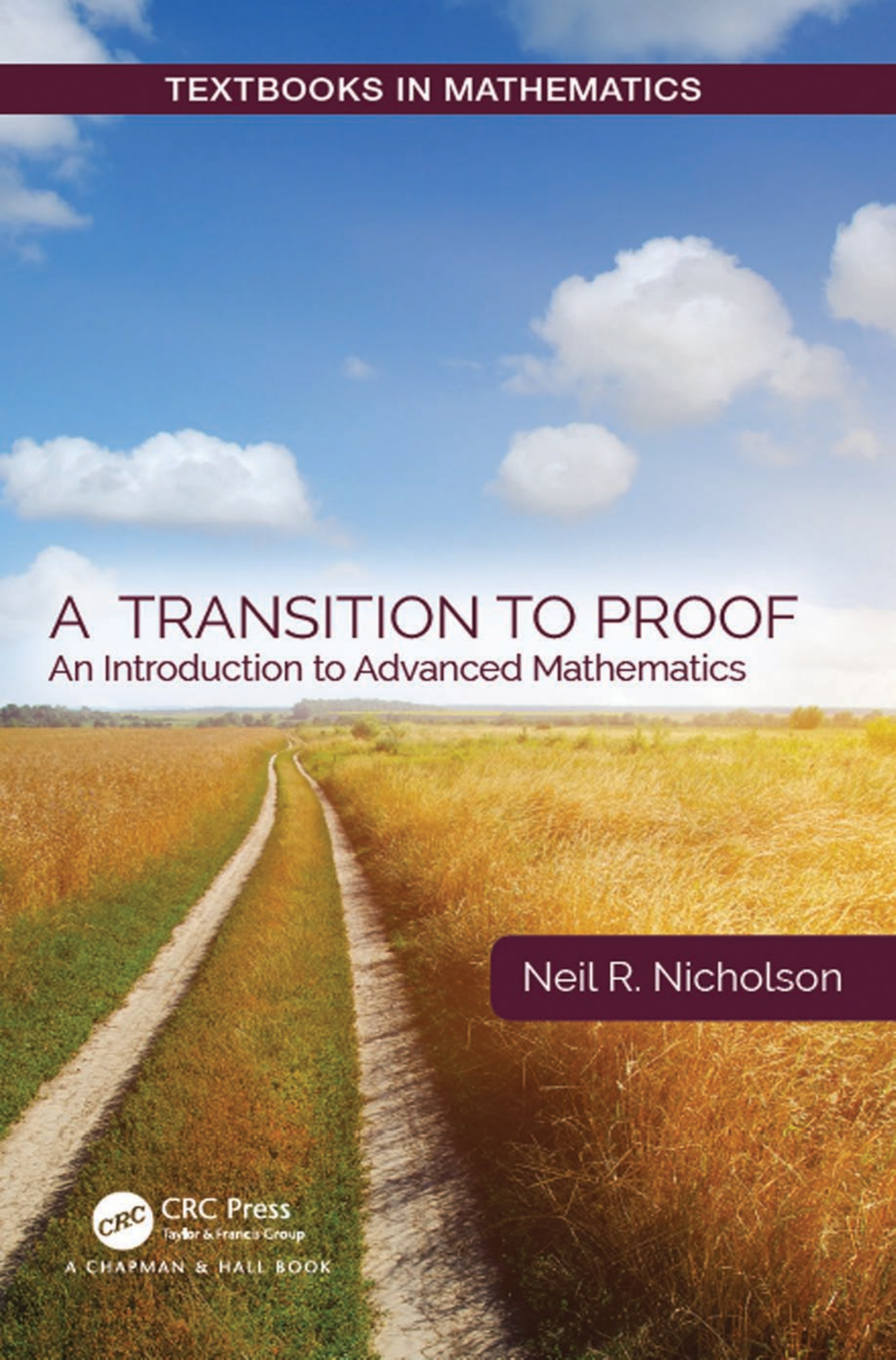

TEXTBOOKS IN MATHEMATICS

A TRANSITION TO PROOF

An Introduction to Advanced Mathematics

Neil R. Nicholson

CRC Press
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A CHAPMAN & HALL BOOK

A Transition to Proof

An Introduction to Advanced Mathematics

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To my mom and dad,
for believing in me since day one.

To Elizabeth and Zeke,
for love and support every day.

And to every cat I've ever known.

Contents

Preface

Why write another textbook aimed at the student "beginning" the study of theoretical mathematics? I imagine my answer is not too different from many other math textbook authors when asked why they are writing another book about *fill in the blank*. Having taught this course many times over, I have developed my approach for the material, tested, tweaked, tested again, adjusted and fine-tuned semester-after-semester. Certain topics in one textbook fully support part of my approach, while others in another text fulfill a few of my other goals. Collections of handouts fill in the remaining gaps, rounding out what I feel is *my course* taught in *my style* emphasizing what I feel are the important aspects of a bridge-to-higher-math course.

All of that work has led me to this place and this text. Is it better than others? Hardly. There are fantastic textbooks on the market introducing the up-and-coming mathematician to theoretical math. Each comes with its own flavor and style, and those texts may speak to a certain instructor's approach. By all means, those instructors can effectively teach this material in their own manner and if that approach is more systematic, axiomatic, inquiry-based or through different subject areas, then this text is not for them.

So who then is this textbook for? Rather than describe the intended instructor who would use this book, because this text is written for students, perhaps it is better to describe the outcomes that you, a *student*, could expect to obtain and why those outcomes are important in your development as a mathematician.[1] If you are the instructor and these outcomes align with your goals for your course, then this book may be the right book for you.

First and foremost, you will *understand mathematical proof* in all its contexts: why proofs are necessary, when to use them and how to write them. Learning mathematics is a journey and you began the trip years ago. As you come to different intersections along the way, it is not just important to know where roads take you, but it is critical to know where those roads came from and why they have led to the point at which you currently stand. You have to understand the entire landscape.

As you learn to write proofs, you will learn to *develop your mathematical voice*. There is not a single correct way to prove a mathematical result. Think of it like creating a persuasive paper. How you piece it together, employ mechanical techniques or choose language make the writing *yours*. You will learn

[1] The "why they are important" is debatable; the views expressed here were chosen by the author for emphasis.

the required fundamentals of proofs, but from there, the approaches vary as to how you create final drafts.

However... there is a lot that goes into creating a mathematical proof before you actually get around to writing it. So, in addition to actually writing correct and personalized proofs, ample discussion of how to figure out the "nuts and bolts" of the proof takes place: *thought processes*, *scratch work* and *ways to attack problems*. You will learn not just how to *write* mathematics but also how to *do* mathematics. Putting these two components together will allow you to *communicate mathematics effectively*.

There must be some vehicle for which to learn these aspects of proof. Here, different concepts from abstract and discrete mathematics play that role; you will be exposed to *fundamental definitions and results* from multiple areas of mathematics. The choice of topics to include in this text was purposeful; symbolic logic, sets, elementary number theory, relations, functions and cardinality are included because they appear throughout mathematics. In learning the theory of these areas, you create a firm foundation for later studies. Additionally, the final chapter of the text introduces point-set topology. It is included to show you that "high level" mathematics is often grounded in basic ideas (in this case, sets and their properties).

The material is developed and presented in a systematic fashion: easier to more challenging. You may find early concepts in chapters "simple." This is intentional. By the end of the chapter, proofs are presented with little scratch work, and at times, are quite challenging to work through. However, if you work through the material as it is presented, you will develop *confidence* in your abilities and build *rigor* into your approach. A strong mathematician does not simply muddle through the "easy" material; a mathematician attacks the "hard" questions *for the love of the puzzle*. It is my intent for this puzzle-solving love to grow as you proceed through this book.

To the student

The pure mathematician is often asked, "What is that used for?" To that I respond bluntly: "I do not know nor do I care." In no way is this meant to belittle the questioner. The question is rightfully fair; why do something if there is not a specific end-goal in sight?

In 1843, William Hamilton wanted to generalize the complex numbers to three dimensions. Three dimensions turned out to be one dimension too few; Hamilton had to generalize the complex numbers into *four* dimensions, creating what became known as the *quaternions* [18]. While the quaternions were mathematically "nice," what could the use of something in four dimensions possibly be? Perhaps Hamilton was a visionary, because it turns out that quaternions are fundamental in making every animated movie or video game. To look realistic, rotating objects and dynamic lighting effects rely on the four-dimensional mathematical objects. Hamilton did math *for math's sake*; the direct use of his work is merely a side effect.

In short, do not approach this material by asking, "What will this be used for?" It is all part of the larger mathematical picture. Someday, if you choose to directly apply mathematics to an outside field, the tools you learn here will surely be necessary. But if you travel the pure mathematical route, then you are learning the machinery necessary to *do* higher mathematics. This is ultimately the reason for learning the ideas you are about to learn.

Knowing where this material will take you, it is perhaps necessary to lay out some basic assumptions, the *prerequisites* for using this book. It is assumed that you will have seen at least two semesters of calculus. It is not the material of those courses that is necessary, though, for understanding this book. It is the development of a certain "mathematical sophistication" that is needed. You ought to be able to read mathematics, follow mathematical arguments and connect-the-dots across multi-step problems. If you have not seen this calculus material but are mathematically astute, you will be able to follow nearly all of this text (with the exception of a few calculus-based exercises).

Supposing you have read this far and are prepared for the course, you may be wondering how to be successful in working through the material. Part of that success comes from how you have been successful in previous mathematics courses: *working problems.* You cannot learn mathematics without doing mathematics; the more problems you work, the better you will become at attacking similar sorts of questions. However, if this is your first taste of theoretical mathematics, then you may not be aware that there is more to learning this material than simply doing problems. You have to *understand* the theory in order to *apply* the theory. What does this mean? You need to spend time *thinking* about the concepts. This is more than just memorization, however. It involves understanding the concepts *deep down inside* and developing an intuitive sense for definitions and results.

This understanding of the material forms the foundation for your success in the course. Once obtained, you can move on to *implementing* the material through the aforementioned problem-working. The problem sets provided are intended to be at times routine and at other times a serious challenge. Work the problems, assess your solutions, tweak, adjust and ultimately craft your final draft. Spend *time* on the problem solving process; a final draft of a proof should not appear quickly. The final draft is the "cremè de la cremè" of your efforts; think of it as the capstone of the learning pyramid. The final draft allows you to put all the pieces together and show mastery of the material.

It is in this capstone-step of the process that you can incorporate your style and direct your solution for *particular audiences.* The audience for whom you write guides how you write. This awareness, and how your style is dependent upon the audience, is one of the last steps to polishing your technique.

To the instructor

You are aware that your students will learn to *construct* and *write* proofs.

The topics covered are likely of particular interest to you. Consider the following road map to the text.

- Chapters 1 – 2: Logic and sets are the "building blocks" for the proof techniques presented here. An understanding of the terminology, notation and language is essential for the later chapters. Logical deductions (Section 1.4) are intended to be a "taste" of mathematical proof. Bookend these with statements and arguments with multiple quantifiers (Section 2.4) and the necessary preliminaries for proof techniques are formed.

- Chapters 3 – 4: Chapter 3 is *the* introduction to a multitude of proof methods: direct, set element, contrapositive, contradiction and cases. Every section of this chapter is critical for the remainder of the text, as are the first two sections of Chapter 4 on induction. Not only are the techniques presented in these sections important, but basic number theoretic concepts (definitions and results) appear here. These are used throughout the rest of the text as well. Section 4.3 includes material used in later sections (such as the Quotient-Remainder Theorem and results on primes). For the inclined instructor, this section could be supplemented with additional materials from number theory.

- Chapter 5 – 8: The second half of the text is the "putting proof techniques to work" part of the book. Sections 5.1 and 5.2, along with all of Chapter 6, should be covered. All remaining sections are optional and can be covered completely, or if time is limited, can be covered by picking-and-choosing the major ideas from each section. Section 5.4 and Chapter 8 open the door to further exploration beyond this textbook, if so desired.

Whatever sections you choose to cover in the text, proceed linearly, following the order presented. Various results and exercises throughout each section call upon items from previous sections. The homework exercises are designed to vary, from routine (building a basic understanding of concepts) through challenging (introducing new concepts and applying them to ideas from the section). The solutions and hints provided at the end of the text should be considered "bare bones." In particular, proofs given in the solutions section are *not* well-written. They are designed to simply show "how the proof works."

Acknowledgments

A multi-years long process cannot occur without the support of friends, family and colleagues. The love and support (and *patience*) of Elizabeth, Zeke, and my parents is unrivaled. Hours tucked away staring at a computer screen, checking for typos (of which there are some still lingering, and for that I apologize) or simply hearing me talk about “where I’m at” with my book . . . I am grateful and I am lucky.

One feeling I have hoped to convey throughout this book is that *math is fun*. Becoming a mathematician is not simply a career choice. It is who you become: how you think, how you act and what types of jokes you tell. I decided I wanted to be “this way” in large part due to the mathematicians at Lake Forest College. Many, many thanks for their inspiration and guidance.

It is tough to classify my North Central College colleagues simply as such. They are friends and mentors who push me to be a better version of myself, both inside and outside of the classroom. Katherine, Karl, Marco, Mary, Matthew, David, Rich: teachers, scholars and wonderful human beings, in the finest sense of the words. And to the large number of students at North Central College who have helped me hone my skills in teaching this material: you will never know how much you have actually taught me. One can only improve at something through making mistakes. I would be remiss without thanking them for pointing out such flaws and suggesting ways to improve.

Lastly, an utmost sincere and heartfelt appreciation goes out my contacts at CRC Press: Bob Ross, Jose Soto, Karen Simon and her editing team and the multitude of reviewers who provided wonderful suggestions. One could not ask for more supportive and knowledgeable guides through the book-writing process.

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1

Symbolic Logic

Logical reasoning is foundational to every field of study. New knowledge is created by drawing conclusions about certain phenomena. In the social sciences, for example, the behavior of a small, specific part of a population may be observed, and then, these observations are generalized to form a hypothesis about the entire population. To test that theory, other specific groups within the larger population are observed to see if the hypothesis is accurate. Should any of these additional test cases yield results that do not agree with the original generalization, the social scientist must change or reject the hypothesis.

This approach of taking the *specific to the general* is what is called **inductive reasoning**. It is fundamental to every scholar, even the mathematician. Much mathematical research is accomplished by looking at specific cases and generalizing those to some basic theory about *all* cases. But the mathematician knows that if something is true for *some* things, that same thing need not be true for *all* things. An example of this comes from one of history's more famous mathematicians: Pierre de Fermat (1601-1665).

Though he made notable contributions in numerous areas of mathematics, the Frenchman Pierre de Fermat is perhaps most highly regarded as the father of modern number theory (the study of properties of the integers). We will see some of his results in later chapters, but it is his conjecture regarding prime numbers that serves us well for this introduction to logic. Fermat claimed to have found a generating function for primes: for any positive integer n, the number $2^{2^n} + 1$ is prime [32].

A quick check for the first few values of n yields easy-to-verify integers: 5 ($n = 1$), 17 ($n = 2$), 257 ($n = 3$), and $65,537$ ($n = 4$). When we let $n = 5$, we obtain the integer 4,294,967,297. Today, a computer can quickly check if this number is prime, but in the 17th century, this would be quite the task (for fun, check, by hand, to see if any of the primes between 2000 and 3000 divide the number; when you've finished, you'll only have about 6400 more primes to check!). Yet, even if we were to see that it *is* prime, would that make *every* number of the form $2^{2^n} + 1$ prime? Certainly not; if we randomly chose a few larger values of n and found that they made $2^{2^n} + 1$ prime, we still could not claim that $2^{2^n} + 1$ is prime for all choices of n.[1]

How then does the mathematician validate such claims? Rather than reasoning from the specific to the general, she reasons from the *general to the*

[1] It turns out that Fermat's conjecture was wrong. It took nearly a century until Leonhard Euler (1707-1783) showed that $4,294,967,297 = 641 \times 6,700,417$ [17].

specific, a process known as **deductive reasoning**. It is fundamental to all of mathematics and is the basis for mathematical proof. No mathematical theory is deemed as true unless it is proven. In order to construct these proofs, we must understand the rules of logic. But where did this concept of treating logical thinking as its own study begin?

It is the mathematician and philosopher Aristotle (384-322 B.C.), a student of Plato's Academy in ancient Greece, who is regarded as the founder of logic [25]. He investigated the laws of reasoning in everyday language. Nearly two millennia later, Gottfried Leibniz (1646-1716, of calculus fame) sought to formalize this reasoning into a symbolic language [3]. It was this idea of his that spurred deeper investigations by Augustus De Morgan (1806-1871) in his work *Formal Logic* [35] and George Boole (1815-1864) in *The Mathematical Analysis of Logic* [5]. De Morgan and Boole took the symbolism of Leibniz and introduced a system of algebra on it to form what has become known as *symbolic logic*. It is the goal of this chapter to present these tools of symbolic logic that will support our discussions into mathematical proofs.

1.1 Statements and Statement Forms

Charles Lutwidge Dodgson, known by his pen name Lewis Carroll, is perhaps best known for his fictional pieces *Alice's Adventures in Wonderland* [10] and *Through the Looking-Glass* [9], yet he was a noted mathematician and logician [13]. Consider the following puzzle from his *Symbolic Logic* [8].

Example 1.1. If we assume the following four sentences as fact, then a conclusion can be made. What is that conclusion?

(1) None of the unnoticed things, met with at sea, are mermaids.
(2) Things entered in the log, as met with at sea, are sure to be worth remembering.
(3) I have never met with anything worth remembering, when on a voyage.
(4) Things met with at sea, that are noticed, are sure to be recorded in the log.

If we try to piece together all of the statements, it quickly becomes quite perplexing. It turns out that the conclusion to Carroll's puzzle that can be made is, "I have never met with a mermaid at sea."

It is easy to see that everyday language is riddled with confusion. Placement of phrases within sentences suddenly matters a great deal and word choice is of the utmost importance. For example, consider the very simple sentence, "At any given moment, there are two points on opposite sides of Earth that have the same temperature." Does this sentence have the same meaning as, "There are two points on opposite sides of Earth that, at any given moment, have the same temperature," or, "There is a moment for which two points on opposite sides of Earth have the same temperature?" It turns out that all three of these sentences have very different meanings.[2]

In the real world, we often have to consider numerous facts simultaneously in order to draw a single conclusion, such as in Example 1.1 (though you may never find yourself meeting mermaids and keeping a log while on a voyage at sea). How can we wade through the complexities of the English language? Is there a way to simplify the process, eliminating the variety of ways to phrase a single thought? Symbolic logic does just this. It uses symbols to aid in reasoning, and the building block of all symbolic logic structures is a *statement*.

Definition 1.1. A *statement* is a declarative sentence that is either true or false but not both.

Notice the key words in the previous definition. A statement must be a *declarative* sentence. Interrogative sentences, exclamations or commands are not statements. Moreover, a statement must *always* be true or *always* be false. It cannot change over time, and it must be one or the other, not both.

Example 1.2. The following are all statements.

(1) Theodore Roosevelt was born on October 27, 1858.

(2) LaTeXis a typesetting system for creating high-quality scientific documents.

(3) The moon is made of cheese.

(4) $12 \times (5 + 6) = 132$

(5) The real-valued function $f(x)$ is continuous at the real number a if

$$\lim_{x \to a} f(x) = f(a).$$

(6) The function $f(x) = \dfrac{1}{x}$ is continuous at $x = 0$.

Each of these is a declarative sentence that is always true or always false.

[2]The original statement is actually true; it is a consequence of a famous topology result named the Borsuk-Ulam theorem. The basics of topology serve as a capstone for this text and appear in Chapter 8.

Statements (1) and (2) of Example 1.2 are true. Theodore Roosevelt was indeed born on October 27, 1858, and LaTeXis a common tool for typesetting scientific (and in particular, mathematical) content. Upon reading (1.2) of Example 1.2, you may have thought, "I know the moon is not made of cheese!" The fact that it is a statement does not depend upon whether or not it is actually true. In this case, (3) just so happens to be a false statement.

Even though (4) of Example 1.2 does not appear as a "properly written sentence in grammatically correct English," it is a statement. It just as easily could have been written expositorily: "Twelve times the quantity five plus six equals one hundred thirty two." The choice to use a shorthand method does not prohibit it from being a statement.

The last two statements of Example 1.2 come from calculus. You should notice that (5) is the definition of a function $f(x)$ being continuous at a particular real number a. There is no dependence upon "if $f(x)$ is this function and a is this value" for (5) to be or not to be a statement. Simply put, this always will be the definition of continuity. Thus, it is a true statement. Statement (6) is false. The function $f(x) = \frac{1}{x}$ has an asymptotic discontinuity at $x = 0$. As with (3), the fact that this sentence is false does not prohibit it from actually being a statement, however.

Example 1.3. The following are not statements.

(1) Make your bed.
(2) Did you see the rainbow?
(3) $x^2 + 3x + 2 = 0$.
(4) The function $f(x)$ is not continuous at $x = 1$.

While it is obvious why (1) and (2) of Example 1.3 are not statements (they are not declarative sentences), it may not be as obvious why (3) or (4) are not. In (3), depending on what value the variable x takes in the expression, the equation may (such as $x = -1$) or may not ($x = 1$) be true. A statement must always be true or always be false. This same reasoning holds for (4) and the choice of function $f(x)$. In Section 2.3 we will develop a method for creating statements from these sorts of mathematical expressions.

Similar to this use of variables in mathematical expressions is the use of open-ended language in everyday English. The following example lists two sentences that are also not statements because they do not have a fixed truth value.

Example 1.4. The two sentences below are not statements.

(1) This year is a leap year.
(2) She aced her exam!

Sentence (1) of Example 1.4 would be true if it were said in 2012 or 2016. In 2017, it would be false. Because it does not have one fixed truth value, it is not a statement. Similarly, (2) is open-ended in that we do not know who “she” is. Generally, this is not a statement. However, if in context (such as a specific class), “she” is a fixed person, then this is indeed a statement. To avoid this confusion, we will assume throughout the text that such context is assumed and sentences like, “I am a cat owner,” and ”They live down this street” are statements.

At this point, you should be questioning the English language. Is this really a statement? Is it not a statement? For our purposes, we brush aside these concerns and introduce a mathematical notation to solidify our approach to logic. We will denote particular statements with variables. For example, let P be the statement, “The chemical symbol for water is H_2O.” We know P is true. However, if one were to vaguely introduce a statement by saying, “Let Q be a statement,” then we do not know if Q is true or if Q is false. We simply know it represents *some* statement but we do not know which one. Expressions like “Q” are called *statement variables*, though we will refer to them simply as statements.

Then, suppose P and Q are statements. Linguistically we can combine them, with the use of certain conjunctions to create new sentences that themselves are statements. The expressions “P and Q” or “P or Q” are part of our everyday vocabulary and we understand their meaning. For example, “The chemical symbol for water is H_2O and the chemical symbol for nitrogen is N,” is itself a statement, but it can also be thought of as two separate statements joined via the conjunction “and.” This is an example of forming new statements from old by using *logical connectives*. The terms “and” and “or” are our first examples of these. Expressions that are formed by combining statement variables with logical connectives are called *statement forms*, though as with statement variables, to avoid confusion, we refer to them simply as statements.

Definition 1.2. Given statements P and Q, the *conjunction of P and Q* is the statement denoted $P \wedge Q$, pronounced “P and Q,” that is true precisely when both P and Q are true and is false when at least one of P or Q is false.

Because every variable present in a statement form can take one of two truth values, a statement form with n different statement variables in it will have 2^n possible truth assignments. To list these out in a systematic and visual manner, we introduce the notion of a *truth table*. They provide an equivalent method for defining logical connectives.

Definition 1.3. A *truth table* for a statement form P with n statement variables P_1, P_2, $\ldots$, P_n is a table with 2^n rows, consisting of columns for, at the minimum, each of the statement variables P_i and for P. Each row of the table consists of a unique assignment of a specific truth value (true or false) to the statement variables, and the resulting truth value of P.

Example 1.5. For statements P and Q, $P \wedge Q$ is the statement defined via the following truth table.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Definition 1.4. Given statements P and Q, the *disjunction of P and Q* is the statement $P \vee Q$, pronounced "P or Q", that is true precisely when at least one of P or Q is true and is false when both P and Q are false. It is exhibited via the following truth table.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

In the English language, the conjunction "or" can considered one of two ways, *inclusively* or *exclusively*. A sports coach may be told, "You will keep your job if your team wins their division or wins 100 games." Would the coach be fired if his team wins 104 games and then proceeds to win the division championship? Of course not! This is the idea of "or" being used in the *inclusive* sense. Alternatively, at a restaurant, upon ordering an entrée, the server may present you with the option of a side: soup or salad. You are allowed to choose one single side but will be charged extra if you choose both. This is "or" being used *exclusively*.

Note that Definition 1.4 is defining "or" in the inclusive sense. Any mathematical use of the word "or" or the symbol "$\vee$" is meant to be inclusive.[3] When exclusivity is required, it will be explicitly stated, as in the following definition.

Definition 1.5. Given statements P and Q, the *exclusive disjunction of P and Q* is the statement $P \veebar Q$, pronounced "P exclusive or Q", that is true when exactly one of P or Q is true and is false when both P and Q are false or both P and Q are true. It is exhibited via the following truth table.

P	Q	$P \veebar Q$
T	T	F
T	F	T
F	T	T
F	F	F

The last basic logical connective is perhaps the simplest: the negation of a single statement.

Definition 1.6. Given a statement P, the *negation of P* is the statement $\sim P$, pronounced "not P", that is true when P is false and is false when P is true. It is exhibited via the following truth table.

P	$\sim P$
T	F
F	T

These four logical connectives allow for the creation of more complicated statement forms from previously defined statement forms or variables. To clarify this language, consider Example 1.6.

Example 1.6. Consider the following.

(1) P is a statement variable in the statement form $\sim P$.
(2) P and Q are statement variables in the statement form $P \wedge Q$ and $P \vee Q$.
(3) The statement $P \wedge \sim(\sim Q \vee \sim R)$ is a statement form with variables P, Q and R.

[3] A classic mathematical joke is based on this idea: A mother has just given birth to a baby when she turns to the baby's father, a mathematician, and asks, "Is it a boy or a girl?" He responds, "Yes!"

As previously mentioned, however, our use of the term "statement form" will be rare. Simply calling $P \wedge \sim(\sim Q \vee \sim R)$ a statement serves our purposes well.

Armed with just these four logical connectives, we can construct truth tables for countless statements. Grouping symbols such as parentheses may be used, but as with arithmetic operations, there is an "order of precedence" for logical connectives. Of highest precedence is $\sim$ while $\wedge$, $\vee$, and $\veebar$ have the same precedence, taken left-to-right (similar to $+$ and $-$ in the order of arithmetic operations). Thus,

$$\sim P \wedge Q \vee \sim R$$

is interpreted as

$$((\sim P) \wedge Q) \vee (\sim R).$$

The following two examples do more than just provide the truth table for the given statements. They exhibit the *process* of creating a truth table. Note that all of the variables of the statement appear in the first columns. From there, the variables are combined, one logical connective at a time, until the desired statement form is constructed. This is a systematic way to not just form the columns of the truth table but also a very simple method for filling in the truth values of the columns. When a column is formed by logically connecting at most two previous columns, filling in the truth values is as simple as pointing to the two (or one, in the case of $\sim$) "columns" the logical connective is "connecting." If the new column is of the form, "Column 1 $\wedge$ Column 2," then proceed down Columns 1 and 2, pointing at truth values in the same row. If you are pointing at two T values, then the result is to place a T in the new column. Otherwise, place an F in the new column (this is how a conjunction "works").

Example 1.7. Construct the truth table for $(P \vee Q) \wedge \sim(P \wedge Q)$.

We begin by listing the columns. First, the variables of the statement are P and Q. From there, we build up one connective at a time. Thus, we need a column for $P \vee Q$ and a column for $P \wedge Q$. Note that we would not jump straight to a column for $\sim(P \wedge Q)$; this involves two connectives. Then, before we place a column for our desired statement, we do need a column for $\sim(P \wedge Q)$. Then, the final statement's truth values will be obtained by looking only at the truth values in the columns for $P \vee Q$ and $\sim(P \wedge Q)$.

P	Q	$P \vee Q$	$P \wedge Q$	$\sim(P \wedge Q)$	$(P \vee Q) \wedge \sim(P \wedge Q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

Regardless of the variables present or the logical connectives used, truth tables are easily constructed in the fashion described above.

Example 1.8. Construct the truth table for $\sim(P \veebar \sim R) \wedge Q$.

P	Q	R	$\sim R$	$P \veebar \sim R$	$\sim(P \veebar \sim R)$	$\sim(P \veebar \sim R) \wedge Q.$
T	T	T	F	T	F	F
T	T	F	T	F	T	T
T	F	T	F	T	F	F
T	F	F	T	F	T	F
F	T	T	F	F	T	T
F	T	F	T	T	F	F
F	F	T	F	F	T	F
F	F	F	T	T	F	F

In the previous two examples, the statement for which the truth table was constructed was sometimes true and sometimes false, depending on the truth assignments to the specific variables. Statement forms that always takes the same truth value, regardless of the truth assignment to its variables, have one of two special names.

Definition 1.7. A statement form that is true for every truth assignment to its variables is called a *tautology*. One that is false for all truth assignments to its variables is called a *contradiction*.

What is the intuitive sense of a tautology? It is a statement that is *always true*. It is possible to construct rather complicated statements that are tautologies, but one in particular is simple to construct. It relies on the fact that a statement is either true or false. Because of this, either the statement or its negation must be true.

Example 1.9. The following truth table shows that, for a statement P, $P \vee \sim P$ is a tautology.

P	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

Using a similar thought process, we have the following contradiction.

Example 1.10. Let P be a statement. Show that $P \wedge \sim P$ is a contradiction.

P	$\sim P$	$P \wedge \sim P$
T	F	F
F	T	F

The third column of the truth table shows $P \wedge \sim P$ is a contradiction.

Consider the possible truth values for $(P \vee Q) \wedge \sim(P \wedge Q)$ in Example 1.7. They are identical to the truth values of $P \veebar Q$, for all possible truth assignments to P and Q. Such statements are called *logically equivalent*.

Definition 1.8. Two statement forms with the same variables are called *logically equivalent*, with equivalence denoted via the symbol $\equiv$, if the statement forms have the same truth value for every possible truth assignment to the statement variables of the statement form.

Example 1.11. The statements $P \veebar Q$ and $(P \vee Q) \wedge \sim(P \wedge Q)$ are equivalent, as verified by the truth tables in Definition 1.5 and in Example 1.7. We write

$$P \veebar Q \equiv (P \vee Q) \wedge \sim(P \wedge Q).$$

In general, to determine if statements are or are not logically equivalent, construct a single truth table with a column for each of the statements. If the truth values are identical in every single row, then the statements are logically equivalent. If they differ in at least one row, the statements are not logically equivalent.

Example 1.12. For statements P and Q, the truth table below shows that $P \vee Q \equiv Q \vee P$.

P	Q	$P \vee Q$	$Q \vee P$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Example 1.12 should feel natural. Saying "red or blue" has the same meaning as saying "blue or red." This is the intuition behind logical equivalence. It is a mathematical way to communicate that two statements "mean the same thing."

It is important to also note that $\equiv$ is not a logical connective. Rather, it is *relational* in nature, much like the symbol $=$ is relational on real numbers. It is not a tool for creating new statements from old ones, like $\wedge$ and $\sim$ on statements or $+$ or $\sqrt[3]{}$ on real numbers. That is, $P \equiv Q$ is not a statement; it is not possible to construct a truth table for it.

The concept of logical equivalence leads to our first theorem. Though we have not discussed mathematical proof, we prove parts of this theorem here. For now, think of a proof as "irrefutable justification" of the results.

Theorem 1.9. *Let P, Q and R be statements, t a tautology and c a contradiction. The following logical equivalences, called logical equivalence laws, hold.*

1. Commutative	$P \wedge Q \equiv Q \wedge P$ $P \vee Q \equiv Q \vee P$
2. Associative	$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$ $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$
3. Distributive	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
4. Identity	$P \wedge t \equiv P$ $P \vee c \equiv P$
5. Negation	$P \vee \sim P \equiv t$ $P \wedge \sim P \equiv c$
6. Double Negative	$\sim(\sim P) \equiv P$
7. Idempotent	$P \wedge P \equiv P$ $P \vee P \equiv P$
8. Universal Complement	$\sim t \equiv c$ $\sim c \equiv t$

Proof Proofs of the negation equivalence (5) and the commutative equivalence (1) appear in Examples 1.10 and 1.12, respectively. We prove here only one of the distributive laws; the remaining proofs are left as exercises.

To prove the first of the two distributive laws, let P, Q and R be statements. For brevity, only the columns for the statement variables and desired statement forms are presented.

P	Q	R	$P \wedge (Q \vee R)$	$(P \wedge Q) \vee (P \wedge R)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	T	T
F	F	T	T	T
F	F	F	F	F

Regardless of the truth assignment for every variable of the statements, the statements $P \wedge (Q \vee R)$ and $(P \wedge Q) \vee (P \wedge R)$ have the same truth value, as exhibited above. Thus,

$$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R),$$

the desired result. □

There is one extremely important logical equivalence law that could have been included in Theorem 1.9. Because of its importance, however, we state it as its own theorem. It is attributed to one of the founding fathers of symbolic logic, the aforementioned Augustus De Morgan.

Theorem 1.10. (De Morgan's Laws) For statements P and Q, we have

$$\sim(P \wedge Q) \equiv \sim P \vee \sim Q$$

and

$$\sim(P \vee Q) \equiv \sim P \wedge \sim Q.$$

Proof We prove the first of De Morgan's Laws, leaving the second as an exercise. For statements P and Q, we have

P	Q	$\sim P$	$\sim Q$	$P \wedge Q$	$\sim(P \wedge Q)$	$\sim P \vee \sim Q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Because $\sim(P \wedge Q)$ and $\sim P \vee \sim Q$ have the same truth value for every truth assignment to P and Q, the result holds. □

The basic logical connectives of *not*, *and* and *or* allow us to create meaningful logically equivalent statements. In the next section, we introduce a pair of logical connectives that open the door to mathematical conjecture, theory and proof.

Exercises

1. Determine which of the following are statements.
   (a) There are 30 days in April.
   (b) What happened on April 23, 1980?
   (c) There are 31 days in April.
   (d) Look out!
   (e) Brian wants to know what brand of bike that is.
   (f) Gasoline is not a liquid.
   (g) That tiger is ferocious!
2. Write the negation of each of the following statements.
   (a) Blue is Caroline's favorite color.
   (b) The smartphone has no more than 32 gigabytes of memory.
   (c) I was not born yesterday,
   (d) It was a home game and they lost.
3. Let statements P, Q and R be defined as follows.

   P: I stayed up past midnight.

   Q: I overslept.

   R: I passed my exam.

   (a) Write each of the following symbolically, using logical connectives and P, Q, and R.
      i. I stayed up past midnight and overslept.
      ii. Even though I stayed up past midnight, I did not oversleep.
      iii. I stayed up past midnight and overslept, but I still passed my exam.
      iv. I either overslept or I passed my exam, but not both.
   (b) Write the following symbolic statements as sentences in English.
      i. $P \wedge Q \wedge R$
      ii. $\sim P \vee \sim R$
      iii. $(P \veebar R) \vee \sim Q$

4. Let P, Q, and R be the following statements.

   P: This pen is out of ink.

   Q: I must complete this assignment by tomorrow morning.

   R: This assignment is worth 10 points.

   In everyday English, exhibit the logical equivalences of Theorem 1.9 for the particular statements.

   (a) Commutative: Q, R
   (b) Associative: P, Q, R
   (c) Distributive: P, Q, R
   (d) Double negative: Q

5. Exhibit, in everyday language, De Morgan's Laws for the statements P and Q of the previous problem.

6. For each of the following, (a) write it in symbolic form (you will need to define the statement variables), (b) write the symbolic negation of each and (c) write the negation of each in everyday English.

   (a) The integer 2 is a prime number but is also an even number.
   (b) Either 7 is even or 7 is prime.
   (c) The number 10 is neither prime nor odd.

7. For each of the following sentences (some of which are not statements), determine if the intention is for the *or* to be considered inclusively or exclusively.

   (a) Would you like cream or sugar in your coffee?
   (b) Flip the switch up or down.
   (c) You get soup or salad with your dinner.
   (d) Was the baby a boy or a girl?[4]

8. Give an example of a statement in the English language that is a tautology.

9. Give an example of a statement in the English language that is a contradiction.

10. Construct a truth table for each of the following statements. Assume that P is a statement.

    (a) P
    (b) $P \wedge \sim P$
    (c) $P \vee \sim(P \vee \sim P)$
    (d) $P \veebar \sim P$

[4]This question is an integral part of a classic mathematical joke. *A couple is in the delivery room, and upon giving birth, the wife asks her husband, a mathematician, "Is it a boy or a girl?" He responds, "Yes."*

11. Construct a truth table for each of the following statements. Assume that P and Q are statements.

    (a) $Q \wedge (P \vee Q)$
    (b) $\sim P \wedge Q$
    (c) $\sim P \vee \sim Q \vee (P \wedge Q)$
    (d) $\sim (P \vee \sim Q) \vee \sim (P \wedge Q)$

12. Construct a truth table for each of the following. Assume that P, Q, and R are statements.

    (a) $P \wedge (Q \vee R)$
    (b) $(\sim P \wedge Q) \wedge R$
    (c) $\sim ((P \vee Q) \vee R)$
    (d) $[(P \wedge \sim Q) \veebar R] \vee (\sim R \vee Q)$

13. Assuming that P, Q and R are statements, determine if any of the following statements is a tautology or a contradiction. Justify your answer using a truth table.

    (a) $\sim (P \vee (\sim P \wedge Q))$
    (b) $P \veebar \sim P$
    (c) $\sim (P \vee Q) \veebar R$

14. Suppose a new logical connective $\times$ is defined in the following truth table. Using this definition, construct a truth table for the following statements, assuming P, Q, and R are statements.

P	Q	$P \times Q$
T	T	F
T	F	T
F	T	F
F	F	T

    (a) $P \vee (Q \times P)$
    (b) $P \times (Q \times R)$
    (c) $(P \times Q) \veebar (P \times R)$
    (d) $(P \veebar Q) \times (P \veebar Q)$

15. Determine if the logical connective $\times$ defined in the previous problem is:

    (a) Associative
    (b) Commutative

16. Show the following logical equivalences hold. Assume that P, Q, and R are statements.

    (a) $P \wedge Q \equiv Q \wedge P$

(b) $P \veebar Q \equiv (P \wedge {\sim}Q) \vee ({\sim}P \wedge Q)$
(c) $P \vee Q \equiv {\sim}({\sim}P \wedge {\sim}Q)$
(d) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$
(e) ${\sim}(P \wedge Q) \equiv {\sim}P \vee {\sim}Q$

17. Show the Associative laws of Theorem 1.9 hold.
18. Show the Identity laws of Theorem 1.9 hold.
19. Show the Negation laws of Theorem 1.9 hold.
20. Show the Double Negative law of Theorem 1.9 holds.
21. Show the Idempotent laws of Theorem 1.9 hold.
22. Show the Universal Complement laws of Theorem 1.9 hold.
23. Determine if each of the following pairs of statements are logically equivalent. Assume all variables represent statements.

    (a) P, Q
    (b) $P \veebar Q$, $(P \vee Q) \vee ({\sim}P \wedge {\sim}Q)$
    (c) ${\sim}({\sim}P) \vee {\sim}P$, $Q \vee {\sim}Q$
    (d) $P \vee (P \wedge {\sim}Q)$, $P \wedge {\sim}Q$

24. Are (a) through (c) below statements? Explain.

    (a) This statement is false.
    (b) The barber shaves everyone who does not shave himself.
    (c) A being with unlimited physical powers can create a wall taller than he is to scale.

1.2 Conditional and Biconditional Connective

Think about decisions you make as you go throughout your day. Maybe you did not know what to wear, so you turned on the weather forecast and thought to yourself, "If it is supposed to be chilly, then I'll wear long sleeves. If the forecast is for warmer weather, then I'll choose a t-shirt to wear." These types of if-then sentences, part of our everyday language, are the basic tool for inference. Mathematicians call these *conditional statements*.

Definition 1.11. A *conditional statement* (or *implication*) is one of the form "if P, then Q," where P and Q are statements, and is denoted $P \Rightarrow Q$. The statement P is called the *hypothesis* (or *assumption*, *premise*, or *antecedent*) of the conditional statement. The statement Q is called the *conclusion* (or *consequence*) of the implication. The conditional statement $P \Rightarrow Q$ is false only when P is true and Q is false. It is exhibited via the following truth table.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

If we consider a real-world example and ask the question, "When is this implication true," we may find ourselves somewhat confused. For example, consider the conditional statement, "If it is raining, then I will carry an umbrella." It could either be raining or not raining and I could either be carrying or not carrying an umbrella. This means there are four possible cases to consider. If it is raining and I have my umbrella open, then it is clear that our original statement is true.

But what about if it is not raining? Is the statement, "If it is raining, then I will carry an umbrella," true if it is not raining yet I do have an umbrella open? Is it true if it is not raining and I do not have my umbrella open? According to Definition 1.11, the original statement is true in both these cases. But *why*? It is in no way intuitive. Looking at it from a different perspective, the reasoning for defining the conditional in this way makes complete sense.

What point-of-view clarifies Definition 1.11? Rather than asking when you would say a conditional statement is *true*, what if we asked when it is clearly *false*? Even four-year old children know the answer to this. Suppose a parent tells her child, "If you eat your broccoli, then you can have dessert." What happens if the child cleans his plate of broccoli but is told that he won't be getting any dessert? Perhaps the child screams, "That's not fair! You lied!" No matter the child's response, he realizes that the original statement was clearly *false*.[5]

Because one can explicitly say a conditional statement is false only when the hypothesis is true and the conclusion is false, mathematicians define implications in this way.

Once a truth table for a statement is established, under the guise of logical equivalence we can ask, "Does this statement *say the same thing* as any other statement?" Analyze the truth table for $P \Rightarrow Q$. It is true in the bottom two rows (when P is false), or in the first and third rows (when Q is true). This inspires the following logical equivalence.

Theorem 1.12. *Let P and Q be statements. Then,*

[5] You can partake in applied symbolic logic by attempting this with your favorite toddler. The author assumes no responsibility for the reactions of the youngster.

$$P \Rightarrow Q \equiv \, \sim P \vee Q.$$

Proof Let P and Q be statements. Then,

P	Q	$P \Rightarrow Q$	$\sim P$	$\sim P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Because $P \Rightarrow Q$ and $\sim P \vee Q$ have the same truth value for all truth assignments for P and Q, they are logically equivalent. $\square$

Theorem 1.12 provides an equivalent way to verbalize conditional statements. Saying "if P, then Q" is logically equivalent to saying, "not P or Q." For example, saying, "It is not raining or I'm carrying an umbrella" has the same logical interpretation as, "If it is raining, then I carry an umbrella." But this alternative pronunciation is just one way to present conditional statements in the English language. Example 1.12 provides some of these equivalent sayings, while others are left as exercises.

Example 1.13. The following three sentences are equivalent to saying, "if P, then Q."

(1) Q if P.
(2) P implies Q.
(3) P is a sufficient condition for Q.

The third equivalence to saying "if P, then Q" of the previous example, "P is a sufficient condition for Q", is a common mathematical expression. Why is its interpretation the same as "if P, then Q?" The latter statement is interpreted as, "Whenever P occurs, Q must happen." View it as a domino-effect; the domino representing P falling over must tip over the domino representing Q. Then, knowing that the P domino falls is sufficient for knowing the Q domino falls over.

A similar mathematical expression involves the word *necessary*. Saying, "P is a necessary condition for Q" is equivalent to saying "if Q, then P." Notice the order of the conditional statement; Q is the antecedent. Why? In this scenario, the Q domino cannot fall over without the P domino falling. Thus, for Q to fall, it is necessary for P to fall.

Example 1.14. Rewrite each of the statements in the standard if-then form of a conditional statement.

(1) Being at least 18 years old is a necessary condition to vote in Iowa.

Equivalent statement: If a person votes in Iowa, then he or she is at least 18 years old.

(2) It is necessary for Joey to have a helmet to compete in the bike race.

Equivalent statement: If Joey competes in the bike race, then he must have a helmet.

(3) Knowing that an integer is divisible by 8 is sufficient to know it is even.

Equivalent statement: If an integer is divisible by 8, then it is even.

It is important to be aware that being necessary for and being sufficient for something to happen are two different things. For example, the third statement of Example 1.14 states that being divisible by 8 is sufficient in knowing an integer is even. Is it necessary? Of course not. There are plenty of even integers that are not divisible by 8, such as 2, 4 and 6.

In Theorem 1.9 we saw that the logical connectives $\vee$ and $\wedge$ are associative. We show below that $\Rightarrow$ is *not* associative. Because of this, the use of grouping symbols is necessary to create properly defined compound statements.

Example 1.15. The statements $P \Rightarrow (Q \Rightarrow R)$ and $(P \Rightarrow Q) \Rightarrow R$ are not logically equivalent, written

$$P \Rightarrow (Q \Rightarrow R) \not\equiv (P \Rightarrow Q) \Rightarrow R.$$

The two statements will have identical truth tables if they are logically equivalent. Consider the table for each.

P	Q	R	$Q \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$	$P \Rightarrow Q$	$(P \Rightarrow Q) \Rightarrow R$
T	T	T	T	T	T	T
T	T	F	F	F	T	F
T	F	T	T	T	F	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	F	T	T	F
F	F	T	T	T	T	T
F	F	F	T	T	T	F

Because $\Rightarrow$ is not associative, statements like $P \Rightarrow Q \Rightarrow R$ require, as mentioned above, either grouping symbols or, as discussed in the previous section, an "order of operations." The connective $\Rightarrow$ has its precedence directly after $\wedge$ and $\vee$. For example, $P \wedge Q \Rightarrow R \vee S$ is interpreted as $(P \wedge Q) \Rightarrow (R \vee S)$. To avoid confusion, we will consistently use grouping symbols. Before proceeding in our development of symbolic logic, we use this as an opportunity for our first discussion on *style*.

To this point in your mathematical upbringing, you have been mostly concerned with simply *getting the right answer*. You may have learned to develop scratch work and then write up your final solution "neatly," but have you ever stopped and pondered what is meant by "neatly presented mathematics?" In calculus, this might be showing the appropriate steps of a lengthy integration, clearly noting substitutions you have used and how those substitutions impact the bounds on the integral. Your page is organized, your handwriting clear and your page contains no "messy stuff." This organization and in-page display is the foundation of proper mathematical presentation, but as we will learn throughout this text, there is much more to writing *good* mathematics.

For now, there are two points about writing a proper solution worth discussing. Both involve keeping your reader informed. When you write, you are writing *for* somebody. You do not want your reader to be confused. Starting every problem with a proper introduction and finishing with a summarizing conclusion are simple ways to bookend your work and keep your reader focused. "This is what we are going to do . . . We have shown what we intended to show." For example, if you are showing two statements are logically equivalent, do not simply present a truth table. Even if the truth table is perfectly correct, your reader does not know why it is there. A short, concise introduction, such as, "We will show A and B are logically equivalent by considering the following truth table," tells your reader exactly what you are about to do and why you are doing it.

Along this same line, if you do not tell your readers why a certain conclusion holds, they are left confused. "She told me she was going to show A and B were logically equivalent, and I see the truth table, but is there *more* to the solution?" A quick sentence squashes all potential issues: "Because A and B have identical truth tables, the desired result holds." You have *justified* your reasoning, something mathematicians demand.

Lastly, as you present your work, keep your reader in mind. This idea circles back to the choice to use grouping symbols when presenting statements such as $P \Rightarrow Q \Rightarrow R$. While it is not required, because of the order of precedence with logical connectives, it allows your reader to more easily follow your work. There is no need for a reader to pause and think, "Does $P \Rightarrow Q \Rightarrow R$ mean $(P \Rightarrow Q) \Rightarrow R$ or $P \Rightarrow (Q \Rightarrow R)$."

Example 1.15 showed that the logical connective $\Rightarrow$ is not associative. Is it commutative? In terms of everyday language, asking if the conditional connective is commutative equates to asking if saying, "If I eat my broccoli, then I get dessert," is the same as saying, "If I had dessert, then I must have eaten my broccoli?" It is not, which we prove after the following definition.

Definition 1.13. The *contrapositive* to the statement $P \Rightarrow Q$ is the statement $\sim Q \Rightarrow \sim P$. The *converse* to $P \Rightarrow Q$ is the statement $Q \Rightarrow P$, and the *inverse* statement to $P \Rightarrow Q$ is $\sim P \Rightarrow \sim Q$.

Example 1.16. Write the contrapositive, converse, and inverse to the statement below.

Statement: If I water the plants daily, then they will bloom.

Contrapositive: If the plants do not bloom, then I did not water them daily.

Converse: If the plants bloom, then I watered them daily.

Inverse: If I do not water the plants daily, then they will not bloom.

Which of the statements in Example 1.16 have the same logical meaning? All of the statements are conditional statements, so it makes sense to determine the sole case when each is false. The original statement is false only when the plants are watered daily and they do not bloom. The contrapositive is not true precisely when the plants do not bloom and they are not *not* watered daily (that is, they are watered daily). Thus, the statement and its contrapositive have the same meaning.

The converse, however, is false only when the plants bloom and they are not watered daily. This is not the same as the original statement. Likewise, the inverse is not true only when the plants are not watered daily and they do not *not* bloom. So, the converse and the inverse have the same meaning. Theorem 1.14 summarizes these ideas.

Theorem 1.14. *A conditional statement is logically equivalent to its contrapositive but not to either its converse or inverse.*

Proof The truth table below proves that $P \Rightarrow Q$ and $\sim Q \Rightarrow \sim P$ always have the same truth value while $P \Rightarrow Q$ and its converse or inverse do not.

P	Q	$\sim P$	$\sim Q$	$P \Rightarrow Q$	$\sim Q \Rightarrow \sim P$	$\sim P \Rightarrow \sim Q$	$Q \Rightarrow P$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	F	F
F	F	T	T	T	T	T	T

$\square$

Note that proof to Theorem 1.14 shows that the converse and the inverse of an implication are themselves logically equivalent. This can also be seen either in the fact that they are contrapositives of one another, and by Theorem 1.14, they must be logically equivalent.

One last thing to note about conditional statements is that $P \Rightarrow Q$ and $Q \Rightarrow P$ are *not* negations of one another. That is, knowing that one is not true does not automatically imply that the other is. Mathematically, we are claiming that $\sim(P \Rightarrow Q)$ and $Q \Rightarrow P$ are not logically equivalent. The truth table below verifies this.

P	Q	$P \Rightarrow Q$	$\sim(P \Rightarrow Q)$	$Q \Rightarrow P$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	F	T

What then do we mean if we mention the *negation* of a conditional statement $P \Rightarrow Q$? The negation of a statement is itself a statement, and it is true precisely when the original statement is false. In the case of $P \Rightarrow Q$, this is when P is true and Q is false.

Definition 1.15. The *negation* of the conditional statement $P \Rightarrow Q$ is the statement $\sim P \wedge Q$.

Sometimes in our everyday conversations we want to state that knowing one piece of information is identical to knowing another. That is, knowing statement P means we know statement Q, and, knowing statement Q means we know statement P. This would require saying two separate conditional statements: $P \Rightarrow Q$ and $Q \Rightarrow P$, respectively. In everyday language this can be a bit cumbersome: "if you eat your broccoli, then you will get dessert, and, if you have had dessert, then you must have eaten your broccoli." Luckily, there is a shorthanded way of saying this, and it uses the logical connective known as the *biconditional*.

Definition 1.16. A *biconditional statement* is one of the form "P if and only if Q" where P and Q are statements, and is denoted $P \Leftrightarrow Q$. It is true when P and Q have the same truth value and false when their truth values are different, exhibited via the following truth table.

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Note the process for creating the truth table for a biconditional. In the spirit of "building" a column by pointing at previous columns, $P \Leftrightarrow Q$ is true if you are pointing at the same truth value (regardless of what the actual truth value is) and false if you are pointing at different truth values.

Example 1.17. Build the truth table for $P \Leftrightarrow (Q \Rightarrow \sim R)$.

Even though this statement involves a new logical connective, we construct the truth table just as before. Begin by creating columns for the variables P, Q, and R. Before creating a column for $P \Leftrightarrow (Q \Rightarrow \sim R)$, we must have a column for $Q \Rightarrow \sim R$, necessitating a column for $\sim R$.

P	Q	R	$\sim R$	$Q \Rightarrow \sim R$	$P \Leftrightarrow (Q \Rightarrow \sim R)$
T	T	T	F	F	F
T	T	F	T	T	T
T	F	T	F	T	T
T	F	F	T	T	T
F	T	T	F	F	T
F	T	F	T	T	F
F	F	T	F	T	F
F	F	F	T	T	F

The statement "P if and only if Q" is the conjunction (in plain English) of two other statements: "P if Q", and "P only if Q." This is not by accident. Both of these latter statements are themselves conditional statements: $Q \Rightarrow P$ and $P \Rightarrow Q$, respectively. The discussion leading into Definition 1.16 introduced the biconditional as such. Theorem 1.17 formalizes this concept. Its proof is left as an exercise. While Theorem 1.17 has repercussions in symbolic logic, its real value will appear in our discussions on proof techniques.

Theorem 1.17. *For statements P and Q, $P \Leftrightarrow Q \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$.*

Pairing Theorem 1.17 with the multitude of ways to verbalize conditional statements creates even more ways to verbalize biconditional statements. This is just one more suggestion as to the power of symbolic logic. While there are dozens of ways to say two statements P and Q have the same meaning, symbolically, any such way will be logically equivalent to $P \Leftrightarrow Q$.

Example 1.18. Rewrite the statement, "An integer is even if and only if it is divisible by 2," two different ways.

While there are many options, we first explicitly use the result of Theorem 1.17:

(1) If an integer is even, then it is divisible by 2, and, if an integer is divisible by 2, then it is even.

Next, refer to the discussion on necessary and sufficient conditions. The given statement is therefore equivalent to:

(2) Being divisible by 2 is a necessary and sufficient condition for an integer to be even.

With this understanding of symbolic logic and the five logical connectives of $\wedge$, $\vee$, $\sim$, $\Rightarrow$ and $\Leftrightarrow$, we are prepared to understand what it means to mathematically *argue*. Having a firm grasp on the notion of *argument* will not only help you better understand mathematical reasoning, it will prepare you to substantiate arguments in the public realm (or point out that certain arguments carry no merit).

Exercises

1. Write the following statements in standard "if, then" form.
   (a) We will go on vacation if your mother does not have to work.
   (b) Your injury will heal as long as you rest it.
   (c) Classes are canceled if it snows more than 12 inches.
   (d) Vertical angles are congruent.
   (e) Linda's cat Rudy hides whenever he hears thunders.
2. Write the following statements in standard "if, then" form.
   (a) It is necessary for you to do your chores in order to earn your allowance.
   (b) Giving the plants water is necessary for their survival.
   (c) The commuter train being on schedule is sufficient for you arriving on time.
   (d) A sufficient condition for traveling from Chicago to St. Louis is taking a direct flight.
3. Write the following statements in standard "if, then" form.
   (a) I always remember my first kiss when I hear this song.
   (b) A failing grade is a consequence of plagiarizing.

   (c) Winning the game follows from outscoring our opponents.
   (d) The video game will reset only if you hit those buttons simultaneously.
   (e) I begin to sweat whenever I think about snakes.

4. Write the following statements in standard "if and only if" form.
   (a) The exam is curved precisely when the average falls below 70%.
   (b) If I go to the pool, then I will sleep well tonight, and if I sleep well tonight, then I went to the pool.
   (c) Passing the final exam is both necessary and sufficient for you to pass this class.
   (d) That dashboard light being on is a consequence of a low oil level, and vice versa.

5. Which of the following are true statements?
   (a) If the earth is flat, then 15 is a negative number.
   (b) A whole number is a perfect square if its square root is also a whole number.
   (c) $x + 7 = 9$ is necessary for knowing $x = 2$.
   (d) $x + 7 = 9$ is sufficient for knowing $x = 2$.
   (e) You sweat only if you run a marathon.
   (f) You sweat if you run a marathon.

6. For each of the following statements, write its converse, inverse and contrapositive statements.
   (a) If the restaurant is closed, we will eat at home.
   (b) The question should be answered if the material is covered in class.
   (c) The tree falling down would be a consequence of a bad storm.
   (d) I want to go only if he does too.

7. Write each of the following statements in standard "if, then" form, and then write their converses and contrapositives.
   (a) Harper apologizing implies that she caused the accident.
   (b) She will win the tournament if she birdies the hole.
   (c) Brad cries whenever he watches this movie.
   (d) To get into the club, it is necessary that you know the secret handshake.
   (e) The cake will set only if you use the correct ratio of ingredients.

8. Create a truth table for each of the following. Assume P, Q, and R are statements.
   (a) $P \Rightarrow \sim Q$

(b) $P \Rightarrow (Q \Rightarrow R)$
(c) $(P \Rightarrow Q) \Rightarrow R.$
(d) $({\sim}P \vee Q) \Rightarrow ({\sim}Q \Rightarrow P)$
(e) $(P \vee Q) \Rightarrow (R \wedge Q)$

9. Create a truth table for each of the following. Assume P, Q, and R are statements.
   (a) $(P \Leftrightarrow Q) \Leftrightarrow R$
   (b) $R \Leftrightarrow (Q \Leftrightarrow R)$
   (c) $(P \Leftrightarrow R) \wedge (P \Leftrightarrow Q)$
   (d) $P \Leftrightarrow (Q \veebar (R \Rightarrow P))$
10. Create a tautology using symbolic statements (P, Q, R, etc.) and the logical connectives $\sim$ and $\Rightarrow$, and show that it is indeed a tautology.
11. Create a contradiction using symbolic statements (P, Q, R, etc.) and the logical connectives $\sim$ and $\Rightarrow$, and show that it is indeed a contradiction.
12. Create a tautology using symbolic statements (P, Q, R, etc.) and the logical connectives $\sim$ and $\Leftrightarrow$, and show that it is indeed a tautology.
13. Create a contradiction using symbolic statements (P, Q, R, etc.) and the logical connectives $\sim$ and $\Leftrightarrow$, and show that it is indeed a contradiction.
14. Show that Theorem 1.17 is true: $(P \Leftrightarrow Q) \equiv (P \Rightarrow Q) \wedge (Q \Rightarrow P)$.
15. State the negation of each of the following.
   (a) If the answer is negative, then I was incorrect.
   (b) The carpet will be replaced only if the basement floods.
   (c) Campus officials will attend if and only if they are invited.
   (d) If I order the salad bar, then I will get a free bowl of soup or a free dessert.
16. State the negation of each of the following using only the logical connectives $\wedge$ and $\sim$. Assume P, Q, and R are statements.
   (a) ${\sim}P \Rightarrow Q$
   (b) $(P \wedge Q) \Rightarrow (Q \vee R)$
   (c) ${\sim}(R \vee {\sim}P) \Rightarrow (P \Rightarrow Q)$
   (d) $P \Leftrightarrow {\sim}Q$
   (e) $(P \Rightarrow Q) \Leftrightarrow (R \Rightarrow P)$
17. Suppose P and Q are statements and that the statement $P \Rightarrow Q$ is false. Can you conclude anything about whether the following are true or false? Justify.

(a) $P \wedge Q$
(b) $P \vee Q$
(c) $P \veebar Q$
(d) $P \Leftrightarrow Q$
(e) $Q \Rightarrow P$

18. Suppose P and Q are statements and that the statement $P \Leftrightarrow Q$ is false. Can you conclude anything about whether the following are true or false? Justify.

(a) $P \wedge Q$
(b) $P \vee Q$
(c) $P \veebar Q$
(d) $P \Rightarrow Q$
(e) $Q \Rightarrow P$

19. What can you conclude about the truth of $P \Rightarrow Q$ if you know the statement in each of the following is false? (Note that these are separate problems; for each, you know that only the given statement is false.) Assume all variables represent statements.

(a) $(P \Rightarrow Q) \Rightarrow {\sim}(P \Rightarrow Q)$
(b) ${\sim}(P \Leftrightarrow Q)$
(c) $R \Rightarrow (P \Rightarrow Q)$

20. Determine, with justification, if each of the following is a tautology, contradiction, or neither. Assume all variables represent statements.

(a) $P \Rightarrow (P \Rightarrow Q)$
(b) $(Q \Leftrightarrow P) \Leftrightarrow (P \Leftrightarrow R)$
(c) ${\sim}(Q \Rightarrow P) \Rightarrow P$
(d) $[P \wedge (Q \Leftrightarrow R)] \Rightarrow {\sim}R$
(e) $[P \Rightarrow (Q \Rightarrow P)] \veebar (Q \Leftrightarrow P)$

21. For each of the following, show that the first statement is logically equivalent to the second. Assume all variables represent statements.

(a) $(P \Rightarrow Q) \wedge P$, Q
(b) $(P \Rightarrow Q) \wedge {\sim}Q$, ${\sim}P$
(c) $P \Rightarrow Q) \wedge (P \Rightarrow R)$, $P \Rightarrow (Q \wedge R)$
(d) $(P \Rightarrow R) \wedge (Q \Rightarrow R)$, $(P \vee Q) \Rightarrow R$
(e) $P \Leftrightarrow Q$, ${\sim}P \Leftrightarrow {\sim}Q$

22. Suppose P and Q are logically equivalent statements, and, Q and R are logically equivalent statements. Must P and R be logically equivalent? Explain.

23. Suppose P, Q, R, S and T are statements. Show that

$$P \Rightarrow (Q \Rightarrow (R \Rightarrow (S \Rightarrow T)))$$

and

$$P \Rightarrow (Q \Rightarrow (R \Rightarrow (S \Rightarrow \sim T)))$$

are not logically equivalent.

1.3 Arguments

Symbolic logic not only is a necessary tool for deductive mathematics but it also introduces us to the world of proof via *arguments*. For example, suppose Katie and Luke are discussing the upcoming weekend's events.

Katie: I heard that if you go bowling tonight, you will get a coupon for free entrance to the zoo on Sunday.

Luke: Really? If I get a coupon like that, I always go.

Katie: You promised that if you ever go to the zoo you would buy me a souvenir.

Luke: I did, didn't I? Well, you know what, after giving it some though, I'm going to go bowling tonight.

If we assume all of these statements are true, should Katie expect a souvenir from the zoo? Of course. That's a logical conclusion to make from the given information. Luke said he was going to go to bowling, which in turn means he will receive the zoo coupon. His first statement, if assumed true, means he will use the coupon and consequently will result in his keeping of a promise a souvenir.

Symbolically, we could have represented this as follows.

B: Luke goes bowling tonight.

C: Luke receives a coupon for free entrance to the zoo on Sunday.

A: Luke uses his coupon for free entrance.

S: Luke buys Katie a souvenir.

Translating the conversation to symbols, we assume the following statements to be true: $B \Rightarrow C$, $C \Rightarrow A$, $A \Rightarrow S$, and B and the conclusion we draw is S.

Symbolically, that this is a "good" argument should seem somewhat natural. We assume the statement B to be true. Then, since B implies C is true, we can conclude that C must be true. By the same reasoning, since we have

$C \Rightarrow A$, the statement A must hold. Consequently, S must be true (since $A \Rightarrow S$ is assumed true). We formalize this idea by defining an *argument*.

Definition 1.18. An *argument* is a finite list of statements P_1, P_2, $\ldots$, P_{n-1}, called the *assumptions* (or *premises* or *hypotheses*) of the argument, and a single statement P_n, called the *conclusion* of the argument. We write such an argument in the following way. The symbol $\therefore$ is read as "therefore."

$$\begin{array}{l} P_1 \\ P_2 \\ \vdots \\ P_{n-1} \\ \hline \therefore P_n \end{array}$$

The argument in the example above "makes sense;" the conclusion seems to logically follow from the assumptions. But this does not need to be the case for every argument. An argument is simply defined as a finite list of assumptions and a single conclusion. There is no requirement of that conclusion to "follow" from the assumptions. For example, consider the following basic argument between two parents.

Parent A: I said that if Cory did the laundry, then he would get to use the car.

Parent B: Well, Cory has the car, so the laundry must be done.

Symbolically, if L represents Cory doing the laundry and C Cory getting to use the car, the argument is

$$\begin{array}{l} L \Rightarrow C \\ C \\ \hline \therefore L \end{array}$$

Being comfortable with implications, we know that this conclusion somehow "isn't right." Assuming both $P \Rightarrow Q$ and Q to be true does not mean P must be true. In fact, if P is false, then the statement $P \Rightarrow Q$ is true regardless of the truth value of Q.

This idea about "good" arguments and "poor" arguments is formalized by classifying arguments as either *valid* or *invalid*.

Definition 1.19. An argument is called *valid* if whenever every assumption of the argument is true, then the conclusion is also true. If there is a case when the assumptions are true but the conclusion is false, then the argument is called *invalid*.

This definition does more than just define valid and invalid arguments. It instructs us how to check an argument's validity. To do so, build a single truth table with a column containing every single assumption and the conclusion of the argument. Then, consider every row of the truth table where *all* assumptions are true. Is the conclusion true in every single case? If so, the argument is valid. If there is even just one row where all the assumptions are true yet the conclusion is false, then the argument is invalid. Let us exhibit this process by showing that the laundry/car example is an invalid argument.

Example 1.19. Show that the following argument is invalid.

$$\begin{array}{l} L \Rightarrow C \\ \underline{C \qquad} \\ \therefore L \end{array}$$

We build a truth table with a column for every assumption and the conclusion.

L	C	$L \Rightarrow C$	
T	T	T	
T	F	F	
F	T	T	$\leftarrow$
F	F	T	

The third row, as indicated by the arrow, shows that this argument is invalid. The assumptions are both true but the conclusion is false.

Using a truth table to show an argument is invalid requires finding a *single* row that satisfies certain conditions (every assumption is true and the conclusion is false). Showing an argument is valid often takes a bit more work.

Example 1.20. Show the argument below is valid.

$$\begin{array}{l} A \Rightarrow C \\ B \Leftrightarrow C \\ \underline{\sim A \qquad} \\ \therefore A \Rightarrow B \end{array}$$

We build a truth table with a column for each assumption and a column for the conclusion.

A	B	C	$A \Rightarrow C$	$B \Leftrightarrow C$	$\sim A$	$A \Rightarrow B$	
T	T	T	T	T	F	T	
T	T	F	F	F	F	T	
T	F	T	T	F	F	F	
T	F	F	F	T	F	F	
F	T	T	T	T	T	T	$\leftarrow$
F	T	F	T	F	T	T	
F	F	T	T	F	T	T	
F	F	F	T	T	T	T	$\leftarrow$

We must consider *every* row where *all* assumptions are true. For this argument there are two such rows to consider, as indicated by the arrows. Once we have identified these rows, we ask: is the conclusion also true in every one of these rows? Here, the conclusion, $A \Rightarrow B$, is true. Thus, the argument is valid.

It is important to note that by claiming an argument is valid, we are making a claim only about the *argument* rather than the individual statements within the argument. This is highlighted by the following two examples, both of which contain valid arguments.

Example 1.21. The following argument is valid. Showing it is valid is left as an exercise.

The moon is made of cheese.
If the moon is made of cheese, then the world is flat.
____
Therefore, the world is flat.

Example 1.22. Investigate the validity of the following argument.

$$\frac{P \wedge \sim P}{\therefore Q}$$

Consider the following truth table:

P	Q	$\sim P$	$P \wedge \sim P$
T	T	F	F
T	F	F	F
F	T	T	F
F	F	T	F

Notice that there are no rows where the hypothesis $P \wedge \sim P$ is true. In order to say that an argument is *invalid*, there must be a row of the truth table in which all the hypotheses are true and the conclusion is false. Since this does not exist, we say that the argument is valid *vacuously*.

Now imagine if we wanted to check the validity of the argument posed at the beginning of this section:

$$\begin{array}{l} B \Rightarrow C \\ C \Rightarrow A \\ A \Rightarrow S \\ \underline{B \qquad\quad} \\ \therefore S \end{array}$$

To do so, we would construct a truth table with $2^4 = 16$ rows. That is manageable, but imagine an argument with 10 variables. The truth table we would need to construct would require $2^{10} = 1024$ rows! Certainly there must be another method. When we reasoned through the "logic" of the above argument, we repeatedly used the same argument; if $P \Rightarrow Q$ and P are true, then Q must logically follow. If we can show that this argument is itself valid, for *any* statements P and Q, then we could call upon it to validate "mini-conclusions" from the larger argument above.

The method, called *logical deductions*, appears in the next section, but one of the main tools for them is what are called *rules of inference*.

Definition 1.20. A *rule of inference* is an argument that is valid.

While any argument that is valid could be considered to be a rule of inference, certain ones are natural and occur frequently in our everyday conversations. For example, the assumptions $P \Rightarrow Q$ and P lead to a conclusion understood by our aforementioned ice-creaming loving toddlers. A promise of "if you eat your broccoli, then you get ice cream" is understood to mean "ice cream will be served" once the broccoli is eaten.

There are a handful of other common arguments, each named as follows. P, Q and R are all statements.