum_n\langle\xi|\mathbf{A}|n\rangle\langle n|\xi\rangle = \int\prod_\alpha \frac{d\xi^*_\alpha d\xi_\alpha}{2\pi i}\exp\left\{-\sum_\alpha \xi^*_\alpha\xi_\alpha\right\}\langle\xi|\mathbf{A}|\xi\rangle\end{aligned} \tag{159}$$

or

$$\mathrm{Tr}\,\mathbf{A} = \int\prod_\alpha \frac{d\xi^*_\alpha d\xi_\alpha}{2\pi i}\exp\left\{-\sum_\alpha \xi^*_\alpha\xi_\alpha\right\}\langle\xi|\mathbf{A}|\xi\rangle \tag{160}$$

From quantum mechanics, the completeness of position eigenstates permits us to represent a state

$$\left|\hat{\psi}\right\rangle = \int d\vec{r}\,\hat{\psi}(\vec{r})\left|\vec{r}\right\rangle \tag{161}$$

Here, the coordinate representation of the state $\left|\hat{\psi}\right\rangle$ is

$$\hat{\psi}(\vec{r}) = \left\langle \vec{r}\middle|\hat{\psi}\right\rangle \tag{162}$$

Similarly, equation (139) implies any state $\left|\hat{\psi}\right\rangle$ of Fock space can be represented:

$$\left|\hat{\psi}\right\rangle = \int \prod_{\alpha} \frac{d\xi_{\alpha}^{*} d\xi_{\alpha}}{2\pi i} \exp\left\{-\sum_{\alpha} \xi_{\alpha}^{*}\xi_{\alpha}\right\} \left|\xi\right\rangle\left\langle \xi\middle|\hat{\psi}\right\rangle = \int \prod_{\alpha} \frac{d\xi_{\alpha}^{*} d\xi_{\alpha}}{2\pi i} \exp\left\{-\sum_{\alpha} \xi_{\alpha}^{*}\xi_{\alpha}\right\} \hat{\psi}(\xi^{*}) \left|\xi\right\rangle \tag{163}$$

where

$$\left\langle \xi\middle|\hat{\psi}\right\rangle = \hat{\psi}\left(\xi^{*}\right) \tag{164}$$

by definition, is the coherent state representation of the state $\left|\hat{\psi}\right\rangle$, with ξ denoting the set $\left\{\xi_{\alpha}^{*}\right\}$. The coherent state representation for bosons often is referred to as the holomorphic representation. This is due to the fact that ψ is an analytic function of the variables ξ_{α}^{*}. Physically, the quantity $\hat{\psi}\left(\xi^{*}\right)$ simply is the wave function of the state $\left|\hat{\psi}\right\rangle$ in the coherent state representation. This implies the probability amplitude to find the system in the coherent state $\left|\xi\right\rangle$.

In the case of holomorphic functions $\hat{\psi}\left(\xi^{*}\right)$, the unit operator can be achieved via (138):

$$\left\langle \xi\middle|\hat{\psi}\right\rangle = \int \prod_{\alpha} \frac{d\xi_{\alpha}^{\prime *} d\xi_{\alpha}^{\prime}}{2\pi i} \exp\left\{-\sum_{\alpha} \xi_{\alpha}^{\prime *}\xi_{\alpha}^{\prime}\right\} \left\langle \xi\middle|\xi^{\prime}\right\rangle\left\langle \xi^{\prime}\middle|\hat{\psi}\right\rangle \tag{165}$$

Therefore, it follows that

$$\hat{\psi}\left(\xi^{*}\right) = \int \prod_{\alpha} \frac{d\xi_{\alpha}^{\prime *} d\xi_{\alpha}^{\prime}}{2\pi i} \exp\left\{-\sum_{\alpha} \left(\xi_{\alpha}^{\prime *} - \xi_{\alpha}^{*}\right)\xi_{\alpha}^{\prime}\right\} \hat{\psi}\left(\xi^{\prime *}\right) \tag{166}$$

It is instructive to note that this simply is a general form in the complex plane for the familiar representation of a Dirac delta function:

$$\delta\left(x - x^{\prime}\right) = \int \frac{dk}{2\pi} \exp\left\{ik\left(x - x^{\prime}\right)\right\} \tag{167}$$

2.3 Grassmann Algebra and Fermions

2.3.1 Grassmann Algebra

The path integral approach is easy to employ for bosonic systems due to commuting functions instead of anticommuting operators [8, 9]. However, such an advantage is not obvious for fermionic systems because the integration variables are anticommuting. We will now discuss a fermionic system within the framework of the fermionic coherent state path integral. When dealing with fields instead of operators, we apply **Grassmann algebra**, which maintains the **Pauli Exclusion Principle**. Grassmann algebra allows us to elaborate all necessary calculation rules to derive the path integral and, subsequently, the Dyson equation for fermionic functionals that are accordingly functionals of Grassmann functions.

We consider the distribution law for Grassmann variables:

$$(\xi_1+\xi_2)\xi_3=\xi_1\xi_3+\xi_2\xi_3 \quad , \quad \xi_1(\xi_2+\xi_3)=\xi_1\xi_2+\xi_1\xi_3 \quad , \quad \lambda(\xi_1\xi_2)=(\lambda\xi_1)\xi_2=\xi_1(\lambda\xi_2) \tag{168}$$

as well as the anticommutative property

$$\{\xi_i,\xi_j\}=\xi_i\xi_j+\xi_j\xi_i=0 \tag{169}$$

This permits us to arrive at the square of any generator vanishing for any k:

$$\xi_k^2=0 \tag{170}$$

This is a particular important property of the anticommutation relation of equation 169.

We can construct a finite dimensional Grassmann algebra from n such elements, which are called generators $\{\xi_k\}$, $k=1,\cdots,n$. Then, from property (170), all elements of the given algebra can now be expressed via a linear combination of these generators:

$$\{1,\xi_{\lambda_1},\xi_{\lambda_1}\xi_{\lambda_2},\cdots,\xi_{\lambda_1},\xi_{\lambda_2}\cdots\cdots\xi_{\lambda_n}\} \tag{171}$$

Here, $0<\xi_k\leq n$, and we assume that the elements, by convention, are ordered by the indices $\lambda_1<\lambda_2<\cdots<\lambda_n$. From (170), there exists no element of the higher products containing more than one ξ_k. So now, any element of the n-dimensional Grassmann algebra can be expressed as a polynomial of first order in the generators:

$$f(\xi_1,\cdots,\xi_n)=f_0+\sum_{\alpha_1}f_{\alpha_1}\xi_1+\sum_{\alpha_1<\alpha_2}f_{\alpha_1\alpha_2}\xi_1\xi_2+\cdots+\sum_{\alpha_1<\alpha_2<\cdots<\alpha_n}f_{\alpha_1\alpha_2\cdots\alpha_n}\xi_{\alpha_1}\xi_{\alpha_2}\cdots\xi_{\alpha_n} \tag{172}$$

where the complex valued coefficients $f_k\in\mathbb{C}$ are complex numbers or complex functions. So, f is a function of the generators and a complex variable; therefore, we refer to objects shown in equation (172) as Grassmann functions.

In order to operate using Grassmann functions, it is necessary to define analog operations of differentiation and integration for Grassmann functions.

2.3.1.1 Differentiation over Grassmann Variables

We define differentiation over a Grassmann variable (generator) as:

$$\frac{d}{d\xi_j}\xi_{\lambda_1}\xi_{\lambda_2}\cdots\xi_{\lambda_n}=\delta_{j\lambda_1}\xi_{\lambda_2}\cdots\cdots\xi_{\lambda_n}-\delta_{j\lambda_2}\xi_{\lambda_1}\xi_{\lambda_3}\cdots\cdots\xi_{\lambda_n}+\cdots+(-1)^{n-1}\delta_{j\lambda_n}\xi_{\lambda_1}\xi_{\lambda_2}\cdots\xi_{\lambda_{n-1}} \tag{173}$$

The derivative over Grassmann numbers is precisely left-sided. In essence, we have to anticommute the variable to the left and apply the rules

$$\frac{d}{d\xi_i}1=0 \quad , \quad \frac{d}{d\xi_i}\xi_j=\delta_{ij} \tag{174}$$

This is a linear operation that is the same for ordinary numbers.

2.3.1.1.1 Grassmann Function Differentiation Rules

To examine differentiation rules for Grassmann functions, we consider some important properties of Grassmann generators. This can be done by first examining the following commutation of two Grassmann numbers with a single Grassmann number:

$$[\eta_1\eta_2,\xi] = \eta_1\eta_2\xi - \xi\eta_1\eta_2 = -\eta_1\xi\eta_2 - \xi\eta_1\eta_2 = \xi\eta_1\eta_2 - \xi\eta_1\eta_2 = 0 \tag{175}$$

From induction, it is obvious that any even number of Grassmann numbers commutes with a single Grassmann number:

$$[\xi_1\xi_2\cdots\xi_{2n},\eta] = 0 \tag{176}$$

Similarly, we assume

$$[\eta_1\eta_2\cdots\eta_{2n},\xi] = 0 \tag{177}$$

Then we examine the case:

$$\begin{aligned}
&[\eta_1\eta_2\cdots\eta_{2n+2},\xi] = \eta_1\eta_2\cdots\eta_{2n}\eta_{2n+1}\eta_{2n+2}\xi - \xi\eta_1\eta_2\cdots\eta_{2n}\eta_{2n+1}\eta_{2n+2} = \\
&= -\eta_1\eta_2\cdots\eta_{2n}\eta_{2n+1}\xi\eta_{2n+2} - \xi\eta_1\eta_2\cdots\eta_{2n}\eta_{2n+1}\eta_{2n+2} = \\
&= \eta_1\eta_2\cdots\eta_{2n}\xi\eta_{2n+1}\eta_{2n+2} - \xi\eta_1\eta_2\cdots\eta_{2n}\eta_{2n+1}\eta_{2n+2} = (\eta_1\eta_2\cdots\eta_{2n}\xi - \xi\eta_1\eta_2\cdots\eta_{2n})\eta_{2n+1}\eta_{2n+2}
\end{aligned} \tag{178}$$

With the swap in positions of η_{2n+2} and ξ in the first term, there is a sign change for that term and

$$\begin{aligned}
&[\eta_1\eta_2\cdots\eta_{2n+2},\xi] = -\eta_1\eta_2\cdots\eta_{2n}\eta_{2n+1}\xi\eta_{2n+2} - \xi\eta_1\eta_2\cdots\eta_{2n}\eta_{2n+1}\eta_{2n+2} = \\
&= \eta_1\eta_2\cdots\eta_{2n}\xi\eta_{2n+1}\eta_{2n+2} - \xi\eta_1\eta_2\cdots\eta_{2n}\eta_{2n+1}\eta_{2n+2} = [\eta_1\eta_2\cdots\eta_{2n},\xi]\eta_{2n+1}\eta_{2n+2}
\end{aligned} \tag{179}$$

By induction from (177), then

$$[\eta_1\eta_2\cdots\eta_{2n+2},\xi] = 0 \tag{180}$$

We also show that the commutation of an arbitrary number of even Grassmann numbers with any other number of Grassmann numbers:

$$[\eta_1\eta_2\cdots\eta_{2n},\xi_1\xi_2\cdots\xi_k] = 0 \tag{181}$$

We examine the case:

$$[\eta_1\eta_2\cdots\eta_{2n},\xi_1\xi_2\cdots\xi_{k+1}] = \eta_1\eta_2\cdots\eta_{2n}\xi_1\xi_2\cdots\xi_k\xi_{k+1} - \xi_1\xi_2\cdots\xi_k\xi_{k+1}\eta_1\eta_2\cdots\eta_{2n} \tag{182}$$

Swapping the positions of ξ_k and ξ_{k+1} in the second term, then

$$[\eta_1\eta_2\cdots\eta_{2n},\xi_1\xi_2\cdots\xi_{k+1}] = \eta_1\eta_2\cdots\eta_{2n}\xi_1\xi_2\cdots\xi_k\xi_{k+1} - \xi_1\xi_2\cdots\xi_k\eta_1\eta_2\cdots\eta_{2n}\xi_{k+1} \tag{183}$$

or

$$[\eta_1\eta_2\cdots\eta_{2n},\xi_1\xi_2\cdots\xi_{k+1}] = [\eta_1\eta_2\cdots\eta_{2n},\xi_1\xi_2\cdots\xi_k]\xi_{k+1} \tag{184}$$

So, from induction and considering (181), then

$$\left[\eta_1\eta_2\cdots\eta_{2n},\xi_1\xi_2\cdots\xi_{k+1}\right]=0 \tag{185}$$

We show that two arbitrary odd numbers of Grassmann numbers anticommute:

$$\left\{\eta_1\eta_2\cdots\eta_{2n+1},\xi_1\xi_2\cdots\xi_{2k+1}\right\}=0 \tag{186}$$

One Grassmann number anticommutes with an odd number of Grassmann variables and we assume that

$$\left\{\eta_1\eta_2\cdots\eta_{2n+1},\xi\right\}=0 \tag{187}$$

Hence,

$$\left\{\eta_1\eta_2\cdots\eta_{2n+3},\xi\right\}=\eta_1\eta_2\cdots\eta_{2n+3}\xi+\xi\eta_1\eta_2\cdots\eta_{2n+3}=\eta_1\eta_2\cdots\eta_{2n+1}\eta_{2n+2}\eta_{2n+3}\xi+\xi\eta_1\eta_2\cdots\eta_{2n+1}\eta_{2n+2}\eta_{2n+3} \tag{188}$$

After swapping Grassmann variables in the first term of the last relation, then

$$\left\{\eta_1\eta_2\cdots\eta_{2n+3},\xi\right\}=\eta_1\eta_2\cdots\eta_{2n+1}\xi\eta_{2n+2}\eta_{2n+3}+\xi\eta_1\eta_2\cdots\eta_{2n+1}\eta_{2n+2}\eta_{2n+3}=\left\{\eta_1\eta_2\cdots\eta_{2n+1},\xi\right\}\eta_{2n+2}\eta_{2n+3} \tag{189}$$

From (187) and by induction, we arrive at

$$\left\{\eta_1\eta_2\cdots\eta_{2n+3},\xi\right\}=0 \tag{190}$$

Therefore, for an arbitrary $n>0$ and for any k by induction, we have

$$\left\{\eta_1\eta_2\cdots\eta_{2n+1},\xi_1\xi_2\cdots\xi_{2k+1}\right\}=0 \tag{191}$$

We now consider

$$\left\{\eta_1\eta_2\cdots\eta_{2n+1},\xi_1\xi_2\cdots\xi_{2k+3}\right\}=\eta_1\eta_2\cdots\eta_{2n+1}\xi_1\xi_2\cdots\xi_{2k+3}+\xi_1\xi_2\cdots\xi_{2k+3}\eta_1\eta_2\cdots\eta_{2n+1} \tag{192}$$

and insert the Grassmann variables ξ_{2k+1} and ξ_{2k+2} in the terms so that

$$\left\{\eta_1\eta_2\cdots\eta_{2n+1},\xi_1\xi_2\cdots\xi_{2k+3}\right\}=\eta_1\eta_2\cdots\eta_{2n+1}\xi_1\xi_2\cdots\xi_{2k+3}+\xi_1\xi_2\cdots\xi_{2k+1}\xi_{2k+2}\xi_{2k+3}\eta_1\eta_2\cdots\eta_{2n+1} \tag{193}$$

After swapping the Grassmann variables in the last term, we have

$$\left\{\eta_1\eta_2\cdots\eta_{2n+1},\xi_1\xi_2\cdots\xi_{2k+3}\right\}=\eta_1\eta_2\cdots\eta_{2n+1}\xi_1\xi_2\cdots\xi_{2k+3}+\xi_1\xi_2\cdots\xi_{2k+1}\eta_1\eta_2\cdots\eta_{2n+1}\xi_{2k+2}\xi_{2k+3} \tag{194}$$

This relation can also be rewritten as

$$\left\{\eta_1\eta_2\cdots\eta_{2n+1},\xi_1\xi_2\cdots\xi_{2k+3}\right\}=\left\{\eta_1\eta_2\cdots\eta_{2n+1},\xi_1\xi_2\cdots\xi_{2k+1}\right\}\xi_{2k+2}\xi_{2k+3} \tag{195}$$

So, by induction from (191), we have

$$\left\{\eta_1\eta_2\cdots\eta_{2n+1},\xi_1\xi_2\cdots\xi_{2k+3}\right\}=0 \tag{196}$$

The previous demonstrations allow us to summarize **some important properties of Grassmann generators**:

- **Two even numbers of Grassmann variables commute with each other:**

$$\left[\eta_1\eta_2\cdots\eta_{2n},\xi_1\xi_2\cdots\xi_{2k}\right]=0 \tag{197}$$

- **Even and odd numbers of Grassmann variables commute with each other:**

$$\left[\eta_1\eta_2\cdots\eta_{2n},\xi_1\xi_2\cdots\xi_{2k+1}\right]=0 \tag{198}$$

- **Two odd numbers of Grassmann variables anticommute with each other:**

$$\left\{\eta_1\eta_2\cdots\eta_{2n+1},\xi_1\xi_2\cdots\xi_{2k+1}\right\}=0 \tag{199}$$

Considering these properties and (172), then we see that two arbitrary Grassmann functions do not commute. Thus,

$$\left[f(\eta),g(\eta)\right]\neq 0 \tag{200}$$

Generally, this may lead to the definition of even and odd Grassmann functions.

We may also characterize an element of Grassmann algebra by introducing an automorphism, P, that acts as a parity operator

$$\mathrm{P}\left(\xi_{\lambda_1}\cdots\xi_{\lambda_n}\right)=(-1)^n\xi_{\lambda_1}\cdots\xi_{\lambda_n} \tag{201}$$

So, if f is an even function, then

$$\mathrm{P}(f)=f \tag{202}$$

and if it is an odd function, then

$$\mathrm{P}(f)=-f \tag{203}$$

Consider again the last relation in (174). In this case, the anticommutation relations (169) imply

$$\left\{\frac{\partial}{\partial\xi_i},\frac{\partial}{\partial\xi_j}\right\}_+=0 \tag{204}$$

So, the operator $\frac{\partial}{\partial\xi_i}$ is nilpotent, $\frac{\partial^2}{\partial\xi_i^2}=0$, thereby showing that the function $f\left(\xi_1,\cdots,\xi_n\right)$ is at most of order 1 in each variable. The last relation in (174) defines a left differentiation

$$\frac{\partial}{\partial\xi_i}\xi_j\xi_i=-\frac{\partial}{\partial\xi_i}\xi_i\xi_j=-\xi_j \quad , \quad (i\neq j) \tag{205}$$

When $\frac{\partial}{\partial\xi_i}$ acts on ξ_i, we shift first ξ_i to the left of the monomial. The **chain rule for differentiation** can then be written

$$\frac{\partial}{\partial\xi_i}f(g)=\frac{\partial g}{\partial\xi_i}\frac{\partial f}{\partial g} \tag{206}$$

However, contrary to ordinary variables, the order of the terms on the right-hand side is important.

Integration and differentiation for Grassmann numbers are identical:

$$\int d\xi_i f(\xi_1,\cdots,\xi_n) = \frac{\partial}{\partial \xi_i} f(\xi_1,\cdots,\xi_n) \tag{207}$$

This is a condition ensuring that two fundamental properties of ordinary integrals over functions vanishing at infinity are satisfied. The integral of an exact differential is zero:

$$\int d\xi_i \frac{\partial}{\partial \xi_i} f(\xi_1,\cdots,\xi_n) = 0 \tag{208}$$

The integral over ξ_i of $f(\xi_1,\cdots,\xi_n)$ is independent of ξ_i so that its derivative vanishes:

$$\frac{\partial}{\partial \xi_i} \int d\xi_i f(\xi_1,\cdots,\xi_n) = 0 \tag{209}$$

Properties in (208) as well as in (209) follow from (207) and the nilpotence of the differential operator $\frac{\partial}{\partial \xi_i}$. If we consider a Grassmann algebra with a single generator

$$f(\xi) = f_0 + f_1\xi \tag{210}$$

then

$$\frac{\partial}{\partial \xi} \int d\xi f = f_1 \tag{211}$$

2.3.1.2 Exponential Function of Grassmann Numbers

We examine the following exponential function of Grassmann variables:

$$\exp\left\{\sum_{\lambda_k} \xi_{\lambda_k}\xi_{\lambda_{k+1}}\cdots\xi_{\lambda_{2k}}\right\} = \prod_{\lambda_k} \exp\left\{\xi_{\lambda_k}\xi_{\lambda_{k+1}}\cdots\xi_{\lambda_{2k}}\right\} \equiv \exp\left\{\sum_{\lambda_k}^{2n}\prod_k \xi_{\lambda_k}\right\} = \prod_{\lambda_k}^{2n} \exp\left\{\prod_k \xi_{\lambda_k}\right\} \tag{212}$$

This permits us to write

$$\left[\exp\left\{\sum_{\lambda} \xi_{\lambda_1}\xi_{\lambda_2}\cdots\xi_{\lambda_{2k}}\right\},\eta_1\cdots\eta_n\right] = \prod_{\lambda_k}\left[\exp\left\{\xi_{\lambda_1}\xi_{\lambda_2}\cdots\xi_{\lambda_{2k}}\right\},\eta_1\cdots\eta_n\right] = \prod_{\lambda_k}\left[1-\xi_{\lambda_1}\xi_{\lambda_2}\cdots\xi_{\lambda_{2k}},\eta_1\cdots\eta_n\right] \tag{213}$$

or

$$\prod_{\lambda_k}\left[1-\xi_{\lambda_1}\xi_{\lambda_2}\cdots\xi_{\lambda_{2k}},\eta_1\cdots\eta_n\right] = \prod_{\lambda_k}\left(\left[1,\eta_1\cdots\eta_n\right]-\left[\xi_{\lambda_1}\xi_{\lambda_2}\cdots\xi_{\lambda_{2k}},\eta_1\cdots\eta_n\right]\right) \tag{214}$$

Because

$$\left[1,\eta_1\cdots\eta_n\right]=0 \quad , \quad \left[\xi_{\lambda_1}\xi_{\lambda_2}\cdots\xi_{\lambda_{2k}},\eta_1\cdots\eta_n\right]=0 \tag{215}$$

then

$$\left[\exp\left\{\sum_{\lambda}\xi_{\lambda_1}\xi_{\lambda_2}\cdots\xi_{\lambda_{2k}}\right\},\eta_1\cdots\eta_n\right]=0 \tag{216}$$

And this takes into consideration the relations

$$\left[\exp\left\{\sum_{\lambda}\xi_{\lambda}^{*}\xi_{\lambda}\right\},\eta\right]=0 \quad , \quad \left[\exp\left\{\sum_{\lambda}\xi_{\lambda}^{*}\xi_{\lambda}\right\},\eta_1\cdots\eta_n\right]=0 \tag{217}$$

Consider the general Hamiltonian determinant in normal order to be an operator of the form

$$\hat{\mathrm{H}}=\frac{1}{n!}\sum_{\lambda_1,\cdots\lambda_n\mu_1,\cdots\mu_n}\langle\lambda_1,\cdots\lambda_n|\mathrm{H}|\mu_1,\cdots\mu_n\rangle\xi_{\lambda_1}^{\dagger}\cdots\xi_{\lambda_n}^{\dagger}\xi_{\mu_n}\cdots\xi_{\mu_1} \tag{218}$$

Therefore, for any operator, there is always an even combination of creation and annihilation operators. Hence,

$$\left[\exp\left\{-i\varepsilon\mathrm{H}\left[\hat{\psi}^{\dagger},\hat{\psi}\right]\right\},\xi_k\right]=0 \tag{219}$$

Also, for any even Grassmann function $f(\xi)$ and any Grassmann function $g(\xi)$, it follows that

$$\left[\exp\left\{f(\xi)\right\},\mathrm{g}(\xi)\right]=0 \tag{220}$$

2.3.1.3 Involution of Grassmann Numbers

We consider each even Grassmann algebra of $n=2l$ for which we introduce an involution operation. This is done by associating one generator ξ_k^{*} with each generator ξ_k and requesting that

$$(\xi_k)^{*}=\xi_k^{*} \quad , \quad (\xi_k^{*})^{*}=\xi_k \quad , \quad (\lambda\xi_k)^{*}=\lambda^{*}\xi_k^{*} \quad , \quad (\xi_{\lambda_1},\xi_{\lambda_2}\cdots\xi_{\lambda_n})^{*}=\xi_{\lambda_n}^{*}\xi_{\lambda_{n-1}}^{*}\cdots\xi_{\lambda_1}^{*} \tag{221}$$

Here, λ is complex valued index. Note that the generators ξ_k and ξ_k^{*} are completely independent. Hence, all rules previously derived are applicable. **Involuted Grassmann numbers as well as objects of the form $\xi_1+\xi_2$ are sometimes called complex Grassmann numbers**.

2.3.1.4 Bilinear Form of Operators

We simplify notation by considering the general element of Grassmann algebra with two generators $\{\xi^{*},\xi\}$, which are analytic functions of ξ^{*} and ξ with bilinear form:

$$f(\xi,\xi^{*})=f_0+f_1\xi^{*}+f_2\xi+f_{12}\xi^{*}\xi \tag{222}$$

Via the differentiation and integration rules, we find that

$$\frac{\partial f}{\partial\xi}=\int d\xi\, f=f_2-f_{12}\xi^{*} \quad , \quad \frac{\partial f}{\partial\xi^{*}}=\int d\xi^{*} f=f_1+f_{12}\xi \quad , \quad \frac{\partial^2 f}{\partial\xi^{*}\partial\xi}=\int d\xi^{*}\, d\xi\, f=-f_{12} \tag{223}$$

From here, we see that

$$\frac{\partial^2 f}{\partial\xi^* \partial\xi} = -\frac{\partial^2 f}{\partial\xi \partial\xi^*} = -f_{12} \tag{224}$$

and so

$$\left\{ \frac{\partial}{\partial\xi^*}, \frac{\partial}{\partial\xi} \right\} = 0 \tag{225}$$

This again confirms the fact that the operator $\frac{\partial}{\partial\xi_i}$ is nilpotent, $\frac{\partial^2}{\partial\xi^2} = 0$, and that the function f is at most of order 1 in each variable. So, **for the two generators $\{\xi^*, \xi\}$ the algebra is generated by the four numbers $\{1, \xi, \xi^*, \xi^*\xi\}$. Further, the linear function in equation (210) will be the coherent state representation of a wave function, and the coherent state representation of an operator in Grassmann algebra will be a function of ξ^* and ξ that should have the form of equation (222).**

2.3.1.5 Berezin Integration

When explaining the definite integral, there is no equivalent for the familiar sum motivating the Riemann integral for ordinary variables. Therefore, integration over Grassmann variables can be defined as linear mapping with the fundamental property of ordinary integrals over functions vanishing at infinity. In this case, the integral of an exact differential form is zero. This constraint means the integral of 1 is zero because 1 is the derivative of ξ. The nonvanishing integral is only ξ^* because ξ is not a derivative. So, the Berezin definite integral over Grassmann numbers is defined as [10]:

$$\int d\xi 1 = 0 \quad , \quad \int d\xi\xi = 1 \tag{226}$$

In the case of a derivative, to apply the second equation in (226), we must first anticommute the variable ξ so as to bring it next to $d\xi$. This definition simply imitates Grassmann integration, which is equivalent to Grassmann differentiation. Because half of the generators ξ_i are defined arbitrarily to be conjugate variables but are otherwise equivalent to the generators ξ_i^*, we define integration over conjugate variables similarly:

$$\int d\xi^* 1 = 0 \quad , \quad \int d\xi^* \xi^* = 1 \tag{227}$$

This implies again that integration is equivalent to differentiation. The aforementioned definitions tailor integration to obey the usual rules of partial integration and in particular

$$\int d\xi \frac{\partial f(\xi)}{\partial\xi} = 0 \tag{228}$$

for any function $f(\xi) = f_0 + f_1\xi$. Note that condition (228) requires the satisfaction of the first equations in (226) and (227). Therefore, the last equations in (226) and (227) are solely for normalization purposes. Then the integral of the function $f(\xi)$:

$$\int d\xi f(\xi) = \int d\xi (f_0 + f_1\xi) = f_1 \tag{229}$$

From here, we see that for the Grassmann integral to have meaning, we have

$$\int d\xi f(\xi+\eta)=\int d\xi f(\xi) \tag{230}$$

where η is a Grassmann variable.

Similarly, we have

$$\int d\xi d\xi^* = 0 \quad , \quad \int d\xi d\xi^*\xi = 0 \quad , \quad \int d\xi d\xi^*\xi^* = 0 \quad , \quad \int d\xi d\xi^*\xi^*\xi = 1 \quad , \quad \int d\xi d\xi^*\xi\xi^* = -1 \tag{231}$$

2.3.1.6 Grassmann Delta Function

The Grassmann delta function can be defined by

$$\delta(\xi,\xi')=\int d\eta \exp\{-\eta(\xi-\xi')\}=\int d\eta\left(1-\eta(\xi-\xi')\right)=-(\xi-\xi') \tag{232}$$

Here, η is a Grassmann variable. We verify that this definition has the desired behavior by using the function $f(\xi)=f_0+f_1\xi$, and we have

$$\int d\xi'\delta(\xi,\xi')f(\xi')=-\int d\xi'(\xi-\xi')(f_0+f_1\xi')=f_0+f_1\xi=f(\xi) \tag{233}$$

2.3.1.7 Scalar Product of Grassmann Algebra

We define the scalar product in such a way that, on the one hand, it imitates the form of a scalar product with bosonic coherent states, and on the other hand, it allows functions of a Grassmann variable to have the structure of a Hilbert space.

$$\begin{aligned}\langle f|g\rangle &= \int d\xi d\xi^* \exp\{-\xi\xi^*\} f^*(\xi^*)g(\xi)=\int d\xi d\xi^*(1-\xi\xi^*)(f_0^*+f_1^*\xi^*)(g_0+g_1\xi)= \\ &= -\int d\xi d\xi^*\xi\xi^* f_0^* g_0+\int d\xi d\xi^*\xi^*\xi f_1^* g_1 = f_0^* g_0 + f_1^* g_1\end{aligned} \tag{234}$$

The results presented for the two generators ξ^* and ξ can be applied to $2k$ generators $\xi_1\cdots\xi_{2k},\xi_1^*\cdots\xi_k^*$.

2.3.2 Fermions

Employing anticommuting Grassmann variables for calculating physical quantities for fermionic systems is a well-established technique [11]. This involves, in particular, the calculation of expectation values of quasifree (Gaussian) fermion states. This notion entails replacing the linear combinations of canonically anticommuting Fermi field operators with complex coefficients by linear combinations with coefficients that are **anticommuting Grassmann numbers** [11]. Consequently, these linear combinations achieve **canonical commutation relations**.

This section provides a mathematical background that imports the **Grassmann calculus** into the description of fermion systems. We set up a formalism that can be viewed as the fermionic analog of a quantum harmonic analysis on phase space. We present basic theorems that will aid in the understanding of **Grassmann algebra**.

Consider the anticommuting fermionic operators $\hat{\psi}^{\dagger}$ and $\hat{\psi}$ that create or annihilate, respectively, a fermion in state $|\xi\rangle$ and that serve as a **basis** for many-particle operators:

$$\hat{\psi}\hat{\psi}^{\dagger}+\hat{\psi}^{\dagger}\hat{\psi}=1 \tag{235}$$

We find

$$\hat{n}^2=\left(\hat{\psi}^{\dagger}\hat{\psi}\right)^2=\hat{\psi}^{\dagger}\hat{\psi}\left(1-\hat{\psi}\hat{\psi}^{\dagger}\right) \tag{236}$$

Then, from $\hat{\psi}^2=0$, we have

$$\hat{n}^2=\left(\hat{\psi}^{\dagger}\hat{\psi}\right)^2=\hat{\psi}^{\dagger}\hat{\psi}\left(1-\hat{\psi}\hat{\psi}^{\dagger}\right)=\hat{\psi}^{\dagger}\hat{\psi}=\hat{n} \tag{237}$$

We see that $n^2=n$. This implies that the **occupation number in each state can only be** $n=0$ **or** $n=1$.

Consider the eigenstates of the **fermionic** creation $\hat{\psi}^{\dagger}$ or annihilation operator $\hat{\psi}$:

$$\hat{\psi}^{\dagger}|0\rangle=|1\rangle \quad , \quad \hat{\psi}|1\rangle=|0\rangle \quad , \quad \hat{\psi}^{\dagger}=|1\rangle\langle 0|=\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} \quad , \quad \hat{\psi}=|0\rangle\langle 1|=\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} \tag{238}$$

where

$$|0\rangle=\begin{pmatrix}0\\1\end{pmatrix} \quad , \quad |1\rangle=\begin{pmatrix}1\\0\end{pmatrix} \tag{239}$$

From here, there should be a problem in dealing with the **anticommuting** behavior of the fermionic creation $\hat{\psi}^{\dagger}$ and annihilation $\hat{\psi}$ operators:

$$\hat{\psi}^{\dagger}\hat{\psi}=\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \quad , \quad \hat{\psi}\hat{\psi}^{\dagger}=\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}=\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} \quad , \quad \hat{\psi}^{\dagger}\hat{\psi}+\hat{\psi}\hat{\psi}^{\dagger}=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}=\hat{1} \tag{240}$$

Therefore, we require Grassmann numbers to define fermionic coherent states where we resort to the **Grassmann numbers** ξ and ξ^* when considering the anticommuting eigenvalues of the operators $\hat{\psi}^{\dagger}$ and $\hat{\psi}$. The set $\{1,\xi,\xi^*,\xi^*\xi\}$, and their linear combinations with complex coefficients, form **Grassmann algebra** (this mimics the behavior of Fock space algebra). Therefore, the regular Fock space is a buildup of the direct sum of all n-dimensional Hilbert spaces (1) and has to be extended in order to contain Grassmann numbers. In this case, the extended Fock space is then formed by building the linear combination of the regular Fock space states and the Grassmann coefficients. Hence, the basis of Grassmann algebra is all distinct products, and we show that Grassmann numbers can be multiplied together and anticommute under multiplication from the following definitions:

$$\xi|0\rangle=|0\rangle\xi \quad , \quad \xi|1\rangle=-|1\rangle\xi \quad , \quad \xi^*\langle 0|=\langle 0|\xi^* \quad , \quad \langle 1|\xi^*=-\xi^*\langle 1| \tag{241}$$

$$\xi|0\rangle\langle 1|+|0\rangle\langle 1|\xi=0 \quad , \quad \langle 0|\xi|0\rangle=-\langle 1|\xi|1\rangle \tag{242}$$

$$\left\{\xi,\hat{\psi}\right\}=\xi|0\rangle\langle 1|+|0\rangle\langle 1|\xi=|0\rangle\xi\left(-\langle 1|\xi\right)+|0\rangle\langle 1|\xi=0 \quad , \quad \xi\langle 1|=-\langle 1|\xi \tag{243}$$

The presence of the sign change when there is a swap is a direct consequence of the anticommutation relations between Grassmann variables and Fock space operators as seen earlier. So, the Grassmann numbers can be multiplied together and anticommute under multiplication:

$$\{\xi,\xi\}=0 \quad , \quad \xi^2=0 \quad , \quad \left(\xi^*\right)^2=0 \tag{244}$$

This guarantees any analytic function will be linear in Grassmann algebra.

Grassmann numbers can be multiplied by complex numbers, and the multiplication by a complex number is distributive:

$$\alpha(\xi_1+\xi_2)=\alpha\xi_1+\alpha\xi_2 \tag{245}$$

Considering the fact that Grassmann numbers occur only inside time-ordered products, then it suffices to define the adjoint in such a way that it also anticommutes. So, the quantities

$$\xi=\xi_1+i\xi_2 \quad , \quad \xi^*=\xi_1-i\xi_2 \tag{246}$$

can be treated as independent Grassmann variables:

$$\{\xi,\xi^*\}=0 \tag{247}$$

Considering that the square and higher powers of a Grassmann number vanish, then the Taylor expansion of a wave function with Grassmann variables has only two terms. For example, say,

$$\exp\{\xi\}=1+\xi$$

2.4 Fermions and Coherent States

Considering the aforementioned, we see that fermionic systems such as electrons in a metal or ultracold fermionic atoms in a magnetic trap can be described by Grassmann variables. So, to construct coherent states for fermions, we must enlarge the fermion Fock space. It is useful to note that an important property needed for setting up path integration is the completeness of the states. We now examine coherent states and define them as eigenstates of the operator $\hat{\psi}$:

$$|\xi\rangle=\exp\{-\xi\hat{\psi}^\dagger\}|0\rangle=(1-\xi\hat{\psi}^\dagger)|0\rangle=|0\rangle-\xi|1\rangle \quad , \quad \langle\xi|=\langle 0|\exp\{-\xi^*\hat{\psi}\}=\langle 0|(1-\xi^*\hat{\psi})=\langle 0|-\langle 1|\xi^* \tag{248}$$

We apply $\hat{\psi}$ or $\hat{\psi}^\dagger$ on these states, and then we have

$$\hat{\psi}|\xi\rangle=\psi|0\rangle-\hat{\psi}\xi|1\rangle=\hat{\psi}|0\rangle+\xi\hat{\psi}|1\rangle=\xi(|0\rangle-\xi|1\rangle)=\xi|\xi\rangle \tag{249}$$

$$\hat{\psi}|\xi\rangle=\xi|\xi\rangle \tag{250}$$

$$\langle\xi|\hat{\psi}^\dagger=\langle\xi|\xi^* \tag{251}$$

Relations (250) and (251) are fundamental properties of fermionic coherent states and are also true for bosons. It is important to note that in addition to the space-time variables, fermionic fields require a spin index $\alpha=\uparrow,\downarrow$. So, if

$$|\xi\rangle=\exp\left\{-\sum_\alpha \xi_\alpha\hat{\psi}_\alpha^\dagger\right\}|0\rangle=\prod_\alpha\left(1-\xi_\alpha\hat{\psi}_\alpha^\dagger\right)|0\rangle \tag{252}$$

then we again prove relations (250) and (251):

$$\hat{\psi}_\alpha\left(1-\xi_\alpha\hat{\psi}_\alpha^\dagger\right)|0\rangle=\hat{\psi}_\alpha|0\rangle-\hat{\psi}_\alpha\xi_\alpha|1\rangle=\xi_\alpha\hat{\psi}_\alpha|1\rangle=\xi_\alpha|0\rangle=(\xi_\alpha-0)|0\rangle=\left(\xi_\alpha-\xi_\alpha^2\hat{\psi}_\alpha^\dagger\right)|0\rangle=\xi_\alpha\left(1-\xi_\alpha\hat{\psi}_\alpha^\dagger\right)|0\rangle \tag{253}$$

So,

$$\hat{\psi}_\alpha|\xi\rangle=\hat{\psi}_\alpha\prod_\beta\left(1-\xi_\beta\hat{\psi}_\beta^\dagger\right)|0\rangle=\hat{\psi}_\alpha\prod_{\beta\neq\alpha}\left(1-\xi_\beta\hat{\psi}_\beta^\dagger\right)\left(1-\xi_\alpha\hat{\psi}_\alpha^\dagger\right)|0\rangle=\prod_{\beta\neq\alpha}\left(\hat{\psi}_\alpha+\xi_\beta\hat{\psi}_\alpha\hat{\psi}_\beta^\dagger\right)\left(1-\xi_\alpha\hat{\psi}_\alpha^\dagger\right)|0\rangle \tag{254}$$

Swapping the positions of the operators $\hat{\psi}_\alpha$ and $\hat{\psi}_\beta^\dagger$ in the first term of the product in the last expression, there is a sign change, and we have

$$\begin{aligned}\hat{\psi}_\alpha|\xi\rangle&=\prod_{\beta\neq\alpha}\left(\hat{\psi}_\alpha-\xi_\beta\hat{\psi}_\beta^\dagger\hat{\psi}_\alpha\right)\left(1-\xi_\alpha\hat{\psi}_\alpha^\dagger\right)|0\rangle=\prod_{\beta\neq\alpha}\left(1-\xi_\beta\hat{\psi}_\beta^\dagger\right)\hat{\psi}_\alpha\left(1-\xi_\alpha\hat{\psi}_\alpha^\dagger\right)|0\rangle=\\&=\prod_{\beta\neq\alpha}\left(1-\xi_\beta\hat{\psi}_\beta^\dagger\right)\xi_\alpha\left(1-\xi_\alpha\hat{\psi}_\alpha^\dagger\right)|0\rangle=\xi_\alpha\prod_{\beta\neq\alpha}\left(1-\xi_\beta\hat{\psi}_\beta^\dagger\right)\left(1-\xi_\alpha\hat{\psi}_\alpha^\dagger\right)|0\rangle=\xi_\alpha\prod_{\alpha}\left(1-\xi_\alpha\hat{\psi}_\alpha^\dagger\right)|0\rangle=\xi_\alpha|\xi\rangle\end{aligned} \tag{255}$$

Therefore $|\xi\rangle$ is an eigenstate, which implies that it is a coherent state. From the coherent state, its adjoint is a left eigenstate of the creation operator:

$$\begin{aligned}\langle\xi|\hat{\psi}_\alpha^\dagger&=\langle 0|\exp\left\{-\sum_\beta\hat{\psi}_\beta\xi_\beta^*\right\}\hat{\psi}_\alpha^\dagger=\langle 0|\prod_\beta\left(1-\hat{\psi}_\beta\xi_\beta^*\right)\hat{\psi}_\alpha^\dagger=\langle 0|\left(1-\hat{\psi}_\alpha\xi_\alpha^*\right)\hat{\psi}_\alpha^\dagger\prod_{\beta\neq\alpha}\left(1-\hat{\psi}_\beta\xi_\beta^*\right)=\\&=\langle 0|\hat{\psi}_\alpha\hat{\psi}_\alpha^\dagger\xi_\alpha^*\prod_{\beta\neq\alpha}\left(1-\hat{\psi}_\beta\xi_\beta^*\right)=\langle 0|\prod_\beta\left(1-\hat{\psi}_\beta\xi_\beta^*\right)\xi_\alpha^*=\langle\xi|\xi_\alpha^*\end{aligned} \tag{256}$$

So

$$\langle\xi|\hat{\psi}_\alpha^\dagger=\langle\xi|\xi_\alpha^* \tag{257}$$

It is useful to note that these coherent states are not orthonormal.

Similarly, we consider the application of a creation operator on a coherent state as for bosons:

$$\hat{\psi}_\alpha^\dagger|\xi\rangle=\hat{\psi}_\alpha^\dagger\exp\left\{-\sum_\beta\xi_\beta\hat{\psi}_\beta^\dagger\right\}|0\rangle=\hat{\psi}_\alpha^\dagger\prod_\beta\left(1-\xi_\beta\hat{\psi}_\beta^\dagger\right)|0\rangle=\hat{\psi}_\alpha^\dagger\left(1-\xi_\alpha\hat{\psi}_\alpha^\dagger\right)\prod_{\beta\neq\alpha}\left(1-\xi_\beta\hat{\psi}_\beta^\dagger\right)|0\rangle \tag{258}$$

Consider that $\left(\hat{\psi}_\alpha^\dagger\right)^2=0$ in the last expression, then

$$\hat{\psi}_\alpha^\dagger|\xi\rangle=\left(\hat{\psi}_\alpha^\dagger+\xi_\alpha\left(\hat{\psi}_\alpha^\dagger\right)^2\right)\prod_{\beta\neq\alpha}\left(1-\xi_\beta\hat{\psi}_\beta^\dagger\right)|0\rangle=\hat{\psi}_\alpha^\dagger\prod_{\beta\neq\alpha}\left(1-\xi_\beta\hat{\psi}_\beta^\dagger\right)|0\rangle=-\frac{\partial}{\partial\xi_\alpha}\left(1-\xi_\alpha\hat{\psi}_\alpha^\dagger\right)\prod_{\beta\neq\alpha}\left(1-\xi_\beta\hat{\psi}_\beta^\dagger\right)|0\rangle \tag{259}$$

or

$$\hat{\psi}_\alpha^\dagger|\xi\rangle=-\frac{\partial}{\partial\xi_\alpha}\left(1-\xi_\alpha\hat{\psi}_\alpha^\dagger\right)\prod_{\beta\neq\alpha}\left(1-\xi_\beta\hat{\psi}_\beta^\dagger\right)|0\rangle=-\frac{\partial}{\partial\xi_\alpha}\prod_\beta\left(1-\xi_\beta\hat{\psi}_\beta^\dagger\right)|0\rangle=-\frac{\partial}{\partial\xi_\alpha}|\xi\rangle \tag{260}$$

So,

$$\hat{\psi}_\alpha^\dagger|\xi\rangle = -\frac{\partial}{\partial\xi_\alpha}|\xi\rangle \tag{261}$$

Similarly,

$$\langle\xi|\hat{\psi}_\alpha = \langle 0|\prod_{\beta\neq\alpha}\left(1-\hat{\psi}_\beta\xi_\beta^*\right)\left(1-\hat{\psi}_\alpha\xi_\alpha^*\right)\hat{\psi}_\alpha = \frac{\partial}{\partial\xi_\alpha}\langle 0|\prod_\beta\left(1-\hat{\psi}_\beta\xi_\beta^*\right) = \frac{\partial}{\partial\xi_\alpha}\langle\xi| \tag{262}$$

or

$$\langle\xi|\hat{\psi}_\alpha = \frac{\partial}{\partial\xi_\alpha}\langle\xi| \tag{263}$$

Also,

$$\langle\xi|\hat{\psi}_\alpha^\dagger = \langle 0|\exp\left\{-\sum_\beta \xi_\beta^*\hat{\psi}_\beta\right\}\hat{\psi}_\alpha^\dagger = \langle 0|\prod_\beta\left(1-\xi_\beta^*\hat{\psi}_\beta\right)\hat{\psi}_\alpha^\dagger = \langle 0|\left(1-\xi_\alpha^*\hat{\psi}_\alpha\right)\hat{\psi}_\alpha^\dagger\prod_{\beta\neq\alpha}\left(1-\xi_\beta^*\hat{\psi}_\beta\right) \tag{264}$$

From where

$$\langle\xi|\hat{\psi}_\alpha^\dagger = \langle 0|\hat{\psi}_\alpha\hat{\psi}_\alpha^\dagger\xi_\alpha^*\prod_{\beta\neq\alpha}\left(1-\xi_\beta^*\hat{\psi}_\beta\right) = \langle 0|\prod_\beta\left(1-\xi_\beta^*\hat{\psi}_\beta\right)\xi_\alpha^* = \langle\xi|\xi_\alpha^* \tag{265}$$

Here, we considered that

$$\langle 0|\left[\hat{\psi}_\alpha,\hat{\psi}_\alpha^\dagger\right]_+ = \langle 0| \tag{266}$$

The inner product of two coherent states:

$$\begin{aligned}
\langle\xi|\xi'\rangle &= \langle 0|\exp\left\{\sum_\alpha \xi_\alpha^*\hat{\psi}_\alpha\right\}\exp\left\{-\sum_\alpha \xi'_\alpha\hat{\psi}_\alpha^\dagger\right\}|0\rangle = \langle 0|\prod_\alpha\left(1+\xi_\alpha^*\hat{\psi}_\alpha\right)\prod_\alpha\left(1-\xi'_\alpha\hat{\psi}_\alpha^\dagger\right)|0\rangle = \\
&= \langle 0|\prod_\alpha\left(1+\xi_\alpha^*\hat{\psi}_\alpha\right)\left(1-\xi'_\alpha\hat{\psi}_\alpha^\dagger\right)|0\rangle = \langle 0|\prod_\alpha\left(1-\xi'_\alpha\hat{\psi}_\alpha^\dagger+\xi_\alpha^*\hat{\psi}_\alpha-\xi_\alpha^*\hat{\psi}_\alpha\xi'_\alpha\hat{\psi}_\alpha^\dagger\right)|0\rangle = \\
&= \prod_\alpha\left(\langle 0|0\rangle-\xi'_\alpha\langle 0|\alpha\rangle+\langle 0|\xi_\alpha^*\hat{\psi}_\alpha|0\rangle+\xi_\alpha^*\xi'_\alpha\langle 0|\hat{\psi}_\alpha\hat{\psi}_\alpha^\dagger|0\rangle\right)
\end{aligned} \tag{267}$$

From

$$\hat{\psi}_\alpha|0\rangle = 0 \quad , \quad \langle 0|\hat{\psi}_\alpha\hat{\psi}_\alpha^\dagger|0\rangle = 1 \tag{268}$$

then

$$\langle\xi|\xi'\rangle = \prod_\alpha\left(1+\xi_\alpha^*\xi'_\alpha\right) = \exp\left\{\sum_\alpha \xi_\alpha^*\xi'_\alpha\right\} \tag{269}$$

Using the overcompleteness relation, we represent a state of the extended Fock space in terms of coherent states as follows:

$$\left|\psi\right\rangle = \int\prod_{\alpha} d\xi_{\alpha}^{*} d\xi_{\alpha} \exp\left\{-\sum_{\alpha}\xi_{\alpha}^{*}\xi_{\alpha}\right\}\left|\xi\right\rangle\left\langle\xi\right|\psi\rangle = \int\prod_{\alpha} d\xi_{\alpha}^{*} d\xi_{\alpha} \exp\left\{-\sum_{\alpha}\xi_{\alpha}^{*}\xi_{\alpha}\right\}\psi\left(\xi^{*}\right)\left|\xi\right\rangle \tag{270}$$

where

$$\left\langle\xi\right|\psi\rangle = \psi\left(\xi^{*}\right) \tag{271}$$

2.4.1 Coherent State Overcompleteness Relation Proof

We show also that the overcompleteness relation for a coherent state basis is reflected in the closure relation:

$$\begin{aligned}\int\prod_{\alpha} d\xi_{\alpha}^{*} d\xi_{\alpha} \exp\left\{-\sum_{\alpha}\xi_{\alpha}^{*}\xi_{\alpha}\right\}\left|\xi\right\rangle\left\langle\xi\right| &= \prod_{\alpha}\int d\xi_{\alpha}^{*}\, d\xi_{\alpha}\left(1-\xi_{\alpha}^{*}\xi_{\alpha}\right)\left(1-\xi_{\alpha}\hat{\psi}_{\alpha}^{\dagger}\right)\left|0\right\rangle\left\langle 0\right|\left(1-\hat{\psi}_{\alpha}\xi_{\alpha}^{*}\right) \\ &= \prod_{\alpha}\left(\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\right) = \hat{1}\end{aligned} \tag{272}$$

Therefore,

$$\int\prod_{\alpha} d\xi_{\alpha}^{*} d\xi_{\alpha} \exp\left\{-\sum_{\alpha}\xi_{\alpha}^{*}\xi_{\alpha}\right\}\left|\xi\right\rangle\left\langle\xi\right| = \hat{1} \tag{273}$$

and this also holds for bosons when

$$\left|\xi\right\rangle = \exp\left\{\chi\sum_{\alpha}\xi_{\alpha}\hat{\psi}_{\alpha}^{\dagger}\right\}\left|0\right\rangle \tag{274}$$

In (273), $\hat{1}$ is the Fock space identity operator. As shown in (273), because the coherent states generally are not orthogonal, the set of coherent states is **overcomplete**. We can then rewrite the **overcompleteness relation** in (273) as follows

$$\int d\left[\xi_{\alpha}^{*}\right] d\left[\xi_{\alpha}\right] \exp\left\{-\sum_{\alpha}\xi_{\alpha}^{*}\xi_{\alpha}\right\}\left|\xi\right\rangle\left\langle\xi\right| = \hat{1} \quad , \quad N = \begin{cases} 2\pi i & , \text{ Bosons} \\ 1 & , \text{ Fermions}\end{cases} \tag{275}$$

so as to include the bosonic case. The integration measure in this case:

$$\int d\left[\xi_{\alpha}^{*}\right] d\left[\xi_{\alpha}\right] \equiv \int\prod_{\alpha}\frac{d\xi_{\alpha}^{*} d\xi_{\alpha}}{N} \tag{276}$$

Relation (273) can be proven exactly as in the case of bosons via Schur's lemma.