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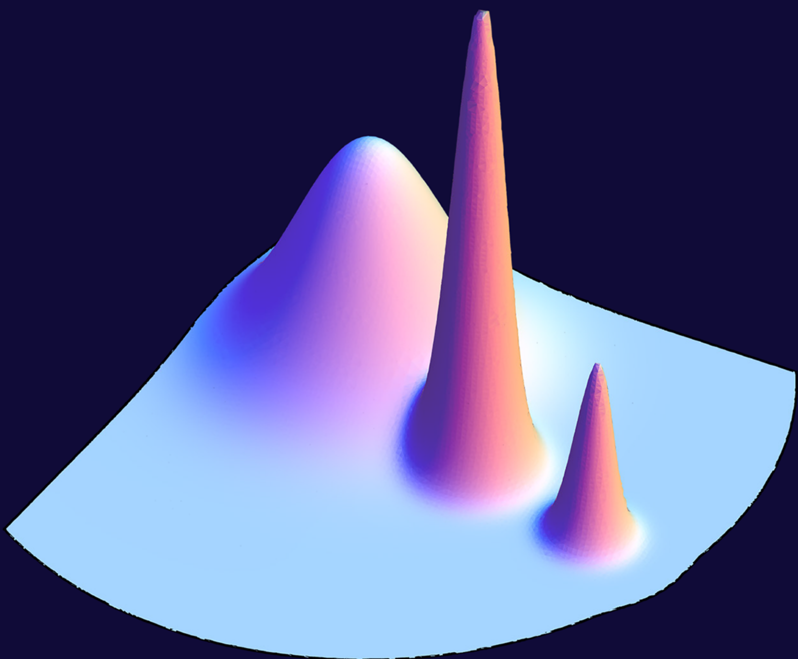

Samia Challal

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Introduction to the Theory of Optimization in Euclidean Space

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Introduction to the Theory of Optimization in Euclidean Space

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Contents

Preface

The book is intended to provide students with a useful background in optimization in Euclidean space. Its primary goal is to demystify the theoretical aspect of the subject.

In presenting the material, we refer first to the intuitive idea in one dimension, then make the jump to n dimension as naturally as possible. This approach allows the reader to focus on understanding the idea, skip the proofs for later and learn to apply the theorems through examples and solving problems. A detailed solution follows each problem constituting an image and a deepening of the theory. These solved problems provide a repetition of the basic principles, an update on some difficult concepts and a further development of some ideas.

Students are taken progressively through the development of the proofs where they have the occasion to practice tools of differentiation (Chain rule, Taylor formula) for functions of several variables in abstract situation. They learn to apply important results established in advanced Algebra and Analysis courses, like, Farkas-Minkowski Lemma, the implicit function theorem and the extreme value theorem.

The book starts, in Chapter 1, with a short introduction to mathematical modeling leading to formulation of optimization problems. Each formulation involves a function and a set of points. Thus, basic properties of open, closed, convex subsets of $\mathbb{R}^n$ are discussed. Then, usual topics of differential calculus for functions of several variables are reminded.

In the following chapters, the study is devoted to the optimisation of a function of several variables f over a subset S of $\mathbb{R}^n$. Depending on the particularity of this set, three situations are identified. In Chapter 2, the set S has a nonempty interior; in Chapter 3, S is described by an equation $g(x) = 0$ and in Chapter 4 by an inequality $g(x) \leqslant 0$ where g is a function of several variables. In each case, we try to answer the following questions:

– If the extreme point exists, then where is it located in S? Here, we look for necessary conditions to have candidate points for optimality. We make the distinction between local and global points.

– Among the local candidate points, which of them are local maximum or local minimum points? Here, we establish sufficient conditions to identify a local candidate point as an extreme point.

– Now, among the local extreme points found, which ones are global extreme points? Here, the convexity/concavity property intervenes for a positive answer.

Finally, we explore how the extreme value of the objective function f is affected when some parameters involved in the definition of the functions f or g change slightly.

Acknowledgments

I am very grateful to my colleagues David Spring, Mario Roy and Alexander Nenashev for introducing the course on optimization, for the first time, to our math program and giving me the opportunity to teach it. I, especially, thank Professor Vincent Hildebrand, Chair of the Economics Department for the useful discussions during the planning of the course content to support students majoring in Economics.

My thanks are also due to Sarfraz Khan and Callum Fraser from Taylor and Francis Group, to the reviewers for their invaluable help, and to Shashi Kumar for the expert technical support.

I have relied on the various authors cited in the bibliography, and I am grateful to all of them. Many exercises are drawn or adapted from the cited references for their aptitude to reinforce the understanding of the material.

Symbol Description

$\forall$	For all, or for each
$\exists$	There exists
$\exists!$	There exists a unique
$\emptyset$	The empty set
s.t	Subject to
$\overset{\circ}{S}$	Interior of the set S
∂S	Closure of the set S
$\overline{S}$	Boundary of the set S
C^S	The complement of S.
i, j, k	$i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$ standard basis of $\mathbb{R}^3$
$B_r(x_0)$	Ball centered at x_0 with radius r
$\mathbb{B}_r(x_0)$	Bordered Hessian of order r at x_0.
$\langle\, .\, ,\, .\, \rangle$	or $[\, .\, ,\, .\,]$ brackets for vectors
∇f	gradient of f
x^*	$= \begin{bmatrix} x_1^* \\ \vdots \\ x_n^* \end{bmatrix}$ column vector identified sometimes to the point $(x_1^*, \ldots, x_n^*)$
$\lVert x \rVert$	$= \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$ norm of the vector x
$M_{m\,n}$	set of matrices of m rows and n columns
A	$= (a_{ij})_{\substack{i=1,\ldots,m,\\ j=1,\ldots,n}}$ is an $m \times n$ matrix
$\lVert A \rVert$	$= \Big(\sum_{i,j=1}^{n} a_{ij}^2\Big)^{1/2}$ norm of the matrix $A = (a_{ij})_{i,j=1,\ldots,n}$
$rankA$	rank of the matrix A
$detA$	determinant of the matrix A
$KerA$	$= \{x : Ax = 0\}$ Kernel of the matrix A
${}^t h$	$= \begin{bmatrix} h_1 & \ldots & h_n \end{bmatrix}$ transpose of $h = \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}$
${}^t h.x^*$	$= \sum_{k=1}^{n} h_k.x_k$ dot product of the vectors h and x^*
$C^1(D)$	set of continuously differentiable functions on D
$C^k(D)$	set of continuously differentiable functions on D up to the order k
$C^\infty(D)$	set of continuously differentiable functions on D for any order k
$H_f(x)$	$= (f_{x_i x_j})_{n \times n}$ Hessian of f
$D_k(x)$	$= \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} & \cdots & f_{x_1x_k} \\ f_{x_2x_1} & f_{x_2x_2} & \cdots & f_{x_2x_k} \\ \vdots & \vdots & \vdots & \vdots \\ f_{x_kx_1} & f_{x_kx_2} & \cdots & f_{x_kx_k} \end{vmatrix}$ leading minor of order k of the Hessian H_f

Author

Samia Challal is an assistant professor of Mathematics at Glendon College, the bilingual campus of York University. Her research interests include homogenization, optimization, free boundary problems, partial differential equations and problems arising from mechanics.

Chapter 1

Introduction

Optimization problems arise in different domains. In Section 1.1 of this chapter, we introduce some applications and learn how to model a situation as an optimization problem.

The points where an optimal quantity is attained are looked for in subsets that can be one dimensional, multi-dimensional, open, closed, bounded or unbounded, ... etc. We devote Section 1.2 to study some topological properties of such subsets of $\mathbb{R}^n$.

Finally, since, the phenomena analyzed are often complex, because of the many parameters that are involved, this requires an introduction to functions of several variables that we study in Section 1.3.

1.1 Formulation of Some Optimization Problems

The purpose of this short section is to show, through some examples, the main elements involved in an optimization problem.

Example 1. ***Different ways in modeling a problem.***

To minimize the material in manufacturing a closed can with volume capacity of V *units*, we need to choose a suitable radius for the container.

i) Show how to make this choice without finding the exact radius.

ii) How to choose the radius if the volume V may vary from one liter to two liters?

Solution: Denote by h and r the height and the radius of the can respectively. Then, the area and the volume of the can are given by

$$\text{area} = A = 2\pi r^2 + 2\pi rh, \qquad \text{volume} = V = \pi r^2 h.$$

i) * The area can be expressed as a function of r and the problem is reduced to find $r \in (0, +\infty)$ for which A is minimum:

$$\begin{cases} \text{minimize } A = A(r) = 2\pi r^2 + \dfrac{2V}{r} & \text{over the set } S \\ S = (0, +\infty) = \{r \in \mathbb{R} \quad / \quad r > 0\}. \end{cases}$$

Note that the set S, as shown in Figure 1.1, is an open unbounded interval of $\mathbb{R}$.

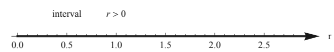

FIGURE 1.1: $S = (0, +\infty) \subset \mathbb{R}$

** We can also express the problem as follows:

$$\begin{cases} \text{minimize } A(r,h) = 2\pi r^2 + 2\pi rh & \text{over the set } S \\ S = \{(r,h) \in \mathbb{R}^+ \times \mathbb{R}^+ \quad / \quad \pi r^2 h = V\}. \end{cases}$$

Here, the set S is a curve in $\mathbb{R}^2$ and is illustrated by Figure 1.2 below:

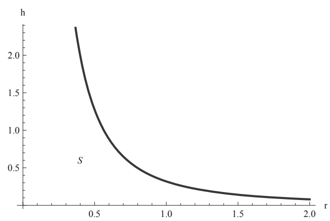

FIGURE 1.2: S is a curve $h = \pi^{-1}/r^2$ in the plane (V=1 liter)

ii) In the case, we allow more possibilities for the volume, for example $1 \leqslant V \leqslant 2$, then we can formulate the problem as a two dimensional problem

$$\begin{cases} \text{minimize } A(r,h) = 2\pi r^2 + 2\pi rh & \text{over the set } S \\ \\ S = \{(r,h) \in \mathbb{R}^+ \times \mathbb{R}^+ \quad / \quad \dfrac{1}{\pi r^2} \leqslant h \leqslant \dfrac{2}{\pi r^2}\}. \end{cases}$$

The set S is the plane region, in the first quadrant, between the curves $h = \dfrac{1}{\pi r^2}$ and $h = \dfrac{2}{\pi r^2}$ (see Figure 1.3).

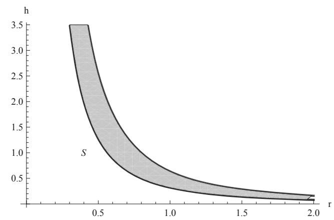

FIGURE 1.3: S is a plane region between two curves

A three dimensional formulation of the same problem is

$$\begin{cases} \text{minimize } A(r,h,V) = 2\pi r^2 + \dfrac{2V}{r} & \text{over the set } S \\ \\ S = \{(r,h,V) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \quad / \quad \pi r^2 h = V, \quad 1 \leqslant V \leqslant 2\} \end{cases}$$

where, the set $S \subset \mathbb{R}^3$ is the part of the surface $V = \pi r^2 h$ located between the planes $V = 1$ and $V = 2$ in the first octant; see Figure 1.4.

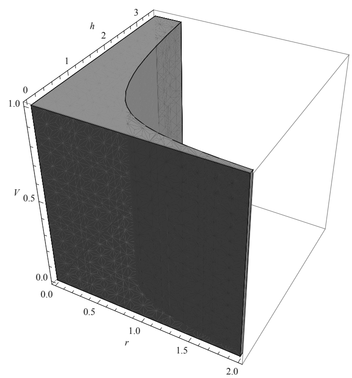

FIGURE 1.4: S is a surface in the space

Example 2. ***Too many variables and linear inequalities.***

Diet Problem. * One can buy four types of aliments where the nutritional content per unit weight of each food and its price are shown in Table 1.1 [5]. The diet problem consists of obtaining, at the minimum cost, at least twelve calories and seven vitamins.

	$type1$	$type2$	$type3$	$type4$
$calories$	2	1	0	1
$vitamins$	3	4	3	5
$price$	2	2	1	8

TABLE 1.1: A diet problem with four variables

Solution: Let u_i be the weight of the food of type i. The total price of the four aliments consumed is given by the relation

$$2u_1 + 2u_2 + u_3 + 8u_4 = f(u_1, u_2, u_3, u_4).$$

To ensure that at least twelve calories and seven vitamins are included, we can express these conditions by writing

$$2u_1 + u_2 + u_4 \geqslant 12 \qquad \text{and} \qquad 3u_1 + 4u_2 + 3u_3 + 5u_4 \geqslant 7.$$

Hence, the problem would be

$$\begin{cases} \text{minimize } f(u_1, u_2, u_3, u_4) \quad \text{ over the set } S = \Big\{(u_1, u_2, u_3, u_4) \in \mathbb{R}^4 : \\ \quad 2u_1 + u_2 + u_4 \geqslant 12, \quad 3u_1 + 4u_2 + 3u_3 + 5u_4 \geqslant 7\Big\}. \end{cases}$$

** The above problem is rendered more complex if more factors (fat, proteins) and types of food (steak, potatoes, fish, ...) were to be considered. For example, from Table 1.2, we deduce that the total price of the seven

	$type1$	$type2$	$type3$	$type4$	$type5$	$type6$	$type7$
$protein$	3	1	2	7	8	5	10
fat	0	1	0	8	15	10	6
$calories$	2	1	0	1	5	7	9
$vitamins$	3	4	3	5	1	2	5
$price$	2	2	1	8	12	10	8

TABLE 1.2: A diet problem with seven variables

aliments consumed is

$$2u_1 + 2u_2 + u_3 + 8u_4 + 12u_5 + 10u_6 + 8u_7 = p(u_1, u_2, u_3, u_4, u_5, u_6, u_7).$$

To ensure that at least twelve calories, seven vitamins, twenty proteins are included, and less than fifteen fats are consumed, the problem would be formulated as

$$\begin{cases} \text{minimize } p(u_1, u_2, u_3, u_4, u_5, u_6, u_7) \quad \text{over the set} \\ S = \Big\{(u_1, u_2, u_3, u_4, u_5, u_6, u_7) \in \mathbb{R}^7 : \\ \qquad 3u_1 + u_2 + 2u_3 + 7u_4 + 8u_5 + 5u_6 + 10u_7 \geqslant 20 \\ \qquad u_2 + 8u_4 + 15u_5 + 10u_6 + 6u_7 \leqslant 15 \\ \qquad 2u_1 + u_2 + u_4 + 5u_5 + 7u_6 + 9u_7 \geqslant 12 \\ \qquad 3u_1 + 4u_2 + 3u_3 + 5u_4 + u_5 + 2u_6 + 5u_7 \geqslant 7.\Big\} \end{cases}$$

Example 3. ***Too many variables and nonlinearities.***

* A company uses x units of capital and y units of labor to produce $x\,y$ units of a manufactured good. Capital can be purchased at 3$/ unit and labor can be purchased at 2$/ unit. A total of 6$ is available to purchase capital and labor. How can the firm maximize the quantity of the good that can be manufactured?

Solution: We need to maximize the quantity $x\,y$ on the set of points (see Figure 1.5)

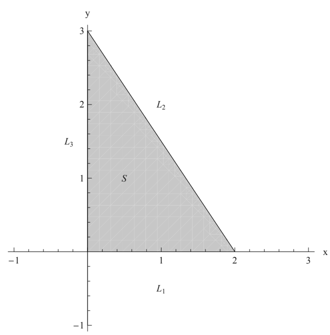

FIGURE 1.5: S is a triangular region in the plane

$$S = \{(x, y) \in \mathbb{R}^2 : \quad 3x + 2y \leqslant 6, \quad x \geqslant 0, \quad y \geqslant 0\}.$$

The set S is the triangular plane region bounded by the sides L_1, L_2 and L_3, defined by: $\quad L_1 = \{(x, 0),\ 0 \leqslant x \leqslant 2\}$,

$$L_2 = \{(0, y),\ 0 \leqslant y \leqslant 3\}, \qquad L_3 = \{(x, (6 - 3x)/2),\ 0 \leqslant x \leqslant 2\}.$$

Here, the objective function $f(x, y) = xy$ is nonlinear and the set S is described by linear inequalities.

** Such a model may work for a certain production process. However, it may not reflect the situation as other factors involved in the production process cannot be ignored. Therefore, new models have to be considered. For Example [7]:

- The Canadian manufacturing industries for 1927 is estimated by:

$$P(l, k) = 33 l^{0.46} k^{0.52}$$

  where P is product, l is labor and k is capital.

- The production P for the dairy farming in Iowa (1939) is estimated by:

$$P(A, B, C, D, E, F) = A^{0.27} B^{0.01} C^{0.01} D^{0.23} E^{0.09} F^{0.27}$$

  where A is land, B is labor, C is improvements, D is liquid assets, E is working assets and F is cash operating expenses.

Each of these nonlinear production function P is optimized on a suitable set S that describes well the elements involved.

As seen above, the main purpose, of this study, is to find a solution to the following optimization problems

$$\text{find } u \in S \qquad \text{such that} \qquad f(u) = \min_S f(v)$$

or

$$\text{find } u \in S \qquad \text{such that} \qquad f(u) = \max_S f(v)$$

where $f : S \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ is a given function and S a given subset of $\mathbb{R}^n$.

It is obvious that establishing existence and uniqueness results of the extreme points, depends on properties satisfied by the set S and the function f. So, we need to know some categories of subsets in $\mathbb{R}^n$ as well as some calculus on multi-variable functions. But, first look at the following remark:

Remark 1.1.1 *The extreme point may not exist on the set S. In our study, we will explore the situations where $\min_S f$ and $\max_S f$ are attained in S.*

For example

$$\min_{(0,1)} f(x) = x^2 \qquad\qquad \text{does not exist.}$$

Indeed, suppose there exists $x_0 \in (0,1)$ such that $f(x_0) = \min_{(0,1)} f(x)$. Then,

$$0 < \frac{x_0}{2} < x_0 \qquad \Longrightarrow \qquad \frac{x_0}{2} \in (0,1)$$

$$f \text{ is a strictly increasing function on } (0,1) \quad \Longrightarrow \quad f(\frac{x_0}{2}) < f(x_0),$$

which contradicts the fact that x_0 is a minimum point of f on $(0,1)$. However, we remark that

$$f(x) > 0 \qquad \forall x \in (0,1).$$

To include these limit cases, usually, instead of looking for a minimum or a maximum, we look for

$$\inf_S f(x) = \inf\{f(x) : x \in S\} \qquad \text{and} \qquad \sup_S f(x) = \sup\{f(x) : x \in S\}$$

where $\inf E$ and $\sup E$ of a nonempty subset E of $\mathbb{R}$ are defined by [2]

$$\sup E = \text{ the least number greater than or equal to all numbers in } E$$

$$\inf E = \text{ the greatest number less than or equal to all numbers in } E.$$

If E is not bounded below, we write $\inf E = -\infty$. If E is not bounded above, we write $\sup E = +\infty$. By convention, we write $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

For the previous example, we have

$$\inf_{(0,1)} x^2 = 0, \qquad\qquad \text{and} \qquad\qquad \sup_{(0,1)} x^2 = 1.$$

1.2 Particular Subsets of $\mathbb{R}^n$

We list here the main categories of sets that we will encounter and give the main tools that allow their identification easily. Even though the purpose is not a topological study of these sets, it is important to be aware of the precise definitions and how to apply them accurately [18], [13].

Open and Closed Sets

In one dimension, the distance between two real numbers x and y is measured by the absolute value function and is given by

$$d(x, y) = |x - y|.$$

d satisfies, for any x, y, z, the properties

$$\begin{array}{ll} d(x,y) \geqslant 0 & d(x,y) = 0 \iff x = y \\ d(y,x) = d(x,y) & \text{symmetry} \\ d(x,z) \leqslant d(x,y) + d(y,z) & \text{triangle inequality.} \end{array}$$

These three properties induce on $\mathbb{R}$ a metric topology where a set $\mathcal{O}$ is said to be **open** if and only if, at each point $x_0 \in \mathcal{O}$, we can insert a **small interval** centered at x_0 that remains included in $\mathcal{O}$, that is,

$$\mathcal{O} \quad \text{is open} \quad \iff \quad \forall x_0 \in \mathcal{O} \quad \exists \epsilon > 0 \quad \text{such that} \quad (x_0 - \epsilon, x_0 + \epsilon) \subset \mathcal{O}.$$

In higher dimension, these tools are generalized as follows:

The distance between two points $x = (x_1, \cdots, x_n)$ and $y = (y_1, \cdots, y_n)$ is measured by the quantity

$$d(x, y) = \|x - y\| = \sqrt{(x_1 - y_1)^2 + \ldots + (x_n - y_n)^2}.$$

d is called the Euclidean distance and satisfies the three properties above. A set $\mathcal{O} \subset \mathbb{R}^n$ is said to be open if and only if, at each point $x_0 \in \mathcal{O}$, we can insert a small **ball**

$$B_\epsilon(x_0) = \{x \in \mathbb{R}^n \ : \ \|x - x_0\| < \epsilon\}$$

centered at x_0 with ϵ that remains included in $\mathcal{O}$, that is,

$$\mathcal{O} \quad \text{is } \textbf{open} \qquad \Longleftrightarrow \qquad \forall x_0 \in \mathcal{O} \quad \exists \epsilon > 0 \quad \text{such that} \quad B_\epsilon(x_0) \subset \mathcal{O}.$$

The point x_0 is said to be **an interior point** to $\mathcal{O}$.

Example 1. As n varies, the ball takes different shapes; see Figure 1.6.

$n = 1 \qquad a \in \mathbb{R} \qquad B_r(a) = (a - r, a + r) \quad : \quad$ an open interval

$n = 2 \quad a = (a_1, a_2) \quad B_r(a) = \{(x_1, x_2) : \ (x_1 - a_1)^2 + (x_2 - a_2)^2 < r^2\}$: an open disk

$n = 3 \qquad a = (a_1, a_2, a_3)$
$B_r(a) = \{(x_1, x_2, x_3) : \ (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 < r^2\}$:
set of points delimited by the sphere centered at a with radius r

$n > 3 \qquad a = (a_1, \dots, a_3) \qquad B_r(a)$ is the set of points delimited by the hyper sphere of points x satisfying $d(a, x) = r$.

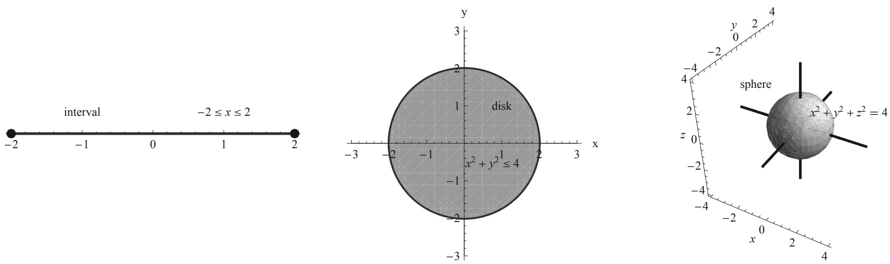

FIGURE 1.6: Shapes of balls in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$

Using the distance d, we define

Definition 1.2.1 *Let S be a subset of $\mathbb{R}^n$.*

– $\overset{\circ}{S}$ *is the **interior** of S, the set of all interior points of S.*

– S *is a **neighborhood** of a if a is an interior point of S.*

– S *is a* ***closed*** *set* $\iff$ C^S *is open.*

– ∂S *is the* ***boundary*** *of* S*, the set of boundary points of* S*, where*

$$x_0 \in \partial S \iff \forall r > 0,\ B_r(x_0) \cap \Big(S\Big) \neq \emptyset \quad \textit{and} \quad B_r(x_0) \cap \Big(C^S\Big) \neq \emptyset.$$

– $\overline{S} = S \cup \partial S$ *is the* ***closure*** *of* S.

– S *is* ***bounded*** $\iff$ $\exists\, M > 0$ *such that* $\|x\| \leqslant M \quad \forall x \in S$.

– S *is* ***unbounded*** *if it is not bounded.*

Example 2. For the sets, $S_1 = [-2, 2] \subset \mathbb{R}$

$S_2 = \{(x, y) :\ x^2 + y^2 \leqslant 4\} \subset \mathbb{R}^2, \qquad S_3 = \{(x, y, z) :\ x^2 + y^2 + z^2 < 4\} \subset \mathbb{R}^3,$

we have

S	$\overset{\circ}{S}$	∂S	$\overline{S}$
S_1	$(-2, 2)$	$\{-2, 2\}$	S_1
S_2	$B_2(0)$	$C_2(0)$: circle	S_2
S_3	$S_3 = B_2(0)$	$S_2(0)$: sphere	$S_3 \cup S_2(0)$

where

$$C_2(0) = \{(x, y) :\ x^2 + y^2 = 4\}, \qquad S_2(0) = \{(x, y, z) :\ x^2 + y^2 + z^2 = 4\}.$$

We have the following properties:

Remark 1.2.1 – $\mathbb{R}^n$ *and* $\emptyset$ *are open and closed sets*

– *The union (resp. intersection) of arbitrary open (resp. closed) sets is open (resp. closed).*

– *The finite intersection (resp. union) of open (resp. closed) sets is open (resp. closed).*

– S *is open* $\iff$ $S = \overset{\circ}{S}$.

– S *is closed* $\iff$ $S = \overline{S}$.

– If f is continuous on an open subset $\Omega \subset \mathbb{R}^n$ (see Section 1.3), then

$$f^{-1}\Big((-\infty, a]\Big) = [f \leqslant a], \quad [f \geqslant a], \quad [f = a] \text{ are closed sets in } \mathbb{R}^n$$

$$f^{-1}\Big((-\infty, a)\Big) = [f < a], \qquad [f > a] \qquad \text{are open sets in } \mathbb{R}^n.$$

Example 3. Sketch the set S in the xy-plane and determine whether it is open, closed, bounded or unbounded. Give $\overset{\circ}{S}$, ∂S and $\overline{S}$.

$$S = \{(x, y) : \quad x \geqslant 0, \quad y \geqslant 0, \quad xy \geqslant 1\}$$

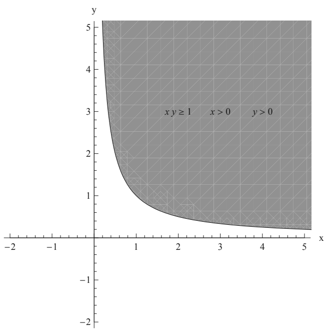

FIGURE 1.7: An unbounded closed subset of $\mathbb{R}^2$

$*$ Note that the set S, sketched in Figure 1.7, doesn't contain the points on the x and y axis. So

$$S = \{(x, y) : \ x > 0, \quad y > 0, \quad xy \geqslant 1\}$$

and can be described using the continuous function $f : (x, y) \longmapsto xy$ on the open set $\Omega = \{(x, y) : \ x > 0, \ y > 0\}$ as

$$S = \{(x, y) \in \Omega : f(x, y) \geqslant 1\} = f^{-1}\Big([1, +\infty)\Big).$$

Therefore, S is a closed subset of $\mathbb{R}^2$. Thus $\overline{S} = S$.

$**$ The set is unbounded since it contains the points $(x(t), y(t)) = (t, t)$ for $t \geqslant 1 \quad (xy = t.t = t^2 \geqslant 1)$ and

$$\|(x(t), y(t))\| = \|(t, t)\| = \sqrt{t^2 + t^2} = \sqrt{2}t \longrightarrow +\infty \qquad \text{as} \qquad t \longrightarrow +\infty.$$

$*\,*\,*$ We have

$$\overset{\circ}{S} = \{(x, y) : \; x > 0, \; y > 0, \; xy > 1\}$$

the region in the 1st quadrant above the hyperbola $y = \dfrac{1}{x}$

$$\partial S = \{(x, y) : \quad x > 0, \quad y > 0, \quad xy = 1\}$$

the arc of the hyperbola in the 1st quadrant.

Example 4. A person can afford any commodities $x \geqslant 0$ and $y \geqslant 0$ that satisfies the budget inequality $x + 3y \leqslant 7$.

Sketch the set S described by these inequalities in the xy-plane and determine whether it is open, closed, bounded or unbounded. Give $\overset{\circ}{S}$, ∂S and $\overline{S}$.

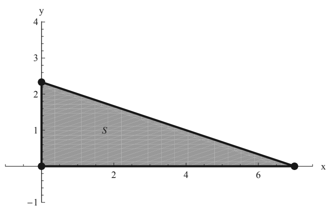

FIGURE 1.8: Closed set as intersection of three closed sets of $\mathbb{R}^2$

$*$ Figure 1.8 shows that S is the triangular region formed by all the points in the first quadrant below the line $x + 3y = 7$:

$$S = \{(x, y) : \quad x + 3y \leqslant 7, \quad x \geqslant 0, \quad y \geqslant 0\}$$

and can be described using the continuous functions

$$f_1 : (x, y) \longmapsto x + 3y, \qquad f_2 : (x, y) \longmapsto x, \qquad f_3 : (x, y) \longmapsto y$$

on $\mathbb{R}^2$ as

$$\begin{aligned} S &= \{(x, y) \in \mathbb{R}^2 : f_1(x, y) \leqslant 7, \quad f_2(x, y) \geqslant 0, \quad f_3(x, y) \geqslant 0\} \\ &= f_1^{-1}\Big((-\infty, 7]\Big) \bigcap f_2^{-1}\Big([0, +\infty)\Big) \bigcap f_3^{-1}\Big([0, +\infty)\Big). \end{aligned}$$

Therefore, S is a closed subset of $\mathbb{R}^2$ as the intersection of three closed subsets of $\mathbb{R}^2$. Thus $\overline{S} = S$.

$**$ The set S is bounded since

$$x + 3y \leqslant 7, \qquad x \geqslant 0, \qquad y \geqslant 0 \qquad \Longrightarrow \qquad 0 \leqslant x \leqslant 7, \qquad 0 \leqslant y \leqslant \frac{7}{3}$$

from which we deduce

$$\|(x,y)\| = \sqrt{x^2 + y^2} \leqslant \sqrt{7^2 + \left(\frac{7}{3}\right)^2} = \frac{7}{3}\sqrt{10} \qquad \forall (x,y) \in S.$$

$***$ We have

$\overset{\circ}{S} = \{(x,y) : \ x > 0, \ y > 0, \ x + 3y < 7\}$ the region S excluding its three sides

$\partial S =$ the three sides of the triangular region.

Convex sets

The category of **convex** sets, deals with sets $S \subset \mathbb{R}^n$ where any two points $x, y \in S$ can be joined by a line segment that remains entirely into the set. Such sets are without holes and do not bend inwards. Thus

$$S \quad \text{is convex} \qquad \Longleftrightarrow \qquad (1-t)x + ty \in S \qquad \forall x,\, y \in S \qquad \forall t \in [0,1].$$

We have the following properties:

Remark 1.2.2 – *$\mathbb{R}^n$ and $\emptyset$ are convex sets*

– *A finite intersection of convex sets is a convex set.*

__**Example 5**__. "*Well known convex sets*" (see Figure 1.9)

$*$ A line segment joining two points x and y is convex. It is described by

$$[x,y] = \{z \in \mathbb{R}^n : \quad \exists t \in [0,1] \quad \text{such that} \quad z = x + t(y-x) = (1-t)x + ty\}.$$

$**$ A line passing through two points x_0 and x_1 is convex. It is described by

$$\mathcal{L} = \{x \in \mathbb{R}^n : \quad \exists t \in \mathbb{R} \quad \text{such that} \quad x = x_0 + t(x_1 - x_0)\}.$$

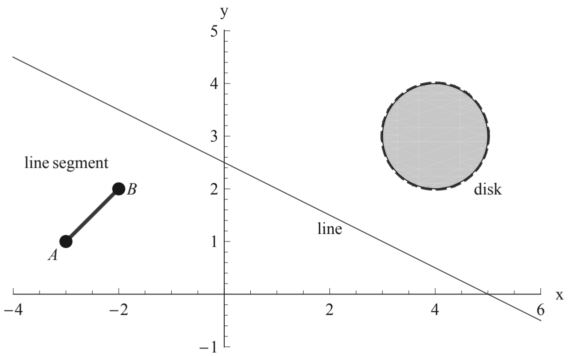

FIGURE 1.9: Convex sets in $\mathbb{R}^2$

$* * *$ A ball $B_r(x_0) = \{x \in \mathbb{R}^n : \quad \|x - x_0\| < r\}$ is convex.

Indeed, let a and b in $B_r(x_0)$ and $t \in [0,1]$, we have

$$\|[(1-t)a + tb] - x_0\| = \|(1-t)(a - x_0) + t(b - x_0)\|$$

$$\leqslant \|(1-t)(a - x_0)\| + \|t(b - x_0)\| = |1-t|\|a - x_0\| + |t|\|b - x_0\|$$

$$< |1-t|r + |t|r = r \quad \text{since} \quad \|a - x_0\| < 1 \ \text{ and } \ \|b - x_0\| < 1.$$

Hence $(1-t)a + tb \in B_r(x_0)$ for any $t \in [0,1]$; that is, $[a,b] \subset B_r(x_0)$.

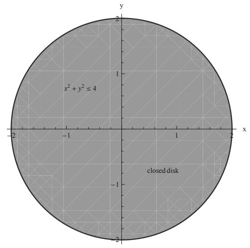

FIGURE 1.10: A closed ball is convex

$* * **$ A closed ball $\overline{B_r(x_0)} = \{x \in \mathbb{R}^n : \ \|x - x_0\| \leqslant r\}$ is convex.

For example, in the plane, the set in Figure 1.10, defined by

$$\{(x,y) : \quad x^2 + y^2 \leqslant 4\} = \overline{B_2((0,0))} \qquad \text{is convex.}$$

The set is the closed disk with center $(0,0)$ and radius 2. It is closed since it includes its boundary points located on the circle with center $(0,0)$ and radius 2. This set is bounded since $\|(x,y)\| \leqslant 2 \quad \forall (x,y) \in \overline{B_2((0,0))}$.

Example 6. "*Convex sets described by linear expressions*"

$*$ For $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $\; b \in \mathbb{R}$, the set of points

$$x = (x_1, \ldots, x_n) \in \mathbb{R}^n : \quad a_1x_1 + a_2x_2 + \ldots + a_nx_n = a.x = b$$

is convex and called hyperplane.

Indeed, consider x^1, x^2 in the hyperplane and $t \in [0,1]$, then

$$a.[(1-t)x^1 + tx^2] = (1-t)a.x^1 + ta.x^2 = (1-t)b + tb = b$$

thus $(1-t)x^1 + tx^2$ belongs to the hyperplane.

As illustrated in Figure 1.11, the graph of an hyperplane is reduced to the point $x_1 = b/a_1$ when $n = 1$, to the line $a_1x_1 + a_2x_2 = b$ in the plane when $n = 2$, and to the plane $a_1x_1 + a_2x_2 + a_3x_3 = b$ in the space when $n = 3$.

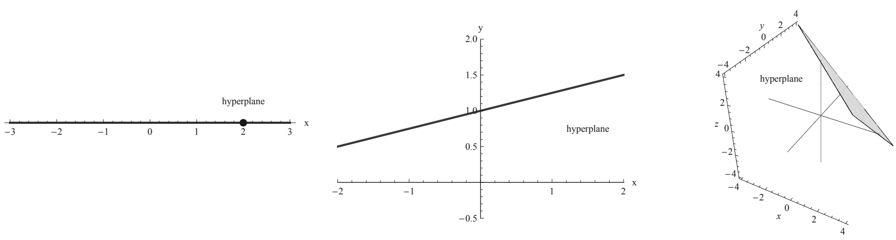

FIGURE 1.11: Hyperplane in $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$

$**$ The set of points in $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ defined by a linear inequality

$$a_1x_1 + a_2x_2 + \ldots + a_nx_n = a.x \leqslant b \qquad \text{(resp. } \geqslant, <, >) \qquad \text{is convex.}$$

Indeed, as above, consider x^1, x^2 in the region $[a.x \leqslant b]$ and $t \in [0,1]$, then

$$a.x^1 \leqslant b \quad \Longrightarrow \quad (1-t)a.x^1 \leqslant (1-t)b \qquad \text{since} \quad (1-t) \geqslant 0$$

$$a.x^2 \leqslant b \quad \Longrightarrow \quad ta.x^2 \leqslant tb \qquad \text{since} \quad t \geqslant 0$$

Adding the two inequalities, we get

$$a.[(1-t)x^1 + tx^2] = (1-t)a.x^1 + ta.x^2 \leqslant (1-t)b + tb = b$$

thus $(1-t)x^1 + tx^2$ belongs to the region $[a.x \leqslant b]$.

The set $a.x \leqslant b$ describes the region of points located below the hyperplane $a.x = b$.

$* * *$ A set of points in $\mathbb{R}^n$ described by linear equalities and inequalities is convex as it can be seen as the intersection of convex sets described by equalities and inequalities.

For example, in Figure 1.12, the set

$$S = \{(x,y) : \ 2x+3y \leqslant 19, \ -3x+2y \leqslant 4, \ x+y \leqslant 8, \ 0 \leqslant x \leqslant 6, \ x+6y \geqslant 0\}$$

can be described as $S = S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5 \cap S_6$ where

$$S_1 = \{(x,y) \in \mathbb{R}^2 : \ x+6y \geqslant 0\} \quad S_2 = \{(x,y) \in \mathbb{R}^2 : \ x \leqslant 6\}$$

$$S_3 = \{(x,y) \in \mathbb{R}^2 : \ x+y \leqslant 8\} \quad S_4 = \{(x,y) \in \mathbb{R}^2 : \ 2x+3y \leqslant 19\}$$

$$S_5 = \{(x,y) \in \mathbb{R}^2 : \ -3x+2y \leqslant 4\} \quad S_6 = \{(x,y) \in \mathbb{R}^2 : \ x \geqslant 0\}.$$

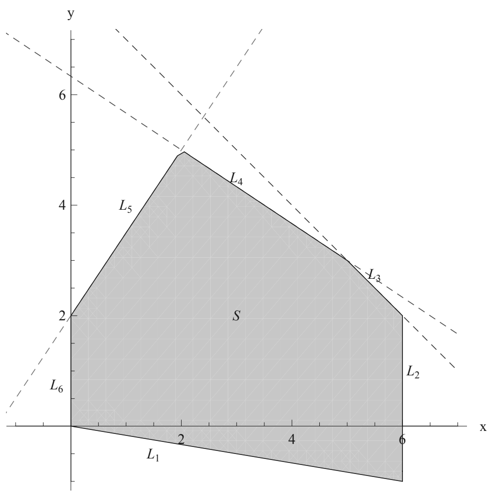

FIGURE 1.12: A convex set described by linear inequalities

S is the region of the plan xy, bounded by the lines

$$L_1: \ x + 6y = 0 \qquad L_2: \ x = 6, \qquad L_3: \ x + y = 8,$$

$$L_4: \ 2x + 3y = 19, \qquad L_5: \ -3x + 2y = 4 \qquad L_6: x = 0.$$

Often, such sets are described using matrices and vectors;

$$S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : \quad \begin{bmatrix} 2 & 3 \\ -3 & 2 \\ 1 & 1 \\ 1 & 0 \\ -1 & -6 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leqslant \begin{bmatrix} 19 \\ 4 \\ 8 \\ 6 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Example 7. "*Well-known non convex sets*"

$*$ The hyper-sphere (see Figure 1.13 for an illustration in the plane)

$$\partial B_r(x^*) = \{x \in \mathbb{R}^n : \quad \|x - x_0\| = r\} \qquad \text{is not convex.}$$

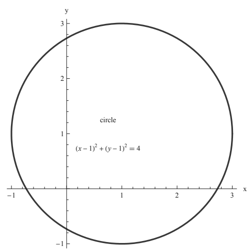

FIGURE 1.13: Circle $\partial B_2((1,1))$ is not convex

Indeed, we have

$$(x_1^*, \dots, x_n^* \pm r) \in \partial B_r(x^*) \quad \text{ since } \quad \|(0, \dots, \pm r)\| = r$$

$$\left\| \frac{1}{2}(x_1^*, \dots, x_n^* + r) + (1 - \frac{1}{2})(x_1^*, \dots, x_n^* - r) \, - \, x^* \right\|$$

$$= \left\| \frac{1}{2}(2x_1^*, \dots, 2x_n^* + r - r) - x^* \right\| = \|x^* - x^*\| = 0 \neq r$$

$$\Longrightarrow \quad \frac{1}{2}(x_1^*, \dots, x_n^* + r) + (1 - \frac{1}{2})(x_1^*, \dots, x_n^* - r) = x^* \notin \partial B_r(x^*).$$

$**$ The domain located outside the hyper-sphere, described by

$$S = \{x \in \mathbb{R}^n : \quad \|x - x^*\| > r\} = \mathbb{R}^n \setminus \overline{B_r(x^*)} \qquad \text{is not convex.}$$