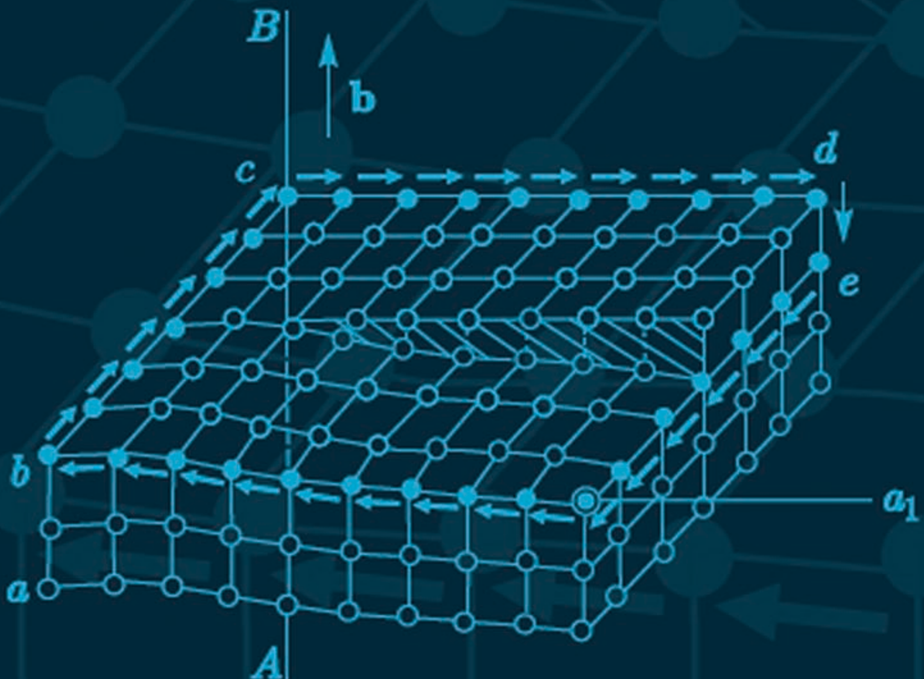
Physico-Mathematical Theory of High Irreversible Strains in Metals



V. M. GRESHNOV

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Contents

Foreword

In the monograph proposed for the first time, an attempt was made to systematically present the scientific results obtained in the study of a single physical process of irreversible deformation and brittle fracture of metals within the framework of the structural phenomenological approach.

The essence of this approach is the integration of macro- and micro-representations, methods of macro- and micro-description of the process. Practical realization of this association took place in the form of a new theory, the exposition of which is the goal of this book.

The theory has a deductive character, that is, it is built on its own postulates. The generalization of the equations of the theory, obtained initially for a uniaxial stressed state, to a volume stress-strain state required further development of the foundations of the classical mathematical theory of plasticity. Two theorems are formulated and proved which are generalizations of the Drucker postulate and the von Mises maximum principle of the classical mathematical theory of plasticity (flow theory) to a viscoplastic medium. Consequently, the theory includes as a special case the classical mathematical theory of plasticity, which follows from the general theory under the assumption that there is no thermodynamic return during deformation.

A special feature of the theory is its construction in finite increments, which made it possible immediately to bypass a number of problems of the classical theory, for example, the problem of dividing large deformations into elastic and irreversible components.

This feature of the theory, together with the presence of evolution equations for the actual structural parameters-the scalar dislocation and microcrack densities-provides the formulation and solution of practical problems in which nonlinear processes of large irreversible deformations occurring under conditions of a nonstationary stress-strain state and a temperature field are considered.

The physico-mathematical theory of strength and plasticity for the first time consistently takes into account the continuous change in the structure of materials during deformation and the accumulation of deformation damage.

The novelty of the theory, in our opinion, is that, for the first time, a scheme for describing plastic deformation and viscous destruction, evolution of structure, creep processes, long-term strength of metals and stress relaxation is proposed in the framework of a unified approach and a unified model.

It is not difficult to see that on this basis it is possible to further expand this scheme and include in it a forecasting model for the residual resource, a model for determining the mechanical characteristics of quasi-samples of standard mechanical properties in deformed semi-finished products by the method of mathematical modeling.

To apply the theory to the development and study of processes of irreversible deformation and viscous destruction, it is necessary to create a specialized software product of a new generation for computer engineering analysis. Existing means, for example *DEFORM-3D*, can not in general be used to implement the model of the deformation process formulated within the framework of the new theory. This is due to the fundamental differences in the algorithms for solving the applied problems of the classical mathematical theory of plasticity and the new theory.

Introduction

Among scientific workers, it is considered a bad practice to go beyond the scope of one's research (one's science) with an attempt to 'shift' another scientific discipline. There are good reasons for this. The modern level of development of scientific disciplines, especially fundamental ones, in which mathematical methods of description and research are widely used, is very high. In order to master the results of the long-term development of a particular science, which is a prerequisite for bold attempts to contribute to its further development, it takes a lot of time and labour. At the same time, as a rule, each scientific discipline has its own characteristic way of thinking, connected with its methods.

Therefore, scientists are sometimes skeptical of attempts by their colleagues to contribute to the development of 'not their own' science.

On the other hand, it is well known that scientific research conducted at the junction of two related disciplines is often the most effective and is accompanied by a combination of scientific directions.

The process of combining disciplines, along with their differentiation, is an objective law of the development of science. A vivid example is the creation of statistical physics, which is a synthesis of molecular–kinetic teaching and thermodynamics. This example demonstrates a synthetic approach to the development of a scientific problem – the unification of the phenomenological and statistical approaches to the study and description of phenomena and processes in macroscopic systems.

The fundamental scientific discipline – the mechanics of a deformable solid (MDS) consists of four sections: theories of elasticity, plasticity, creep, fracture mechanics of materials and their applications. At present, MDS is a highly developed science with a powerful mathematical apparatus that allows solving a wide range of applied problems related to the design of various designs and technologies for processing materials. The applied value of the theory in recent years has increased significantly in connection with the intensive development of computational mechanics and computers.

The main applied task of all four sections of the discipline is the calculation of stress fields, strains, strain rates and deformation damage of solids of arbitrary geometric shape loaded with an arbitrary system of external forces. Information about these fields is obtained as a solution of the initial-boundary value problem of mathematical physics.

Since deformation of continuous solids is one of the forms of their mechanical motion, the mathematical formulation of the initial-boundary problem includes differential equations of motion and kinematic relations, which are closed by the defining equations. The latter are models of deformable solids: models of elasticity, plasticity, creep, material with damage. Therefore, the task of MDs is the development of these models – the laws of deformation of solids.

For centuries and now, despite the creation and increasing use of ceramics, plastics, glasses, composites, the main structural material in all branches of engineering industry are metals that have a unique combination of strength, elasticity, ductility and viscosity. Therefore, specialists of MDS pay great attention to the development of models of metallic materials. The subject of the study in this work are metals and their alloys.

MDS, like general mechanics, is a phenomenological science. The first hypothesis in the phenomenological approach to the study of the motion of matter is the hypothesis of a continuous medium, i.e., as a material model, a continuous deformable medium is assumed. However, it is well known that the deformation behaviour of materials (the properties of materials) is determined by their internal structure. In this case, during deformation, the structure changes significantly, and, consequently, at every stage of deformation we are dealing, generally speaking, with a new material.

The failure to take into account the structure of the material and its evolution under deformation is the cause of problems that have been clearly formulated by the second half of the last century, which have held back the development of MDS for years. These problems are associated with the non-linearity of the processes of large irreversible strains, taking into account the history of deformation and the evolution of the structure of the material. In more detail, these problems are formulated in the first and second chapters of the book, on the content and current state of the mathematical theories of plasticity, creep, and ductile fracture of metals. Numerous attempts to solve these problems, undertaken for decades within the framework of a purely phenomenological approach, have so far not yielded meaningful results.

Irreversible deformation and destruction of metals are a subject of investigation of another fundamental discipline – the physics of strength and plasticity, which is a section of the physics of solids. This discipline uses microstructural and statistical approaches to the study of the motion of matter. It uses a discrete atomic model of the material, studies and takes into account the evolution of the structure under deformation. The laws of deformation of materials are established in the form of some model representations, based on the analysis of micromechanisms of processes occurring in the atomic model of a material when it is loaded by external forces.

The physics of strength and plasticity develops uniaxial deformation laws. It lacks methods of obtaining generalized laws necessary for solving practical problems. The desire to construct a more accurate physical deformation model determines the presence in the equations of a large number of parameters, the physical meaning of which often remains unclear. This makes it practically impossible to use physical models to calculate real processes.

The natural direction of the further development of MDS and the physics of strength and plasticity is the unification of micro- and macro-representations about irreversible deformations and fractures, methods of micro- and macro-description of these physical processes. Therefore, in spite of the partial division of thoughts formulated at the beginning of this introduction, the author, with a small group of assistants (undergraduates and graduate students), began systematic research on the development of the formulated direction in the early 1990s. The results obtained over the past years are generalized and presented to the reader in this monograph.

Taking into account the orientation of the book, the author found it necessary to preface a new material – the physical and mathematical theory of irreversible strainss of metals – with an exposition of the elementary foundations of mechanics and the physics of strength and plasticity of metals, believing that this should help mechanical engineers and physicists get acquainted with its content.

Section I

Current State of Mechanics and Physics of Strength and Plasticity of Metals

Fundamentals of mechanics of strength and plasticity of metals

1.1. Basic concepts, postulates and method in the classical mathematical theory of plasticity (flow theory)

Metals and metallic alloys are the main structural materials in engineering. This is due to a rational combination of the characteristics of strength, elasticity, ductility and toughness. The ductility of metals is essential.

The physical process of plastic deformation underlies one of the oldest methods of metalworking – plastic forming of metals. The ability to plastic deformation determines the performance of metal structures, including parts of machines and mechanisms. The scientific basis for the design of metal structures and machine components, as well as the technological processes of plastic shaping (forging and stamping), along with theories of elasticity and creep, is the theory of plasticity.

The applied task of these scientific disciplines is the calculation of the parameters of the above objects, ensuring their optimal or rational functioning. This problem is solved by determining the stresses and strains in the workpiece being machined, the structure, the part.
It is believed that the theory of plasticity originates from the work of Saint-Venant, published in 1871 [1]. The main stages in the formation and development of the theory of plasticity, as well as biographical information about scientists who contributed to the development of the theory, are given in the interesting concluding section 'A Brief Historical Reference on the Chapters' of the widely known textbook on the mechanics of plastic forming of metals by Prof. V.L. Kolmogorov [2].

The object of studying the theory of plasticity as a fundamental scientific discipline is a special form of mechanical motion of deformable solids – plastic deformation and its models. This form of motion is not considered by theoretical mechanics, in which the model of a solid non-deformable body is adopted [3]. The classical mathematical theory of plasticity is, like mechanics in general, a purely phenomenological discipline, has its own axiomatics, that is, the theory is built on the basis of its own postulates (principles). The construction is carried out within the framework of the classical mechanics of Galileo–Newton and the paradigm that is unified with analytical mechanics.

The main concepts are [4]: *physical space, time, mass, force* (primary concepts in mechanics), *region V of physical space with boundary S, continuous medium, elementary volume (material particle); displacement vector, velocity and acceleration of particle motion, kinetic, internal and free energy, entropy and particle temperature; tensors of deformations, strain rates and stresses, loading processes in the stress space* $\sigma_{ij}(t)$ *and deformation in the strain space* $\varepsilon_{ij}(t)$, *where* σ_{ij} *is the stress tensor,* ε_{ij} *is the strain tensor, and t is the time.*

Thermodynamic concepts and relations are introduced as applied to a particle of a continuous medium. The particle is identified with a mathematical point.

The loading process is called simple if the stress state of the particle changes over time in such a way that the end of the stress vector in the six-dimensional image space of the symmetrical stress tensor moves along the ray emanating from the origin, i.e., the straight line is the path of the stress vector. With other trajectories (curvilinear, broken) – the loading is complex. Cyclic loading is characterized by a periodic change in the sign of the stress, can be simple and complex.

Definitions of the basic concepts are given *a priori*. Therefore, the relations between their quantitative measures (mathematical objects) derived on the basis of certain initial postulates can acquire the meaning of substantive (physical) laws only under the condition of experimental justification of the derivations derived or consequences arising from them.

Initial postulates: *macroscopic continuity, homogeneity and isotropy of the deformed body, homogeneity of stress and strain states in an elementary (representative) volume* ΔV *(material particle)* in which all the vector and scalar quantities characterizing the thermomechanical state are average (integral) in the sense of the averages in statistical physics, *macrophysical definability* [5]*, fluidity conditions, loading functions, a single curve.*

The basic applied problem of the theory of plasticity by definition of the stress–strain state of a body, which in general has an arbitrary shape and is loaded with an arbitrary system of external forces, is solved by setting the *initial-boundary value problem of plasticity theory* [2]. It includes the following equations.

1. Equilibrium equations, into which the equations of motion of a continuous medium (Newton's second law) are transformed, if mass forces are neglected:

$$\sigma_{ij,j} = 0^{1)} i, j = x, y, z. \tag{1.1}$$

2. Geometric Cauchy relations for small strains:

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{1.2}$$

where $\varepsilon_{ij} = \varepsilon_{ij}^{e} + \varepsilon_{ij}^{p}$, ε_{ij}^{e}, ε_{ij}^{p} are the elastic and plastic components of the strain tensor; u_i are the projections onto the coordinate axes of the displacement vector of the particle. The strains must satisfy the compatibility conditions [2]

$$\frac{\partial^2 \varepsilon_{ij}}{\partial x_{kl}} + \frac{\partial^2 \varepsilon_{kl}}{\partial x_i \partial x_j} = \frac{\partial^2 \varepsilon_{il}}{\partial x_k \partial x_j} + \frac{\partial^2 \varepsilon_{kj}}{\partial x_i \partial x_l}.$$

3. The equation of heat conductivity

$$c\rho\frac{dT}{dt}=(\lambda T_{,i})_{,i}+\sigma_{ij}\dot{\varepsilon}_{ij}, \tag{1.3}$$

where c, λ are the coefficients of heat capacity and thermal conductivity of the material being deformed; ρ is its density; T is the thermodynamic temperature; $\dot{\varepsilon}_{ij}=d\varepsilon_{ij}/dt$ is the plastic strain rate tensor.

The paper deals with the theory of plastic deformation of compact metallic materials. If the incompressibility condition $\varepsilon_x + \varepsilon_y + \varepsilon_z = 0$ and ρ = const is adopted, then the strain tensors and strain rates coincide with their deviators.

When the boundary-value problem is formulated in the velocities, equations (1.2) are replaced by the Saint-Venant geometric relations for the strain rates

$$\dot{\varepsilon}_{ij}=\frac{1}{2}(\upsilon_{i,j}+\upsilon_{j,i}), \tag{1.4}$$

where υ_i are the projections onto the coordinate axes of the velocity vector of the material particle.

The system of ten differential equations (1.1)–(1.3) contains 16 unknown (sought) functions of coordinates and time: $\sigma_{ij}(x_i, t)$, $\varepsilon_{ij}(x_i, t)$, $T(x_i, t)$, $u_i(x_i, t)$. It is closed by the *defining equations describing* the relationship of stresses to strains (or strain rates) and the mechanical properties of the material. To obtain a particular solution, the system is supplemented by boundary conditions.

The defining relations are a mathematical model of the plastic deformation of a material (the law of deformation). Therefore, one of the main tasks of the mathematical theory of plasticity as a fundamental discipline is the establishment of laws of deformation of materials in various thermomechanical conditions and structural states.

The most general formulation of the laws of deformation is given on the basis of certain extreme principles [6–8]. It is assumed that the loading process (the change in the state of the elementary volume of the deformed medium) can be described by a finite set of pairs of parameters σ_{ij}, T, ε_{ij}^p, q, k, where q is the hardening parameter, which is usually taken to be the plastic strain, referred to the unit volume, $q=A=\int\sigma_{ij}d\varepsilon_{ij}^p$, or the plastic deformation intensity accumulated by the particle $q=\int d\varepsilon^p$ ($d\varepsilon^p$ is the intensity of the increment of plastic deformations); k is the parameter associated with the yield strength of the material. Time implicitly enters through q.

Since the temperature T is a scalar quantity, it is excluded from the number of determining parameters and is taken into account by the dependence $k(T)$, while, as a rule, an isothermal deformation process is considered. A mixed problem *can also be posed, including the mechanical and thermal problems*. The basis for the formulation and solution of the second problem is equation (1.3).

It is postulated that the current (actual) thermomechanical state of the elementary volume of a body under loading by its system of external forces can be described in a six-dimensional stress space by a certain function of the determining parameters

$$f(\sigma_{ij}, \varepsilon_{ij}^{p}, q, k) = 0, \tag{1.5}$$

which is called the *loading function*. It is also assumed that for small strains, the total strains can be represented as the sum of reversible (elastic) and irreversible (plastic) components. For fixed parameters, the function (1.5) describes a hypersurface in the six-dimensional space of a symmetric bivalent stress tensor σ_{ij} (stress space). If the conditions for the commencement of the Huber–von Mises fluidity and the independence of f from the first and third invariants of the stress tensor are accepted, then the loading function takes a particular and specific form

$$f = \frac{3}{2} s_{ij} s_{ij} - \sigma_T^2 = 0, \tag{1.6}$$

where σ_T is the yield strength of the isotropic material.

The surface described by (1.6) is called the *surface of the beginning of plasticity*, when $\varepsilon_{ij}^{p} = 0$, $q = 0$. In the case of an elastoplastic material, it separates the region of elastic stress states, where the stresses and strains are related by Hooke's law (inside the area bounded by the surface) from the stress states at which the elementary volume passes into a plastic state (the stressed states are on the surface). Consequently, in the elastic region $f < 0$. For a rigid-plastic body in the inner region, bounded by the hypersurface of plasticity (1.6), the material is absolutely rigid. When the hardening material is deformed in (1.6), instead of σ_T, according to the Huber-von Mises plasticity condition, the current stress intensity σ.

With hardening σ, the surface of plasticity, which in this case is called the *loading surface*, also increases and changes. The change in the loading surface describes the hardening of the material to be deformed. In the general case of deformation of a material, it can change the shape and position in the space σ_{ij}. In an ideal plastic material (σ_T = const), the loading surface does not change (fixed) during deformation and is called the *yield surface*.

The following properties of the loading surface are postulated: it is closed, but in some directions it can extend to infinity; does not pass through the origin; any ray drawn from the origin crosses it only once, that is, it does not have concave sections.

The loading surface, or part of it, at each point of which there is a single outward normal and, therefore, it is differentiable with respect to σ_{ij}, is said to be *smooth*, and the loading points on it are *regular*.

The concepts of *loading*, *unloading* and *neutral loading* for regular points are introduced. If the stress state of the material particle described by the six-dimensional vector σ_{ij} belongs to the loading surface Σ (hence, the plasticity condition (1.6) is satisfied) and the loading vector $d\sigma_{ij}$ is directed outwards, this process is accompanied by an increase in the plastic deformation of the particle $d\varepsilon_{ij}^p > 0$, the change of the loading surface (in the case of the hardening body it assumes the position Σ') and is called the *loading* (Fig. 1.1). After loading, the stressed state of the particle is described by the vector σ'_{ij} (shown by the dashed line).

With the direction of the vector $d\sigma_{ij}$ inside Σ, the resulting stress state (point A in Fig. 1.1) is in the elastic region. In this ij case, $d\varepsilon_{ij}^p = 0$, the stresses and strains are related by the generalized Hooke's law, and we are dealing with the *unloading process*. The loading process is called *neutral* if $d\sigma_{ij}$ is directed along the tangent

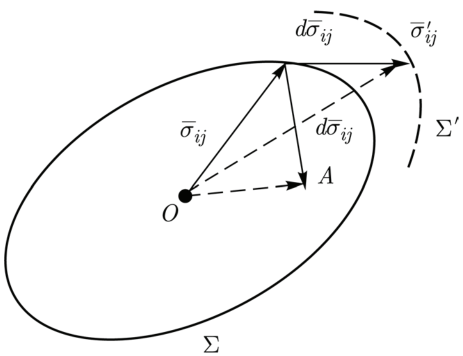

Fig. 1.1. Definition of the loading and unloading processes of an elastoplastic hardening body.

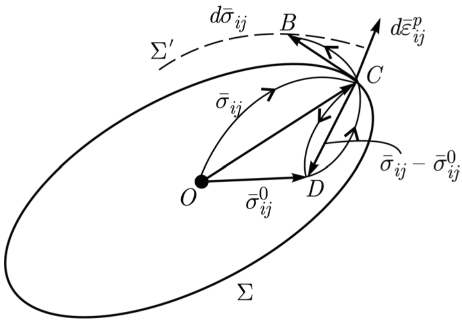

Fig. 1.2. Postulated loading and unloading cycle of the hardening elastoplastic body, on the basis of which the principle of maximum work of plastic deformation is formulated.

to the surface Σ. In the case of a hardening body $d\varepsilon^{p}_{ij} = 0$ and stresses are related with the strains also related by Hooke's law.

The basis for constructing the plasticity model of a hardening body is the Drucker postulate (Drucker, D.C.) [6], which states that the *work of additional stresses on the increments of plastic deformations caused by them during the loading and unloading cycle is positive. In the stress space* we consider the cycle shown in Fig. 1.2. The material particle from the original natural state, in which there are no stresses and deformations in the particle (the point *O* is the origin), is loaded along a certain path to the yield point (point *C*) and unloaded to point *D*. This state is accepted in the argument for the original. In this state, the particle is loaded to the point *C*, is loaded with the vector $d\boldsymbol{\sigma}_{ij}$ (point *B*) and unloaded to the initial state (point *D*). In this case, the loading and unloading paths are arbitrary and lie in the elastic region except for the addition loading by the vector $d\sigma_{ij}$.

From the considered cycle, taking into account the fact that the work of additional stresses on reversible elastic strains under conditions of a closed deformation path is zero, according to the postulate it follows that

$$\left[\left(\sigma_{ij} - \sigma^{0}_{ij}\right) + d\sigma_{ij}\right] d\varepsilon^{p}_{ij} > 0. \tag{1.7}$$

If we take the point *O* as the initial and final states, that is, $\sigma^{0}_{ij} = 0$ then (1.7) takes the form

$$\left(\sigma_{ij} + d\sigma_{ij}\right) d\varepsilon_{ij}^{p} > 0. \tag{1.7a}$$

If the initial state is taken as the state at the yield point (point C in Fig. 1.2), when $\sigma_{ij}^{0} = \sigma_{ij}$, then it follows from (1.7) that

$$d\sigma_{ij} d\varepsilon_{ij}^{p} > 0. \tag{1.8}$$

This expression is considered as a condition for stable deformation beyond the elastic limit of a hardening elastoplastic body in the general case for a volume stress–strain state. We note at once that inequality (1.8) imposes a restriction on the possibility of describing plastic deformation. It excludes from consideration materials with an incident (non-monotonic) deformation diagram when $d\sigma/d\varepsilon^{p} < 0$. Therefore, the description of the deformation of materials with a non-monotonic diagram is one of the problems of the classical theory.

If $\sigma_{ij}^{0} \neq \sigma_{ij}$, then the difference $\sigma_{ij} - \sigma_{ij}^{0}$ can be arbitrarily greater than $d\sigma_{ij}$, and then

$$\left(\sigma_{ij} - \sigma_{ij}^{0}\right) d\varepsilon_{ij}^{p} > 0. \tag{1.9}$$

From this inequality follows the *principle of the maximum work of plastic deformation* for a hardening plastic body:

$$\sigma_{ij} d\varepsilon_{ij}^{p} > \sigma_{ij}^{0} d\varepsilon_{ij}^{p}. \tag{1.10}$$

For any given value of the components of the plastic deformation increment, the increment of the plastic deformation work $\sigma_{ij} d\varepsilon_{ij}^{p}$ *has a maximum value for the actual stress state* σ_{ij} *in comparison with all possible stress states* σ_{ij}^{0} *satisfying the condition* $f(\sigma_{ij}) < 0$.

Like other differential principles of mechanics, the principle of the maximum work of plastic deformation allows in this case to separate the true stress states from all possible ones in deformation of the body.

The increment in the work of plastic deformation is a function of the stress components. The latter are not independent arguments, since during plastic deformation they must simultaneously satisfy the plasticity condition (1.6). Therefore, the maximum of the function of increment of the work of plastic deformation, declared by the above-stated principle, is conditional. In mathematics, several methods for solving problems on the conditional extremum of a function of several variables have been developed. In the theory of plasticity, the condition for the maximum of the function $dA = \sigma_{ij} d\varepsilon_{ij}^{p}$, in the presence of the coupling equation $f(\sigma_{ij}) = 0$ (plasticity condition), is written using the Lagrange multiplier method [9] as

$$\frac{\partial}{\partial \sigma_{ij}}\left(\sigma_{ij} d\varepsilon_{ij}^{p} - d\lambda f\right) = 0, \tag{1.11}$$

where $d\lambda$ is the Lagrange multiplier.

After differentiation, an equation is obtained which is one of the vertices of the mathematical theory of plasticity and is called the *associated flow law (with the plasticity condition)*:

$$d\varepsilon_{ij}^{p} = d\lambda \frac{\partial f}{\partial \sigma_{ij}}. \tag{1.12}$$

It follows from (1.12), firstly, that the loading function f is a *plastic potential* in the stress space and, secondly, that the vector $d\varepsilon_{ij}^{p}$ is directed along the normal to the loading surface, since $\dfrac{\partial f}{\partial \sigma_{ij}} = \mathbf{n}$ ($\mathbf{n}$ is the unit vector of the normal to the surface f), and $d\lambda$ is a scalar.

It is proved that if the first invariant of the stress tensor is included explicitly or implicitly as arguments in f, then the plastic deformation proceeding according to the law (1.12) satisfies the incompressibility condition $d\varepsilon_{ii}^{p} = 3d\varepsilon_{0} = 0$, that is, the tensor $d\varepsilon_{ij}^{p}$ coincides with its deviator.

Substitution of the components of the increment of plastic deformations (1.12) into the expression for the intensity of the increment of plastic strains

$$d\varepsilon^{p} = \left(2/3 d\varepsilon_{ij}^{p} d\varepsilon_{ij}^{p}\right)^{1/2}$$

the value of $d\lambda$ is defined as

$$d\lambda = \sqrt{\frac{3}{2}} d\bar{\varepsilon}^{p} \Bigg/ \left(\frac{\partial f}{\partial \sigma_{ij}} \cdot \frac{\partial f}{\partial \sigma_{ij}}\right)^{\frac{1}{2}}. \tag{1.13}$$

Taking into account the decomposition of the stress tensor into a deviator and spherical tensors:

$$\sigma_{ij} = s_{ij} + \delta_{ij}\left(\frac{1}{3}\delta_{kl}\sigma_{kl}\right),$$

$$s_{ij} = \sigma_{ij} - \delta_{ij}\left(\frac{1}{3}\delta_{kl}\sigma_{kl}\right),$$

the derivative of the loading function with respect to the components of the tensor σ_{ij} is replaced by the derivative with respect to the components of the stress deviator s_{ij}:

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial s_{ij}}\frac{\partial s_{ij}}{\partial \sigma_{ij}} = \frac{\partial f}{\partial s_{ij}} - \delta_{ij}\left(\frac{1}{3}\delta_{kl}\frac{\partial f}{\partial s_{kl}}\right), \tag{1.14}$$

where δ_{ij} is the Kronecker symbol ($\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$).

The equation of the associated flow law (1.12) takes the form

$$d\varepsilon_{ij}^{p} = d\lambda\left(\frac{\partial f}{\partial s_{ij}} - \frac{1}{3}\delta_{ij}\delta_{kl}\frac{\partial f}{\partial s_{kl}}\right). \tag{1.15}$$

If a certain loading function f is adopted for a particular material, then (1.15) are the equations (law) of its plastic deformation in the general case of a volume stress–strain state.

Instead of the principle (1.10), for the formulation of the plasticity model of a hardening body in stress space one can also use the principle of the maximum dissipation rate of mechanical work $D = \sigma_{ij} \cdot \dot{\varepsilon}_{ij}^{p}$. *For fixed parameters* ε_{ij}^{p}, q *of the loading function f, for any given value of the velocity components of the deformation* $\dot{\varepsilon}_{ij}^{p}$, *we have the inequality*

$$\sigma_{ij}\dot{\varepsilon}_{ij}^{p} > \sigma_{ij}^{0}\dot{\varepsilon}_{ij}^{p}, \tag{1.16}$$

where σ_{ij} *are the real values of the stress components corresponding to a given value* $\dot{\varepsilon}_{ij}^{p}$; σ_{ij}^{0} *are the components of any possible stress state allowed by the given loading function* $f\left(\sigma_{ij}^{0}, \varepsilon_{ij}^{p}, q, k\right) < 0$.

This is the *von Mises maximum principle*. It can be seen that (1.16) can be obtained by differentiating (1.10) with respect to time at constant stresses. In this case, the associated flow law will be written as

$$\dot{\varepsilon}_{ij}^{p} = \mu^{0}\frac{\partial f}{\partial \sigma_{ij}}, \tag{1.17}$$

where $\mu^{0} = \sqrt{\dfrac{\dot{\varepsilon}_{hk}^{p}\dot{\varepsilon}_{hk}^{p}}{\partial f/\partial\sigma_{mn}\,\partial f/\partial\sigma_{mn}}} = \dfrac{d\lambda}{dt}$.

The construction of a model of a hardening plastic body can also be based on the definition of a *dissipative function* [6]

$$D = \sigma_{ij}\dot{\varepsilon}_{ij} = D(\dot{\varepsilon}_{ij}, \varepsilon_{ij}, q, k), \tag{1.18}$$

which in the six-dimensional space of the symmetric strain rate tensor for fixed parameters ε_{ij}, q and k describes the *surface of an equal level of dissipation of mechanical work per unit volume per unit time*. In this case, for fixed parameters ε_{ij}, q, along with the real values $\dot{\varepsilon}_{ij}$, we introduce into consideration possible $\dot{\varepsilon}_{ij}^{0}$, for which

$$D\left(\dot{\varepsilon}_{ij}^{0}, \varepsilon_{ij}, q, k\right) \le D\left(\dot{\varepsilon}_{ij}, \varepsilon_{ij}, q, k\right). \tag{1.19}$$

Similarly to the von Mises maximum principle (1.16), *we formulate the principle of the maximum dissipation rate* in the strain rate space – the *Ziegler principle*:

$$\sigma_{ij}\dot{\varepsilon}_{ij} \ge \sigma_{ij}\dot{\varepsilon}_{ij}^{0}. \tag{1.20}$$

The associated flow law in this case has the form

$$\sigma_{ij} = \lambda\frac{\partial D}{\partial \dot{\varepsilon}_{ij}}, \quad \lambda = D\Bigg/\left(\frac{\partial D}{\partial \dot{\varepsilon}_{ij}}\dot{\varepsilon}_{ij}\right). \tag{1.21}$$

Specific material models, as in the case of determining the loading function *f*, are determined by the assumption of the structure of the function *D*. Using the defining relations of the form (1.21), the boundary problem of plasticity is posed and solved at velocities.

Thus, the plastic body model is introduced in two ways: either through the definition of the loading function *f*, or through the definition of the dissipative function *D*. In both cases, the corresponding extreme principles – the principles of maximum – are formulated.

It is important to note that the indicated paths are equivalent, since the definition of the function D is possible if the model of the plastic body is given by the relations (1.5), (1.17) [8].

In conclusion of a brief review of the basics of the mathematical theory of plasticity, let us draw the reader's attention to the following fact which has an epistemological significance. There are clear parallels between the method of constructing Lagrange's analytical mechanics, which deals with the study of general laws of motion of systems of solids, and the method of plastic deformation mechanics (the mathematical theory of plasticity), which deals with the study of general laws of deformation of deformable bodies: the Lagrange function is a loading function; the principle of least action, for example Hamilton – the principle of maximum work of plastic deformation; the Lagrange motion equations are the equations of the associated flow law. These parallels clearly indicate a certain unity of the methodology of mechanics as a whole as a phenomenological science having a deductive character.

The material presented is the basis of one direction of the general mathematical theory of plasticity, called the *flow theory*. A second trend is developing – *the theory of processes* [10], the founder of which is one of the most outstanding mechanics scientists of the 20th century, A.A. Il'yushin [5]. The main difference between the theory of processes and the flow theory lies in the method of geometric interpretation of the deformation process and in the method of constructing the defining relations. The account and analysis of the theory of processes is not included in the tasks of this book.

Particular models of the plasticity theory are a consequence of different formulations of the plasticity condition and, respectively, the loading functions.

Let us consider in retrospect some of these models that have great practical importance.

1.2. The defining relations of the theory of plasticity (particular laws of metal deformation)

1.2.1. The tensor defining relations

Widely used in calculations and mathematical modelling, including technological shaping operations of pressure metal working (PMW), are the defining relationships of the *theory of plasticity of isotropic material with isotropic hardening*.

In this theory, the Huber–von Mises plasticity condition and, accordingly, the loading function are taken in the form

$$f(\sigma_{ij}) = \frac{3}{2} s_{ij} s_{ij} - \left[\Phi(q)\right]^2 = 0, \tag{1.22}$$

where q is the Udquist hardening parameter, $q = \int d\varepsilon^p$ (integration is carried out along the strain path).

The differentiation of the function (1.22) with respect to the components s_{ij} and multiplication by δ_{ij} leads to the result

$$\delta_{ij} \frac{\partial f}{\partial s_{ij}} = 3\delta_{ij} s_{ij} = 0.$$

Then, substituting (1.22) into the associated flow law (1.15), we obtain

$$d\varepsilon_{ij}^p = d\lambda \frac{\partial f}{\partial s_{ij}} = 3d\lambda s_{ij}. \tag{1.23}$$

A comparison of (1.23) with (1.12) yields

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial s_{ij}} = 3s_{ij}. \tag{1.24}$$

Thus, in the case of a loading function of the form (1.22), (1.24) holds and, according to (1.13), the Lagrange multiplier is defined as

$$d\lambda = \frac{1}{2} \frac{d\varepsilon^p}{\sigma}, \tag{1.25}$$

where $\sigma = \sqrt{\frac{3}{2} s_{ij} s_{ij}}$ is the stress intensity.

The substitution of (1.24) and (1.25) into the equation of the associated flow law (1.12) gives the defining equations of the isotropic flow theory, having the form

$$d\varepsilon_{ij}^p = \frac{3}{2} \frac{d\varepsilon^p}{\sigma} s_{ij}. \tag{1.26}$$

Characteristics of the mechanical properties of a particular material enter the flow law (1.26) by means of the ratio $d\varepsilon^p/\sigma$, which, in fact, is a phenomenological coefficient in the phenomenological equation